Quasi-star-free Languages on Infinite Words*

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Abstract

Quasi-star-free languages were first introduced and studied by Barrington, Compton, Straubing and Thérien within the context of circuit complexity in 1992, and their connections with propositional linear temporal logic were established by Ésik and Ito recently. While these results are all for finite words, in this paper we consider the languages on infinite words.

1 Introduction

Characterizations of different subclasses of regular languages have been a constantly active research area since Büchi characterized regular languages by monadic second order logic in [3]. One of the most important characterizations among them is the characterization of star free languages: in [11, 17, 9, 7, 13, 19, 18, 4], star free languages on finite and infinite words were characterized by aperiodic monoids, monadic first order logic and linear temporal logic.

Quasi-star-free languages were first studied by Barrington, Compton, Straubing and Thérien in [2]. Their motivation was to characterize the regular languages that can be recognized by constant-depth Boolean circuits using OR,AND and NOT gates(AC^0). They found that these languages are precisely the quasi-star-free languages. And they give a characterization in terms of quasi-aperiodic semigroups and in terms of first order logic FO[C] which uses only the numerical predicates $x < y$ and $x \equiv r (\mathrm{mod}\ d)$. Recently, Ésik and Ito proved in [5] that FO[C] and propositional linear temporal logic with cyclic counting(LTL[C])

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have the same expressive power. While these results are all for finite words, we extend them to the case of infinite words in this paper.

This paper is organized as follows. In section 2 we give some preliminaries about regular languages on finite and infinite words. Then in section 3, we give some definitions of quasi-star-free languages on finite words($\mathrm{QSF^F}$), and summarize the results of $\mathrm{QSF^F}$ in [2, 5]. In section 4, we define quasi-star-free languages on infinite words ($\mathrm{QSF^I}$), and extend the results of $\mathrm{QSF^F}$ to $\mathrm{QSF^I}$. Finally in section 5, we give some conclusions and remarks on this paper.

2 Preliminaries

2.1 Regular languages on finite words

In this subsection, at first we present some basic facts of semigroups and formal languages on finite words (cf. [12, 6, 14, 10] for more information), then after recalling the definitions of monadic first order logic (FO[<]) and linear temporal logic(LTL) interpreted on finite words, we introduce the classical results of star free languages on finite words.

Let A be a finite alphabet, and $L \subseteq A^*$ be regular.

2.1.1 Monoids and formal languages on finite words

Let M be a finite monoid. We say that morphism $\phi : A^* \to M$ recognizes L if there is $X \subseteq M$ such that $L = X\phi^{-1}$. And we say that monoid M recognizes L if there is a morphism $\phi : A^* \to M$ recognizing L. Moreover we say that congruence $\approx$ on A^* recognizes L if the natural morphism $\phi : A^* \to A^*/\approx$ recognizes L.

The syntactic congruence of L, $\approx_L$, is defined by: $u \approx_L v$ iff ($xuy \in L$ iff $xvy \in L$ for all $x, y \in A^*$); the syntactic monoid of L, $M(L)$, is defined by the quotient monoid $A^*/\approx_L$; and the syntactic morphism of L, $\eta_L : A^* \to M(L)$, is defined by $u\eta_L = [u]$, where $[u]$ denotes the equivalence class of $\approx_L$ containing u. Syntactic congruence is the coarsest congruence of A^* recognizing L, i.e. for any congruence $\approx$ recognizing L, $u \approx v$ implies $u \approx_L v$ for all $u, v \in A^*$.

A morphism $\phi : A^* \to M$ recognizes L iff there is a morphism $\theta : Im(\phi) \to M(L)$ (where $Im(\phi)$ is the image of ϕ) such that for all $u \in A^*$, $u(\phi\theta) = u\eta_L$. Furthermore, a morphism $\phi : A^* \to M$ recognizes L iff there are morphisms $\phi' : A^* \to M'$ and $\theta : Im(\phi) \to M'$ such that ϕ' recognizes L and for all $u \in A^*$, $u(\phi\theta) = u\phi'$.

L is star free if L can be constructed from singleton languages $\{a\}(a \in A)$ and the language A^* by finite applications of operations of union, complementation, and concatenation.

L is noncounting if there is some $n_0 \in N$ satisfying that for all $n \geq n_0$, $xy^nz \in L$ iff $xy^{n+1}z \in L$ for all $x, y, z \in A^*$.

A monoid M is aperiodic if there is some $n_0 \in N$ satisfying that for all $n \geq n_0$, $m^n = m^{n+1}$ for all $m \in M$.

L is aperiodic if $M(L)$ is aperiodic. It is easy to show that L is aperiodic iff there is an aperiodic monoid M recognizing L.

It is not hard to show that L is noncounting iff L is aperiodic. In the remainder of this paper, we don't distinguish between the "noncounting" and "aperiodic" properties of regular languages on finite words.

2.1.2 First order logic and linear temporal logic on finite words

Let FO[<] denote first order logic on words with binary predicate $<$ and unary predicates $P_a(a \in A)$. The formulas of FO[<] are defined by the following rules:

$$\varphi := P_a(x) \mid x < y \mid \varphi_1 \vee \varphi_2 \mid \neg\psi \mid (\exists x)\psi$$

The semantics of FO[<] are defined as follows: let X be a variable set and φ be a formula with free variables in X; $u \in A^*$ and $\eta : X \to \{0, ..., |u|\}$, i.e., η maps variables in X to "positions" in u.

- $(u, \eta) \models P_a(x)$, if $u[|x|] = a$, where $u[|x|]$ is the letter of u at position $x\eta$(the first position is 0, the last position is $|u|$, and by convention the letter at position $|u|$ is ε);
- $(u, \eta) \models x < y$, if $x\eta < y\eta$;
- $(u, \eta) \models \varphi_1 \vee \varphi_2$, if $(u, \eta) \models \varphi_1$ or $(u, \eta) \models \varphi_2$;
- $(u, \eta) \models \neg\psi$, if not $(u, \eta) \models \psi$;
- $(u, \eta) \models (\exists x)\psi$,if there exists a function $\eta\prime : X \to \{0, ..., |u|\}$, which agrees with η on $X - \{x\}$ and possibly differs from η on x, such that $(u, \eta\prime) \models \psi$.

Let φ be an FO[<] sentence and $u \in A^*$. We write $u \models \varphi$ if there is an $\eta : X \to \{0, ..., |u|\}$ such that $(u, \eta) \models \varphi$.

Remark 2.1 *The semantics of FO[<] defined in [5] had a subtle inaccuracy: the assignments of variables were defined by function* $\lambda : X \to [|u|]$*, where* $[|u|] = \{0, ..., |u| - 1\}$*. But then for the empty string* ε*, the assignments would become into* $\lambda : X \to \emptyset$*, since* $[|\varepsilon|] = [0] = \emptyset$*.*

We avoid the accuracy by defining the assignments as $\eta : X \to \{0, ..., |u|\}$*, and thus formulas of FO[<] can be interpreted on the empty string* ε*.* □

It is natural to define the boolean operations "$\wedge$", "$\to$",etc. in a standard way. Here we introduce several other abbreviations for FO[<]: $Last(x)$ for $\forall y(\neg(x < y))$; $True$ for $\varphi \vee \neg\varphi$, where φ is a fixed sentence; and $False$ for $\neg True$.

A language $L \subseteq A^*$ is definable in FO[<] if there is an FO[<] sentence φ such that for all $u \in A^*$, $u \models \varphi$ iff $u \in L$.

Associate each letter a in A with a propositional constant p_a. Then formulas of linear temporal logic (LTL,[15]) over alphabet A are defined by the following rules:

$$\varphi := p_a \mid \varphi_1 \vee \varphi_2 \mid \neg\psi \mid X\psi \mid \varphi_1 U \varphi_2$$

The semantics of LTL formulas on finite words are defined as follows: Let φ be an LTL formula, $u \in A^*$. Denote the suffix of u starting from the i-th position (the first position is 0) as u^i, where $0 \leq i \leq |u|$, and the suffix starting from the $|u|$-th position is empty string ε.

- $u \models p_a$, if $u = av$, for some $v \in A^*$;
- $u \models \varphi_1 \vee \varphi_2$, if $u \models \varphi_1$ or $u \models \varphi_2$;
- $u \models \neg\varphi_1$, if not $u \models \varphi_1$;
- $u \models X\varphi_1$, if $|u| > 0$ and $u^1 \models \varphi_1$;
- $u \models \varphi_1 U \varphi_2$, if there is $0 \leq i \leq |u|$ such that $u^i \models \varphi_2$ and for all $0 \leq j < i$, $u^j \models \varphi_1$.

We introduce several abbreviations for LTL, let $True \equiv p_a \vee \neg p_a$, where a is any letter in A, and let $False \equiv \neg True$. Moreover, let End denote the formula $\wedge_{a\in A} \neg p_a$, so that for all $u \in A^*$, $u \models End$ iff $u = \varepsilon$.

Remark 2.2 *When interpreted on finite words, the LTL formulas $\neg X\varphi$ and $X\neg\varphi$ are not equivalent while on infinite words they are (See Section 2.2.2 for LTL interpreted on infinite words). For instance, $\varepsilon \models \neg X p_a$ while not $\varepsilon \models X\neg p_a$, where ε is the empty string.* □

A language $L \subseteq A^*$ is LTL definable iff there is an LTL formula φ such that for all $u \in A^*$, $u \models \varphi$ iff $u \in L$.

2.1.3 Classical results of star free languages on finite words

The classical results of star free languages on finite words are summarized in the following proposition:

Proposition 2.3 *Let $L \subseteq A^*$ be regular. The following conditions are equivalent [11, 17, 9, 7, 4]:*

- *L is star free;*
- *L is aperiodic;*
- *$M(L)$ contains no nontrivial group (i.e. contains no subsets which form a nontrivial group under the product of $M(L)$);*
- *L is FO[<] definable;*
- *L is LTL definable.* □

2.2 Regular languages on infinite words

Similar to the case of finite words, in this subsection at first we present some basic facts of semigroup and formal languages on infinite words (cf. [1, 20, 21, 4, 16]), then we interpret monadic first order logic ($FO[<]$) and linear temporal logic (LTL) on infinite words, at last we introduce the classical results of star free languages on infinite words.

Let A be a finite alphabet and $L \subseteq A^\omega$ be regular, i.e., $L = \bigcup_{i=1}^{m} X_i Y_i^\omega$, where $X_i \subseteq A^*$, $Y_i \subseteq A^+$ are regular languages on finite words.

2.2.1 Monoids and formal languages on infinite words

Let M be a finite monoid. L is recognized by morphism $\phi : A^* \to M$ if for all $m, n \in M$, $\left(m\phi^{-1}\right)\left(n\phi^{-1}\right)^\omega \cap L \neq \emptyset$ implies $\left(m\phi^{-1}\right)\left(n\phi^{-1}\right)^\omega \subseteq L$. A monoid M recognizes L iff there is a morphism $\phi : A^* \to M$ recognizing L. Moreover we say that a congruence $\approx$ on A^* recognizes L if the natural morphism $\phi : A^* \to A^*/\approx$ recognizes L.

The syntactic congruence of L, $\approx_L$, is defined by: for all $u, v \in A^*$, $u \approx_L v$ iff for all $x, y, z \in A^*$, ($xuyz^\omega \in L$ iff $xvyz^\omega \in L$) and ($x\left(yuz\right)^\omega \in L$ iff $x\left(yvz\right)^\omega \in L$). The syntactic monoid of L, $M(L)$, is defined by the quotient monoid $A^*/\approx_L$. The syntactic morphism of L, $\eta_L : A^* \to M(L)$, is defined by $u\eta_L = [u]$, where $[u]$ is the equivalence class of $\approx_L$ containing u. Syntactic congruence is the coarsest congruence recognizing L.

Proposition 2.4 *Let $L \subseteq A^\omega$ be regular. A morphism $\phi : A^* \to M$ recognizes L iff there is a morphism $\theta : Im(\phi) \to M(L)$ such that for all $u \in A^*$, $u\phi\theta = u\eta_L$.*

Proof.
"⇒" part:
Define $\theta : Im(\phi) \to M(L)$ as follows:

$$m\theta = u\eta_L, \text{where } u \in A^*, u\phi = m$$

θ is well defined since $u\phi = v\phi$ implies that $u\eta_L = v\eta_L$ (syntactic congruence is the coarsest one).

It is easy to verify that $\phi\theta = \eta_L$
"⇐" part:
It is sufficient to prove that for all $m, n \in Im(\phi)$

$$\phi^{-1}(m)[\phi^{-1}(n)]^\omega \bigcap L \neq \emptyset \text{ implies } \phi^{-1}(m)[\phi^{-1}(n)]^\omega \subseteq L$$

Since $\phi^{-1}(m)[\phi^{-1}(n)]^\omega \bigcap L$ is a nonempty regular language, there is an ultimately periodic ω-word $xy^\omega \in \phi^{-1}(m)[\phi^{-1}(n)]^\omega \bigcap L$. So xy^ω has a decomposition: $w_0 w_1^\omega$ such that

$$w_0 \in \phi^{-1}(m)[\phi^{-1}(n)]^p, w_1 \in [\phi^{-1}(n)]^q \text{ for some } p, q \geq 0$$

It is easy to see that $\phi^{-1}(m)[\phi^{-1}(n)]^\omega \subseteq [w_0\phi\phi^{-1}][w_1\phi\phi^{-1}]^\omega$, thus it is sufficient to prove that $[w_0\phi\phi^{-1}][w_1\phi\phi^{-1}]^\omega \subseteq L$, i.e.,$[w_0\phi\phi^{-1}][w_1\phi\phi^{-1}]^\omega \bigcap \bar{L} = \emptyset$.

To the contrary, suppose that $[w_0\phi\phi^{-1}][w_1\phi\phi^{-1}]^\omega \bigcap \bar{L} \neq \emptyset$.

Since $[w_0\phi\phi^{-1}][w_1\phi\phi^{-1}]^\omega \bigcap \bar{L}$ is regular, then there is an ultimately periodic word $\alpha = \alpha_0\alpha_1^\omega \in [w_0\phi\phi^{-1}][w_1\phi\phi^{-1}]^\omega \cap \bar{L}$.

$\alpha_0\alpha_1^\omega$ has a decomposition $\alpha'_0{\alpha'_1}^\omega$ such that $\alpha'_0 \in w_0\phi\phi^{-1}[w_1\phi\phi^{-1}]^r$ and $\alpha'_1 \in [w_1\phi\phi^{-1}]^s$ for some $r, s \geq 0$.

From the assumption $\phi\theta = \eta_L$ we know that $\alpha'_0\eta_L = \alpha'_0\phi\theta = (w_0{w_1}^r)\phi\theta = (w_0{w_1}^r)\eta_L$, and $\alpha'_1\eta_L = \alpha'_1\phi\theta = ({w_1}^s)\phi\theta = ({w_1}^s)\eta_L$. Thus $w_0{w_1}^r({w_1}^s)^\omega \in L$ iff $\alpha \in L$, i.e., $w_0{w_1}^\omega \in L$ iff $\alpha \in L$, i.e., $xy^\omega \in L$ iff $\alpha \in L$, a contradiction. □

Corollary 2.5 *A morphism $\phi : A^* \to M$ recognizes L iff there are morphisms $\phi' : A^* \to M'$ and $\theta : Im(\phi) \to M'$ such that ϕ' recognizes L and for all $u \in A^*$, $u(\phi\theta) = u\phi'$.* □

L is star free if L can be constructed from the language A^ω by finite applications of operations of union, complementation and concatenation on the left by star free languages of A^*.

L is noncounting if there is $n_0 \in N$ such that for all $n \geq n_0$ and $x, u, y, z \in A^*$, ($xu^nyz^\omega \in L$ iff $xu^{n+1}yz^\omega \in L$) and ($x(yu^nz)^\omega \in L$ iff $x(yu^{n+1}z)^\omega \in L$).

L is aperiodic if its syntactic monoid $M(L)$ is aperiodic. And it is easy to show that L is aperiodic iff it is recognized by an aperiodic monoid.

It is not hard to prove that L is noncounting iff L is aperiodic. In the remainder of this paper, for regular languages on infinite words, we don't distinguish between the "noncounting" and "aperiodic" properties.

2.2.2 First order logic and linear temporal logic on infinite words

FO[<] and LTL formulas can also be interpreted on infinite words.

For FO[<]: Let X be the variable set and φ be a formula with free variables in X; $u \in A^\omega$ and $\eta : X \to N$, i.e., η maps variables in X to "positions" in u.

- $(u, \eta) \models P_a(x)$, if $u[|x|] = a$,where $u[|x|]$ is the $x\eta$th letter of u;
- $(u, \eta) \models x < y$, if $x\eta < y\eta$;
- $(u, \eta) \models \varphi_1 \vee \varphi_2$, if $(u, \eta) \models \varphi_1$ or $(u, \eta) \models \varphi_2$;
- $(u, \eta) \models \neg\psi$, if not $(u, \eta) \models \psi$;
- $(u, \eta) \models (\exists x)\psi$, if there exists a function $\eta\prime : X \to N$, which agrees with η on $X - \{x\}$ and possibly differs from η on x, such that $(u, \eta\prime) \models \psi$.

Let φ be an FO[<] sentence and $u \in A^\omega$. We write $u \models \varphi$ if there is an $\eta : X \to N$ such that $(u, \eta) \models \varphi$.

For LTL: Let φ be an LTL formula, $u \in A^\omega$. Denote the suffix of u starting from i-th position (the first position is 0) as u^i, then

- $u \models p_a$, if $u = av$, for some $v \in A^\omega$;
- $u \models \varphi_1 \vee \varphi_2$, if $u \models \varphi_1$ or $u \models \varphi_2$;
- $u \models \neg\varphi_1$, if not $u \models \varphi_1$;
- $u \models X\varphi_1$, if $u^1 \models \varphi_1$;
- $u \models \varphi_1 U \varphi_2$, if there is $i \geq 0$ such that $u^i \models \varphi_2$ and for all $0 \leq j < i$, $u^j \models \varphi_1$.

L is definable in FO[<] if there is an FO[<] sentence φ such that for all $u \in A^\omega$, $u \models \varphi$ iff $u \in L$.

L is definable in LTL if there is an LTL formula φ such that for all $u \in A^\omega$, $u \models \varphi$ iff $u \in L$.

2.2.3 Classical results of star free languages on infinite words

Similar to the finite words, there are the following classical results of star free languages on infinite words.

Proposition 2.6 *Let $L \subseteq A^\omega$ be regular. The following conditions are equivalent [13, 19, 18, 9, 7]:*

- *L is star free;*
- *L is aperiodic;*
- *$M(L)$ contains no nontrivial group;*
- $L = \bigcup_{i=1}^{m} X_i Y_i^\omega$,*where $X_i \subseteq A^*$, $Y_i \subseteq A^+$ are star free and $Y_i Y_i \subseteq Y_i$;*
- *L is FO[<] definable;*
- *L is LTL definable.* □

3 Quasi-star-free languages on finite words

3.1 Quasi-star-free languages on finite words

Definition 3.1 *Let $L \subseteq A^*$ be regular. L is* quasi-star-free *if there is some $d \geq 1$ such that L can be constructed from singleton languages $\{a\}(a \in A)$ and the language $(A^d)^*$ by finite applications of operations of union, complementation, and concatenation.* □

If $L \subseteq A^*$ is star free, it is quasi-star free as well.

The family of quasi-star-free languages on finite words is denoted by $\mathrm{QSF}^{\mathrm{F}}$.

Definition 3.2 *Let $L \subseteq A^*$ be regular. L is* quasi-noncounting *if there is some $d \geq 1$ such that there is some $n_0 \in N$ satisfying that for all $n \geq n_0$, and for all $x, y, z \in A^*$ with $|y| = 0 \bmod d$; $xy^n z \in L$ iff $xy^{n+1}z \in L$.* □

Let $L \subseteq A^*$ be regular and $\eta_L : A^* \to M(L)$ be its syntactic morphism. we denote $(A^d)^*\eta_L$ by $M(L)^{(d)}$. Then we have the following definition:

Definition 3.3 *Let $L \subseteq A^*$ be regular and $\eta_L : A^* \to M(L)$ be its syntactic morphism. L is* quasi-aperiodic *if there is $d \geq 1$ such that $M(L)^{(d)}$ is aperiodic.* □

A language of A^* is quasi-noncounting iff it is quasi-aperiodic. Thus in the remainder of this paper, we don't distinguish between the "quasi-noncounting" and "quasi-aperiodic" properties of regular languages on finite words.

3.2 Logic with cyclic counting interpreted on finite words

FO[<] can be extended with unary predicates $C_d^r (d \geq 1,\ 0 \leq r < d)$ adjoined. C_d^r are interpreted on finite words as follows:

Let $u \in A^*$, $\eta : X \to \{0, ..., |u|\}$, then $(u, \eta) \models C_d^r(x)$ if $x\eta \equiv r \bmod d$.

Denote this extended logic of FO[<] as FO[C].

LTL can be extended with "U"(Until) operator of LTL replaced by new "Until" operators with cyclic counting, namely $U^{(d,r)}$ for all $d \geq 1$ and $0 \leq r < d$. The semantics of $\varphi_1 U^{(d,r)} \varphi_2$ is defined as follows:

Let $u \in A^*$, then $u \models \varphi_1 U^{(d,r)} \varphi_2$ if there is i such that $0 \leq i \leq |u|$, $i \equiv r \bmod d$ and $u^i \models \varphi_2$; moreover, for all j such that ($0 \leq j < i$ and $j \equiv r \bmod d$), $u^j \models \varphi_1$.

Denote this extended LTL by LTL[C].

Similar to FO[<] and LTL, we can define the languages defined by FO[C] sentences and LTL[C] formulas.

The expressive power of FO[C] is strictly stronger than that of FO[<]. For instance, language $(\{a\}A)^*(a \in A$ and $|A| > 1)$ isn't aperiodic, then according to Proposition 2.3, it can't be defined in FO[<], while it can be defined by FO[C] sentence $\forall x \left(Last(x) \to C_2^0(x)\right) \wedge \forall x \left(C_2^0(x) \wedge \neg Last(x) \to P_a(x)\right)$.

It is obvious that for $u \in A^*$, $u \models \varphi_1 U \varphi_2$ iff $u \models \varphi_1 U^{(1,0)} \varphi_2$. Then the expressive power of LTL[C] is at least as strong as that of LTL. In fact, LTL[C] is more expressive than LTL. For instance, language $(\{a\}A)^*(\{a\} \in A$ and $|A| > 1)$ can't be defined in LTL, while it can be defined by LTL[C] formula $p_a U^{(2,0)} End$.

Remark 3.4 *In [5], LTL[C] is defined by adjoining additional constants $Ig_{d,r}(d \geq 1, 0 \leq r < d)$ into LTL, and $U^{(d,r)}$ are just derived temporal operators of $Ig_{d,r}$ and "U". Nevertheless, since $u \models Ig_{d,r}$ iff $|u| \equiv r \bmod d$, LTL[C] defined in [5] can't be interpreted on infinite words. Consequently we directly adjoin $U^{(d,r)}$ into LTL since $U^{(d,r)}$ can be interpreted on infinite words naturally. When interpreted on finite words, $Ig_{d,r}$ can be derived from $U^{(d,r)}$ as follows:*

$$Ig_{d,r} \equiv True U^{(d,r)} End$$

□

3.3 Theorem on quasi-star-free languages on finite words

We summarize the results of quasi-star-free languages on finite words in [2, 5] into the following proposition:

Proposition 3.5 *Let $L \subseteq A^*$ be regular. The following conditions are equivalent:*

(i) L is quasi-star-free;

(ii) L is quasi-aperiodic;

(iii) For all $t \geq 0$, $A^t\eta_L$ contains no nontrivial group;

(iv) L is definable in FO[C];

(v) L is definable in LTL[C]. □

Remark 3.6 *(i), (ii),(iii) and (iv) of Proposition 3.5 were proved equivalent in [2], and (iv) and (v) were proved equivalent in [5]. As a matter of fact, (i),(iii),(iv) of Proposition 3.5 and the following condition (ii′)(Theorem 3(d) in [2]), instead of (ii), were proved equivalent in [2],*

(ii′) L is recognized by a morphism $\psi : \{0,1\}^* \to M wr Z_r$, where M is a finite aperiodic monoid and where the composition $\psi\pi : \{0,1\}^* \to Z_r$ takes both 0 and 1 to the generator 1 of Z_r (see [2] for the exact meaning of (ii′))

And it is not hard to prove that (ii) and (ii′) are equivalent. □

4 Quasi-star-free languages on infinite words

4.1 Quasi-star-free languages on infinite words

Similar to the case of finite words, we define that an ω-language is quasi-star-free, quasi-noncounting and quasi-aperiodic in this subsection.

Definition 4.1 *Let $L \subseteq A^\omega$ be regular. L is* quasi-star-free *if L can be constructed from the language A^ω by finite applications of operations of union, complementation, and concatenation on the left by quasi-star-free languages of A^*.* □

If an ω-language $L \subseteq A^\omega$ is star free, it is quasi-star-free as well. The family of quasi-star-free languages on infinite words is denoted by $\mathrm{QSF}^{\mathrm{I}}$.

Proposition 4.2 *Let $L \subseteq A^\omega$ be quasi-star-free, then there is some $d \geq 1$ such that all those quasi-star-free languages of A^*, used in the construction of L (namely, used in the operations of left concatenation during the construction of L), can be constructed from singleton languages $\{a\}(a \in A)$ and the language $(A^d)^*$ by finite applications of operations of union, complementation and concatenation.*

Proof. Let $L_1, ..., L_k$ be the quasi-star-free languages of A^* used in the construction of L.

Then there are $d_i (1 \leq i \leq k)$ such that $L_i (1 \leq i \leq k)$ can be constructed from singleton languages $\{a\}(a \in A)$ and the language $(A^{d_i})^*$.

Let d be the least common multiple of $d_1, ..., d_k$. Then

$$(A^{d_i})^* = \bigcup_{r=0}^{d'_i - 1} (A^d)^* A^{rd_i} = \bigcup_{r=0}^{d'_i - 1} (A^d)^* \left(\bigcup_{a \in A} \{a\} \right)^{rd_i}, \text{ where } d'_i = \frac{d}{d_i}.$$

Consequently $L_i (1 \leq i \leq k)$ can be constructed from singleton languages $\{a\}(a \in A)$ and the language $(A^d)^*$ by finite applications of operations of union, complementation and concatenation. □

Definition 4.3 *Let $L \subseteq A^\omega$ be regular. L is quasi-noncounting if there is some $d \geq 1$ such that there is $n_0 \in N$ satisfying that for all $n \geq n_0$ and $u, x, y, z \in A^*$ with $|u| = 0 \bmod d$, ($xu^n yz^\omega \in L$ iff $xu^{n+1} yz^\omega \in L$) and ($x(yu^n z)^\omega \in L$ iff $x(yu^{n+1} z)^\omega \in L$).* □

Definition 4.4 *Let $L \subseteq A^\omega$ be regular and $\eta_L : A^* \to M(L)$ be its syntactic morphism. Then L is quasi-aperiodic if there is some $d \geq 1$ such that $M(L)^{(d)}$ is aperiodic.* □

Proposition 4.5 *Let $L \subseteq A^\omega$ be regular. L is quasi-noncounting iff it is quasi-aperiodic.*

Proof.

"⇒" part:

Suppose that there is some $d \geq 1$ such that there is some $n_0 \in N$ satisfying that for all $n \geq n_0$, and for all $x, u, y, z \in A^*$ with $|u| \equiv 0 \bmod d$;

$$\left(xu^n yz^\omega \in L \text{ iff } xu^{n+1} yz^\omega \in L\right) \text{ and } \left(x\left(yu^n z\right)^\omega \in L \text{ iff } x\left(yu^{n+1} z\right)^\omega \in L\right).$$

Now we prove that $M(L)^{(d)}$ is aperiodic.

Let $m \in M(L)^{(d)}$. Then there is some $u \in (A^d)^*$ such that $u\eta_L = m$. Thus for any $n \geq n_0$, and for all $x, y, z \in A^*$;

$$\left(xu^n yz^\omega \in L \text{ iff } xu^{n+1} yz^\omega \in L\right) \text{ and } \left(x\left(yu^n z\right)^\omega \in L \text{ iff } x\left(yu^{n+1} z\right)^\omega \in L\right).$$

Consequently for any $n \geq n_0$, $(u^n)\eta_L = (u^{n+1})\eta_L$, i.e., $m^n = m^{n+1}$.

"⇐" part:

Suppose that there is some $d \geq 1$ such that $M(L)^{(d)}$ is aperiodic, i.e., there is some $n_0 \in N$ satisfying that for all $n \geq n_0$ and $m \in M(L)^{(d)}$; $m^n = m^{n+1}$.

Now we prove that L is quasi-noncounting.

Let $n \geq n_0$ and $x, u, y, z \in A^*$ with $|u| \equiv 0 \bmod d$. Then $u\eta_L \in M(L)^{(d)}$, so $(u^n)\eta_L = (u^{n+1})\eta_L$. From the definition of η_L, we have that

$$\left(xu^n yz^\omega \in L \text{ iff } xu^{n+1} yz^\omega \in L\right) \text{ and } \left(x\left(yu^n z\right)^\omega \in L \text{ iff } x\left(yu^{n+1} z\right)^\omega \in L\right).$$

□

As a result of Proposition 4.5, in the remainder of this paper, we don't distinguish between "quasi-noncounting" and "quasi-aperiodic" properties of regular languages on infinite words.

4.2 Logic with cyclic counting interpreted on infinite words

FO[C] and LTL[C] defined in Section 3.2 can be interpreted on infinite words as follows:

For FO[C]: Let $u \in A^\omega$ and $\eta : X \to N$, then

$$(u, \eta) \models C_d^r(x) \text{ if } x\eta \equiv r \bmod d.$$

For LTL[C]: Let $u \in A^\omega$, then

$u \models \varphi_1 U^{(d,r)} \varphi_2$ if there is $i \geq 0$ such that ($i \equiv r \bmod d$) and ($u^i \models \varphi_2$), and (for all $0 \leq j < i$ and $j \equiv r \bmod d$; $u^j \models \varphi_1$).

Similar to the case of finite words, we can define the languages defined by FO[C] sentences and LTL[C] formulas.

When interpreted on infinite words, the expressive power of FO[C](LTL[C] resp.) is strictly stronger than FO[<](LTL resp.). E.g., language $(\{a\}A)^\omega (a \in A$ and $|A| > 1)$ isn't aperiodic, then according to Proposition 2.6, it can't be defined in FO[<](LTL resp.), while it can be defined by FO[C] sentence $\forall x \left(C_2^0(x) \to P_a(x)\right)$ (LTL[C] formula $\neg \left(True U^{(2,0)} \neg p_a\right)$ resp.)

4.3 Theorem on quasi-star-free languages on infinite words

We extend Proposition 3.5 for $\mathrm{QSF^F}$ to the following theorem for $\mathrm{QSF^I}$.

Theorem 4.6 *Let $L \subseteq A^\omega$ be regular. The following conditions are equivalent:*

(i) L is quasi-star-free;

(ii) L is quasi-aperiodic;

(iii) For all $t \geq 0$, $A^t \eta_L \subseteq M(L)$ contains no nontrivial group;

(iv) $L = \bigcup_{i=1}^{m} X_i (Y_i)^\omega$, where $X_i, Y_i \in QSF^F$, $Y_i \subseteq A^+$ and $Y_i Y_i \subseteq Y_i$;

(v) L is definable in FO[C];

(vi) L is definable in LTL[C].

Before the proof of Theorem 4.6, we give some definitions and lemmas.

Let $A^{(d)}$ denote the alphabet consisting of all letters $\langle u \rangle$, where $u \in A^d$. For any $x \in \left(A^d\right)^*$, we denote the corresponding element of $\left(A^{(d)}\right)^*$ as $\langle x \rangle$.

Let $L \subseteq A^*$ and $u \in A^*$, define $Lu^{-1} = \{x \,|\, x \in A^*, xu \in L\}$.

Let $L \subseteq A^*$and $d \geq 1$, define

$$L^{(d)} = \begin{cases} \{\langle u_0\rangle \ldots \langle u_{k-1}\rangle \,|u_0\ldots u_{k-1} \in L, k \geq 1, \forall\, 0 \leq i < k \left(u_i \in A^d\right)\} & \text{if } \varepsilon \notin L \\ \{\varepsilon\} \bigcup \{\langle u_0\rangle \ldots \langle u_{k-1}\rangle \,|u_0\ldots u_{k-1} \in L, k \geq 1, \forall\, 0 \leq i < k \left(u_i \in A^d\right)\} & \text{othewise} \end{cases}.$$

Let $L \subseteq A^*$ and $u \in A^*$, define $L^{(d,u)} = \left(Lu^{-1}\right)^{(d)}$.

Let $L \subseteq A^\omega$ and $d \geq 1$, define

$$L^{(d)} = \left\{\langle u_0\rangle \ldots \langle u_k\rangle \ldots \,|u_0\ldots u_k\ldots \in L, \forall\, i \geq 0 \left(u_i \in A^d\right)\right\}.$$

Lemma 4.7 *Let $L \subseteq A^\omega$ be regular. Define $\phi : \left(A^{(d)}\right)^* \to M(L)^{(d)}$ by $\langle x\rangle \phi = x\eta_L$ for $\langle x\rangle \in \left(A^{(d)}\right)^*$. Then ϕ recognizes $L^{(d)}$.*

Proof.

We define morphism $\theta : Im(\phi) \to M\left(L^{(d)}\right)$ such that $\phi\theta = \eta_{L^{(d)}}$, and thus according to Proposition 2.4, ϕ recognizes $L^{(d)}$.

Define θ by: for $m \in Im(\phi)$, $m\theta = \langle w\rangle \eta_{L^{(d)}}$, where $\langle w\rangle \in \left(A^{(d)}\right)^*$ and $\langle w\rangle \phi = m$.

At first, we prove that θ is well defined. Let $\langle w_1\rangle \phi = \langle w_2\rangle \phi = m$, i.e. $w_1\eta_L = w_2\eta_L = m$. Then for all $x, y, z \in A^*$, ($xw_1yz^\omega \in L$ iff $xw_2yz^\omega \in L$) and ($x(yw_1z)^\omega \in L$ iff $x\,(yw_2z)^\omega \in L$), thus for all $\langle x\rangle, \langle y\rangle, \langle z\rangle \in \left(A^{(d)}\right)^*$, ($\langle x\rangle \langle w_1\rangle \langle y\rangle \langle z\rangle^\omega \in L^{(d)}$ iff $\langle x\rangle \langle w_2\rangle \langle y\rangle \langle z\rangle^\omega \in L^{(d)}$) and ($\langle x\rangle (\langle y\rangle \langle w_1\rangle \langle z\rangle)^\omega \in L^{(d)}$ iff $\langle x\rangle (\langle y\rangle \langle w_2\rangle \langle z\rangle)^\omega \in L^{(d)}$), i.e. $\langle w_1\rangle \approx_{L^{(d)}} \langle w_2\rangle$, $\langle w_1\rangle \eta_{L^{(d)}} = \langle w_2\rangle \eta_{L^{(d)}}$, so θ is well defined.

Evidently for all $\langle w\rangle \in \left(A^{(d)}\right)^*$, $\langle w\rangle \phi\theta = \langle w\rangle \eta_{L^{(d)}}$. □

Lemma 4.8 *Suppose that $L = \bigcup_{i=1}^{m} X_i\,(Y_i)^\omega$, where $X_i, Y_i \in QSF^F$, $Y_i \subseteq A^+$ and $Y_iY_i \subseteq Y_i$. Then there is $d \geq 1$ such that all those X_i and Y_i can be constructed from the singleton languages $\{a\}(a \in A)$ and the language $\left(A^d\right)^*$.*

Proof. Since $X_i, Y_i \in \mathrm{QSF}^{\mathrm{F}}$, then there are d_{X_i} and d_{Y_i} such that X_i and Y_i are constructed from the singleton languages $\{a\}$ and the language $\left(A^{d_{X_i}}\right)^*$.

Let $d = lcm\{d_{X_i}, d_{Y_i}\,|1 \leq i \leq m\}$. Then similar to the proof of Proposition 4.2, we can prove that X_i and Y_i can be constructed from singleton languages $\{a\}$ and the language $\left(A^d\right)^*$. □

Lemma 4.9 *Suppose that $L \subseteq \left(A^{(d)}\right)^*$ is star free for some $d \geq 1$, then $L' = \{x\,|x \in (A^d)^*, \langle x\rangle \in L\}$ is quasi-star-free.*

Proof.

Since $L \subseteq \left(A^{(d)}\right)^*$ is star free, it can be constructed from singleton languages $\{\langle u\rangle\}$ ($u \in A^d$) and the language $\left(A^{(d)}\right)^*$ by union, complementation and concatenation.

By replacing $\{\langle u\rangle\}(u = a_0\ldots a_{d-1})$ by $\{a_0\}\ldots\{a_{d-1}\}$; $\left(A^{(d)}\right)^*$ by $\left(A^d\right)^*$; $L_1 \bigcup L_2$ by $L_1' \bigcup L_2'$; $\left(A^{(d)}\right)^* - L_1$ by $\left(A^d\right)^* - L_1'$(namely $A^* - \left(\left(A^* - \left(A^d\right)^*\right) \bigcup L_1'\right)$);

and L_1L_2 by $L'_1L'_2$ during the construction procedure of L, we can get the construction procedure of L'(where $L_1, L_2 \subseteq \left(A^{(d)}\right)^*$ and L'_1, L'_2 are the languages of $\left(A^d\right)^*$ corresponding to L_1 and L_2 respectively). Thus L' can be constructed from singleton languages $\{a\}$ and the language $\left(A^d\right)^*$ by union, complementation and concatenation. Consequently it is quasi-star-free by definition. □

Lemma 4.10 *Let $L \subseteq A^\omega$. Then L is definable in FO[C] iff there is some $d \geq 1$ such that $L^{(d)}$ is definable in FO[<].* □

Lemma 4.11 *Let $L \subseteq A^\omega$. Then L is definable in LTL[C] iff L is definable in FO[C].* □

Remark 4.12 *The proofs of Lemma 4.10 and Lemma 4.11 are totally similar to the proofs of the same results for finite words(Proposition 6.5, Proposition 6.7 and Theorem 7.5 in [5]). Consequently we omit the proofs of them here.* □

Now we prove Theorem 4.6.

Proof of Theorem 4.6.

At first we prove the equivalence of (ii) and (iii). According to Lemma 4.11, (v) and (vi) are equivalent. Then if we have proved the equivalence of (i),(ii),(iv) and (v), the proof would be completed. We prove the equivalence of (i),(ii), (iv) and (v) by proving the equivalence of (i),(ii),(v) and equivalence of (ii),(v) respectively.

(ii)⇒(iii):

Suppose that $L \subseteq A^\omega$ is quasi-aperiodic, i.e. $M(L)^{(d)}$ is aperiodic for some $d \geq 1$. Now we show that for all $t \geq 0$, $A^t\eta_L$ contains no nontrivial group.

To the contrary suppose that there is some $t \geq 0$ such that $A^t\eta_L$ contains a nontrivial group. Obviously $t \geq 1$. Select an element m of order $k > 1$ from the group, then $G = \left\{m, ...m^k\right\}$ is also a nontrivial group in $A^t\eta_L$. Hence there are $u, v \in A^t$ such that $u\eta_L = m$, $v\eta_L = m^k$.

Consider $A^{tkd}\eta_L \subseteq M(L)^{(d)}$. It is easy to see that $m^i = \left(v^{k(d-1)}\left(u^iv^{k-i}\right)\right)\eta_L \in A^{tkd}\eta_L$, thus $G \subseteq A^{tkd}\eta_L \subseteq M(L)^{(d)}$, $M(L)^{(d)}$ contains a nontrivial group. Because a monoid is aperiodic iff it contains no nontrivial group, we have that $M(L)^{(d)}$ isn't aperiodic, a contradiction.

(iii)⇒(ii):

The main idea is from the proof of Theorem 3 in [2].

Suppose that $M(L)$ is finite and for all $t \geq 0$, $A^t\eta_L$ contains no nontrivial group.

For each nontrivial group G contained in $M(L)$ pick a nonempty word v_G such that $v_G\eta_L$ is the identity of G. Let d be a common multiple of the lengths of all these v_G. Now we show that $M(L)^{(d)}$ is aperiodic.

To the contrary suppose that $M(L)^{(d)}$ isn't aperiodic. Because a monoid is aperiodic iff it contains no nontrivial group, then there is a nontrivial group in $M(L)^{(d)}$. Select an element m of order $k > 1$ from the group, then $G = \left\{m, ..., m^k\right\}$ is also a nontrivial group in $M(L)^{(d)}$. Select some $v \in \left(A^d\right)^*$ such that $v\eta_L = m$. From the selection of d, we know $|v|$(the length of v) is a multiple of $|v_G|$, thus there is some power w of v_G such that $|v| = |w|$. Let $t = k|v|$, then $m^j = \left(v^j w^{k-j}\right)\eta_L \in A^t\eta_L$, so $G \subseteq A^t\eta_L$, a contradiction.

Therefore we have proved the equivalence of (ii) and (iii).

Now we prove the equivalence of (i), (ii), (v).

(i)⇒(ii):

Suppose that L can be constructed from language A^ω by finite applications of operations of union, complementation, and concatenation on the left by quasi-star-free languages of A^*. Then according to Proposition 4.2, there is $d \geq 1$ such that quasi-star-free languages of A^* used in the construction of L can be constructed from singleton languages $\{a\}$ and the language $\left(A^d\right)^*$.

Now we prove that $M(L)^{(d)}$ is aperiodic by induction on the constructing procedure of L.

Induction base: $L = A^\omega$, then $M(L) = \{e\}$, where e is the identity of $M(L)$. Obviously $M(L)^{(d)} = \{e\}$, then it is aperiodic.

Induction step:

Case $L = A^\omega - L_1$: From induction hypothesis, $M(L_1)^{(d)}$ is aperiodic. Since it is not hard to see that $M(L) = M(L_1)$ and $\eta_L = \eta_{L_1}$ from the definition of syntactic monoid and syntactic morphism of ω-languages, $M(L)^{(d)}$ is aperiodic as well.

Case $L = L_1 \bigcup L_2$: From induction hypothesis, $M(L_i)^{(d)}$ $(i = 1, 2)$ are aperiodic, then according to Proposition 4.5, there are n_i $(i = 1, 2)$ such that for all $n \geq n_i$ and $u, x, y, z \in A^*$ with $|u| \equiv 0 \bmod d$, ($xu^nyz^\omega \in L_i$ iff $xu^{n+1}yz^\omega \in L_i$) and ($x(yu^nz)^\omega \in L_i$ iff $x(yu^{n+1}z)^\omega \in L_i$).

Let $n_0 = \max\{n_1, n_2\}$. Now we show that for all $n \geq n_0$ and $u, x, y, z \in A^*$ with $|u| \equiv 0 \bmod d$, ($xu^nyz^\omega \in L$ iff $xu^{n+1}yz^\omega \in L$) and ($x(yu^nz)^\omega \in L$ iff $x(yu^{n+1}z)^\omega \in L$). Then according to Proposition 4.5 we conclude that $M(L)^{(d)}$ is aperiodic.

Suppose that $xu^nyz^\omega \in L$, then $xu^nyz^\omega \in L_i$ for some $i = 1, 2$. Thus $xu^{n+1}yz^\omega \in L_i$ since $n \geq n_0 \geq n_i$, so $xu^{n+1}yz^\omega \in L$. The proof of $xu^{n+1}yz^\omega \in L$ implies $xu^nyz^\omega \in L$ is similar.

Suppose that $x(yu^nz)^\omega \in L$, then $x(yu^nz)^\omega \in L_i$ for some $i = 1, 2$. Thus $x(yu^{n+1}z)^\omega \in L_i$ since $n \geq n_0 \geq n_i$, so $x(yu^{n+1}z)^\omega \in L$. The proof of $x(yu^{n+1}z)^\omega \in L$ implies $x(yu^nz)^\omega \in L$ is similar.

Case $L = L_1L_2$: where $L_1 \subseteq A^*$ and $L_2 \subseteq A^\omega$. According to Proposition 3.5, L_1 is quasi-aperiodic, then there is n_1 such that for all $n \geq n_1$, $xy^nz \in L_1$ iff $xy^{n+1}z \in L_1$ for all $x, y, z \in A^*$ with $|y| = 0 \bmod d$. From induction hypothesis, $M(L_2)^{(d)}$ is aperiodic, thus there is n_2 such that for all $n \geq n_2$, $u, x, y, z \in A^*$ with $|u| = 0 \bmod d$, ($xu^nyz^\omega \in L_2$ iff $xu^{n+1}yz^\omega \in L_2$) and ($x(yu^nz)^\omega \in L_2$ iff $x(yu^{n+1}z)^\omega \in L_2$).

Let $n_0 = n_1 + n_2 + 1$. It is sufficient to show that for all $n \geq n_0$ and $u, x, y, z \in A^*$ with $|u| = 0 \bmod d$, ($xu^nyz^\omega \in L$ iff $xu^{n+1}yz^\omega \in L$) and ($x(yu^nz)^\omega \in L$ iff $x(yu^{n+1}z)^\omega \in L$) in order to prove that $M(L)^{(d)}$ is aperiodic according to Proposition 4.5.

(a) Suppose that $n \geq n_0$,$u, x, y, z \in A^*$ with $|u| = 0 \bmod d$, and $xu^nyz^\omega \in L$. We show that $xu^{n+1}yz^\omega \in L$.

Since $xu^n yz^\omega \in L = L_1L_2$, $xu^n yz^\omega$ has a decomposition vw such that $v \in L_1$ and $w \in L_2$. There are the following cases:

- $v = x_1$, $w = x_2u^n yz^\omega$ with $x = x_1x_2$;
- there are $h, k \geq 0$, $u_1, u_2 \in A^*$ such that $v = xu^h u_1$, $w = u_2u^k yz^\omega$ with $n = h + k + 1$, $u = u_1u_2$;
- $v = xu^n y_1$, $w = y_2z^\omega$ with $y = y_1y_2$;
- there are $p \geq 0$, $z_1, z_2 \in A^*$ such that $v = xu^n yz^p z_1$, $w = z_2z^\omega$ with $z = z_1z_2$.

Here we take the second case as an example, the discussions of the other cases are similar. In the second case, because $h + k + 1 \geq n_1 + n_2 + 1$, then $h \geq n_1$ or $k \geq n_2$, thus $xu^{h+1}u_1 \in L_1$ or $u_2u^{k+1}yz^\omega \in L_2$, then $xu^{n+1}yz^\omega \in L_1L_2 = L$.

The proof of $xu^{n+1}yz^\omega \in L$ implies $xu^n yz^\omega \in L$ is similar to (a).

(b) Suppose that $n \geq n_0$,$u, x, y, z \in A^*$ with $|u| = 0 \bmod d$, and $x(yu^n z)^\omega \in L$. We show that $x(yu^{n+1}z)^\omega \in L$.

Since $x(yu^n z)^\omega \in L = L_1L_2$, $x(yu^n z)^\omega$ has a decomposition vw such that $v \in L_1$ and $w \in L_2$. There are the following cases:

- $v = x_1$, $w = x_2(yu^n z)^\omega$ with $x = x_1x_2$;
- there are $p \geq 0$, $y_1, y_2 \in A^*$ such that $v = x(yu^n z)^p y_1$, $w = (y_2u^n z)(yu^n z)^\omega$ and $y = y_1y_2$;
- there are $p, h, k \geq 0$, $u_1, u_2 \in A^*$ such that $v = x(yu^n z)^p(yu^h u_1)$, $w = (u_2u^k z)(yu^n z)^\omega$ with $n = h + k + 1$, $u = u_1u_2$;
- there are $p \geq 0$, $z_1, z_2 \in A^*$ such that $v = x(yu^n z)^p(yu^n z_1)$, $w = z_2(yu^n z)^\omega$, $z = z_1z_2$;

Here we take the third case as an example, the discussions of the other cases are similar.

Since $n \geq n_0 = n_1 + n_2 + 1 \geq n_i (i = 1, 2)$, then $x(yu^{n+1}z)^p(yu^h u_1) \in L_1$ and $(u_2u^k z)(yu^{n+1}z)^\omega \in L_2$. Because $h + k + 1 \geq n_1 + n_2 + 1$, we have $h \geq n_1$ or $k \geq n_2$. Thus $x(yu^{n+1}z)^p(yu^{h+1}u_1) \in L_1$ or $(u_2u^{k+1}z)(yu^{n+1}z)^\omega \in L_2$. Consequently

$$x(yu^{n+1}z)^p(yu^{h+1}u_1)(u_2u^k z)(yu^{n+1}z)^\omega \in L_1L_2$$

or

$$x(yu^{n+1}z)^p(yu^h u_1)(u_2u^{k+1}z)(yu^{n+1}z)^\omega \in L_1L_2.$$

Namely, $x(yu^{n+1}z)^\omega \in L_1L_2 = L$.

The proof of $x(yu^{n+1}z)^\omega \in L$ implies $x(yu^n z)^\omega \in L$ is similar to (b).

(ii)⇒(v):

Suppose that L is quasi-aperiodic, then there is $d \geq 1$ such that $M(L)^{(d)}$ is aperiodic, then according to Lemma 4.7, $L^{(d)}$ is aperiodic, thus L is definable in FO[C] according to Lemma 4.10.

(v)⇒(i):

Suppose that $L \subseteq A^\omega$ is definable in FO[C], then according to Lemma 4.10, there is $d \geq 1$ such that $L^{(d)} \subseteq \left(A^{(d)}\right)^\omega$ can be expressed in FO[<]. According to Proposition 2.6, $L^{(d)}$ is star-free, i.e. it can be constructed from $\left(A^{(d)}\right)^\omega$ by union, complementation and concatenation on the left by star free languages of $\left(A^{(d)}\right)^*$.

By replacing $L_1 \bigcup L_2$, $\left(A^{(d)}\right)^\omega - L_1$, and $L_1 L_2$ by $L_1' \bigcup L_2'$, $\left(A^d\right)^\omega - L_1'$ and $L_1' L_2'$ respectively during the construction of $L^{(d)}$ (where L_1', L_2' are languages of $\left(A^d\right)^*$ or $\left(A^d\right)^\omega$ corresponding to L_1 and L_2 respectively), we can get the construction procedure for L. Moreover, according to Lemma 4.9, languages of $\left(A^d\right)^*$ used in the left concatenation during the construction of L must be quasi-star-free. Then we can conclude that L can be constructed from A^ω(namely $\left(A^d\right)^\omega$) by union, complementation and concatenation on the left by quasi-star-free languages of A^*, i.e., L is quasi-star-free.

Therefore we have proved the equivalence of (i),(ii),(v).

Now we prove the equivalence of (ii),(iv) and complete the proof of the theorem.

(ii)⇒(iv):

Suppose that L is quasi-aperiodic, i.e. there is $d \geq 1$ such that $M(L)^{(d)}$ is aperiodic.

According to Lemma 4.7, $L^{(d)}$ is aperiodic. Thus by Proposition 2.6, $L^{(d)} = \bigcup\limits_{i=1}^{m} X_i Y_i^\omega$, where $X_i \subseteq \left(A^{(d)}\right)^*$, $Y_i \subseteq \left(A^{(d)}\right)^+$ are star free, and $Y_i Y_i \subseteq Y_i$.

Let $X_i' = \left\{x \,\middle|\, x \in \left(A^d\right)^*, \langle x \rangle \in X_i\right\}$, $Y_i' = \left\{y \,\middle|\, y \in \left(A^d\right)^*, \langle y \rangle \in Y_i\right\}$, then $L = \bigcup\limits_{i=1}^{m} X_i' \left(Y_i'\right)^\omega$. Evidently $Y_i' Y_i' \subseteq Y_i'$. Since $X_i, Y_i \subseteq \left(A^{(d)}\right)^*$ are star free, then according to Lemma 4.9, X_i' and Y_i' are quasi-star-free.

(iv)⇒(ii):

Suppose that $L = \bigcup\limits_{i=1}^{m} X_i \left(Y_i\right)^\omega$, where $X_i \subseteq A^*, Y_i \subseteq A^+$ are quasi-star-free languages, and $Y_i Y_i \subseteq Y_i$. Then according to Lemma 4.8, there is $d \geq 1$ such that X_i, Y_i can be constructed from singleton languages $\{a\}(a \in A)$ and the language $\left(A^d\right)^*$.

Because X_i is quasi-star-free, according to Proposition 3.5, X_i is quasi-aperiodic, i.e. there is $n_0 \in N$ such that for all $n \geq n_0$ and $x, y, z \in A^*$ with $|y| \equiv 0 \bmod d$, $xy^n z \in X_i$ iff $xy^{n+1} z \in X_i$. Denote this n_0 as $n_0(X_i)$. Similarly we have $n_0(Y_i)$ for Y_i. Moreover, since X_i, Y_i are quasi-star-free, $X_i Y_i$ is quasi-star-free as well, and we let $n_0(X_i Y_i) \geq n_0(X_i) + n_0(Y_i) + 1$ for $X_i Y_i$ such that for all $n \geq n_0(X_i Y_i)$ and $x, y, z \in A^*$ with $|y| \equiv 0 \bmod d$, $xy^n z \in X_i Y_i$ iff $xy^{n+1} z \in X_i Y_i$.

Let $N_0 = 1 + 2 \max\{n_0(X_i Y_i) | 1 \leq i \leq m\}$. It is sufficient to show that for all $n \geq N_0$ and $u, x, y, z \in A^*$ with $|u| = 0 \bmod d$, $(xu^n yz^\omega \in L$ iff $xu^{n+1} yz^\omega \in L)$ and $(x(yu^n z)^\omega \in L$ iff $x(yu^{n+1} z)^\omega \in L)$ in order to prove that L is quasi-aperiodic (according to Proposition 4.5).

(a) Suppose that $n \geq N_0$, $u, x, y, z \in A^*$, $|u| = 0 \bmod d$, and $xu^nyz^\omega \in L$, we show that $xu^{n+1}yz^\omega \in L$.

Because $L = \bigcup_{i=1}^{m} X_i\,(Y_i)^\omega$, $xu^nyz^\omega \in X_i\,(Y_i)^\omega$ for some i. Then there is $p, p', q, q' \geq 0$, $z_1, z_2 \in A^*$ such that $z = z_1z_2$, $xu^nyz^{p'}z_1 \in X_iY_i^p$ and $z_2z^{q'}z_1 \in Y_i^q$. If $p = 0$, then $xu^{n+1}yz^{p'}z_1 \in X_i$ since $n \geq N_0 \geq n_0(X_iY_i) \geq n_0(X_i)$, $xu^{n+1}yz^\omega = \left(xu^{n+1}yz^{p'}z_1\right)\left(z_2z^{q'}z_1\right)^\omega \in X_i\,((Y_i)^q)^\omega = X_iY_i^\omega \subseteq L$. In the case of $p > 0$, $X_iY_i^p \subseteq X_iY_i$ follows from that assumption $Y_iY_i \subseteq Y_i$, so $xu^{n+1}yz^{p'}z_1 \in X_iY_i$ since $n \geq N_0 \geq n_0(X_iY_i)$; then $xu^{n+1}yz^\omega = \left(xu^{n+1}yz^{p'}z_1\right)\left(z_2z^{q'}z_1\right)^\omega \in X_iY_i\,((Y_i)^q)^\omega = X_i(Y_i)^\omega \subseteq L$.

The proof of $xu^{n+1}yz^\omega \in L$ implies $xu^nyz^\omega \in L$ is similar to (a).

(b) Suppose that $n \geq N_0$, $u, x, y, z \in A^*$, $|u| = 0 \bmod d$, and $x(yu^nz)^\omega \in L$, we show that $x(yu^{n+1}z)^\omega \in L$.

Because $L = \bigcup_{i=1}^{m} X_i\,(Y_i)^\omega$, $x\,(yu^nz)^\omega \in X_iY_i^\omega$ for some i. Then there are $p, p', q, q' \geq 0$, $v_1, v_2 \in A^*$ such that $x(yu^nz)^{p'}v_1 \in X_iY_i^p$, $v_2(yu^nz)^{q'}v_1 \in Y_i^q$, $v_1v_2 = yu^nz$.

Here we prove for the case of $p > 0$, the case of $p = 0$ can be proved similarly.

Suppose that $p > 0$.

Since $Y_iY_i \subseteq Y_i$, we have $X_iY_i^p \subseteq X_iY_i$, $Y_i^q \subseteq Y_i$.

Because $n \geq N_0 \geq n_0(X_i, Y_i) \geq n_0(Y_i)$, we have that $x(yu^{n+1}z)^{p'}v_1 \in X_iY_i$ and $v_2(yu^{n+1}z)^{q'}v_1 \in Y_i$.

Now we discuss the following three cases of v_1 and v_2.

- $v_1 = y_1$, $v_2 = y_2u^nz$, $y = y_1y_2$;
- $v_1 = yu^nz_1$, $v_2 = z_2$, $z = z_1z_2$;
- $v_1 = yu^hu_1$, $v_2 = u_2u^kz$, with $h + k + 1 = n$ and $u = u_1u_2$.

Here we take the third case as the example, the discussions of other cases are similar.

Case $v_1 = yu^hu_1$, $v_2 = u_2u^kz$, with $h + k + 1 = n$ and $u = u_1u_2$:

Since $n \geq N_0 \geq 1 + 2n_0(X_iY_i)$, we have $h \geq n_0(X_iY_i)$ or $k \geq n_0(X_iY_i)$.

If $h \geq n_0(X_iY_i)$, then

$$x(yu^{n+1}z)^{p'}(yu^{h+1}u_1) \in X_iY_i,\ (u_2u^kz)(yu^{n+1}z)^{q'}(yu^{h+1}u_1) \in Y_i.$$

Thus

$$x(yu^{n+1}z)^\omega = \left(x(yu^{n+1}z)^{p'}(yu^{h+1}u_1)\right)\left((u_2u^kz)(yu^{n+1}z)^{q'}(yu^{h+1}u_1)\right)^\omega \in X_iY_i^\omega.$$

If $k \geq n_0(X_iY_i)$, then $(u_2u^{k+1}z)(yu^{n+1}z)^{q'}(yu^hu_1) \in Y_i$. Thus

$$x(yu^{n+1}z)^\omega = \left(x(yu^{n+1}z)^{p'}(yu^hu_1)\right)\left((u_2u^{k+1}z)(yu^{n+1}z)^{q'}(yu^hu_1)\right)^\omega \in X_iY_i^\omega.$$

The proof of $x(yu^{n+1}z)^\omega \in L$ implies $x(yu^nz)^\omega \in L$ is similar to (b). $\square$

5 Conclusions and Remarks

In this paper quasi-star-free languages on infinite words ($\mathrm{QSF^I}$) are defined and studied. Quasi-star-free languages on finite words($\mathrm{QSF^F}$) have been studied in [2, 5], and our work in this paper is an extension of those results for $\mathrm{QSF^F}$ in [2, 5].

The extension of results of $\mathrm{QSF^F}$ to $\mathrm{QSF^I}$ should be more useful for the characterizations of the expressive power of temporal logics since temporal logics are usually interpreted on infinite words in order to describe temporal properties of concurrent systems. One of the examples is the characterizations of expressive power of fragments of linear μ-calculus [8](known as νTL). The "next" operators within the scope of the fixed points of νTL formulas act like the FO[C] predicates "$C_d^r(x)$" and LTL[C] operators "$U^{(d,r)}$", e.g. νTL formula $\nu Q.p_a \wedge XXQ$ defines language $(\{a\}A)^\omega$, which can be defined by FO[C] sentence $\forall x(C_2^0(x) \rightarrow p_a(x))$ and LTL[C] formula $\neg(TrueU^{(2,0)}\neg p_a)$ respectively, as we have noticed in Section 4.2.

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