

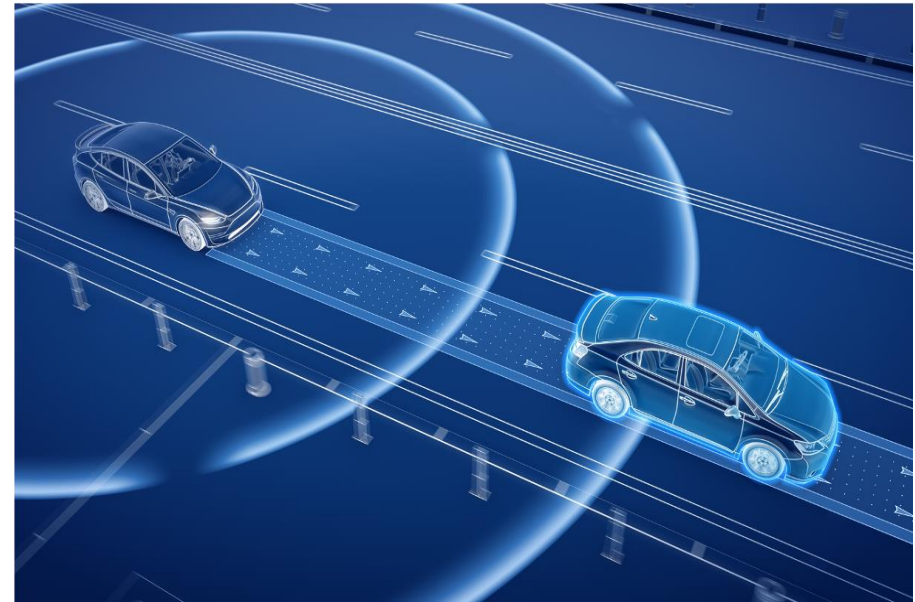
A Framework for Safe Probabilistic Invariance Verification of Stochastic Dynamical Systems

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July 26, 2024

Background

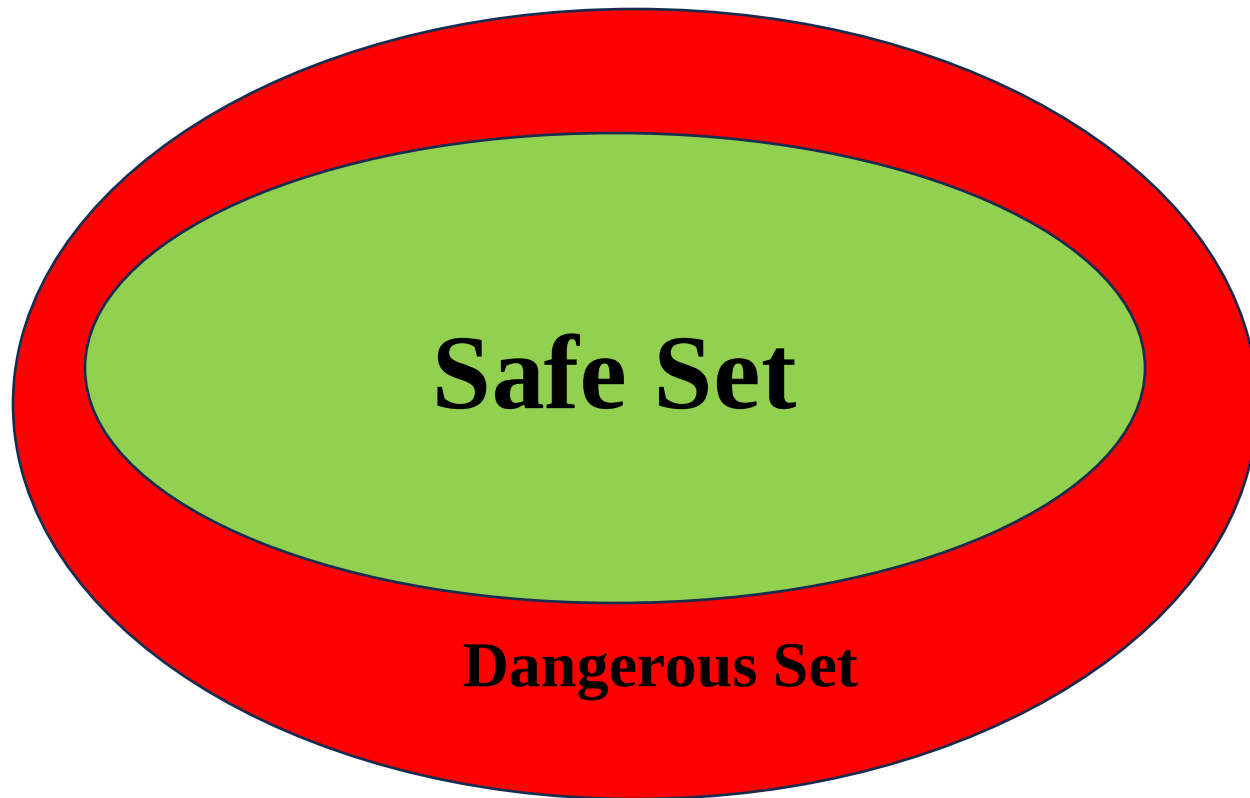


Safety Is Paramount!

A safety property asserts that some “bad thing” does not happen during execution [Leslie Lamport, 1977]

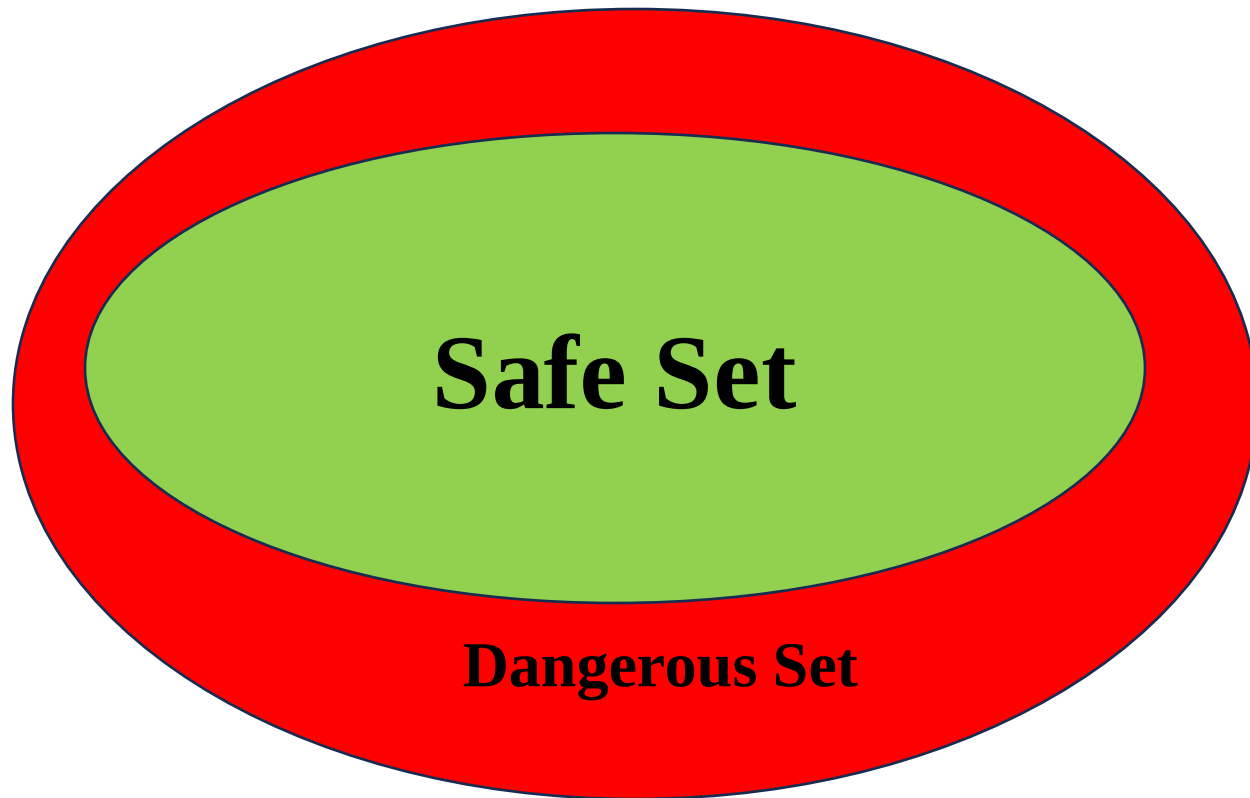
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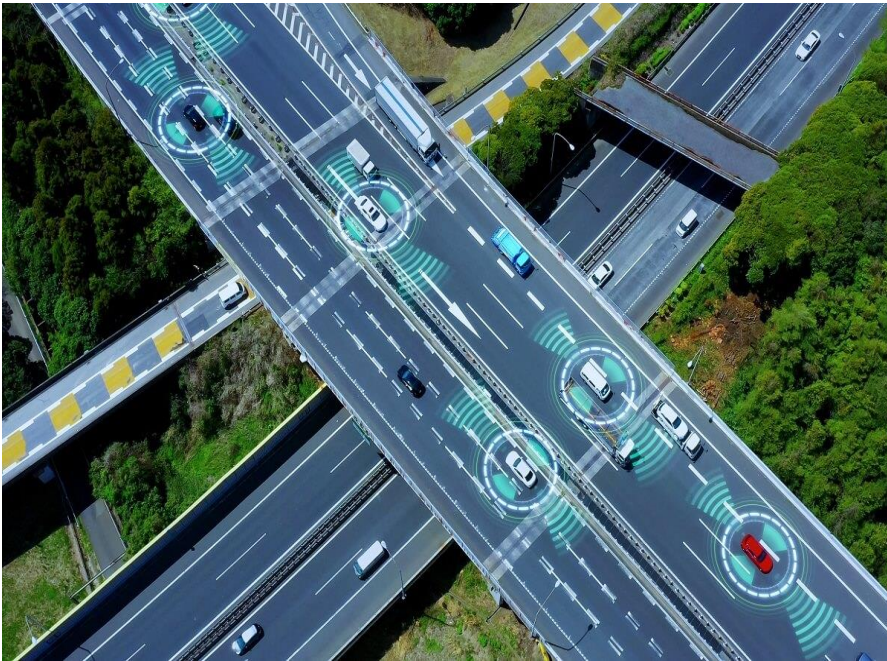
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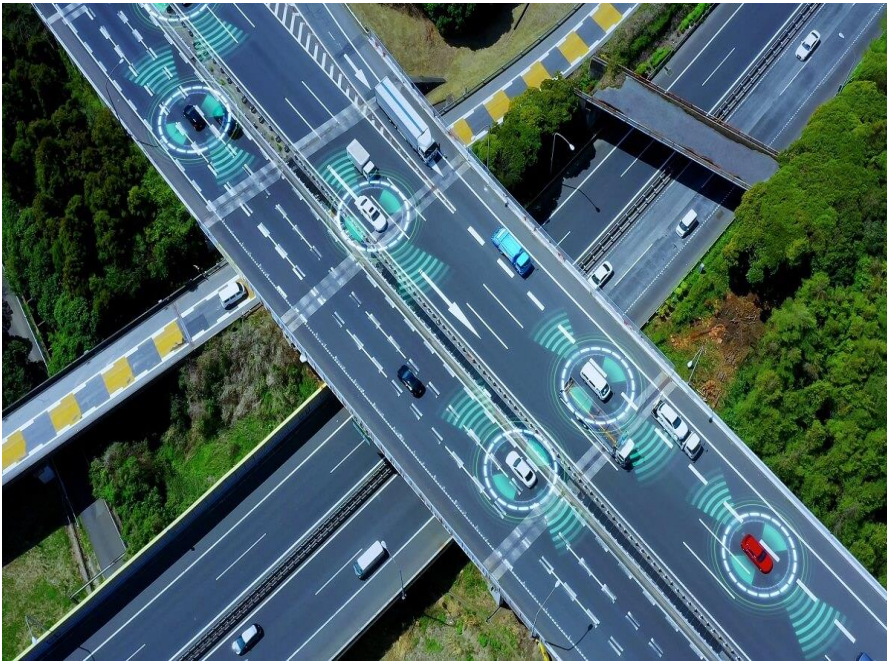
Dynamical systems:

- **Discrete-time Systems:** $x(k + 1) = f(x(k))$
- **Continuous-time Systems:** $\frac{dx}{dt} = f(x)$

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How do we guarantee safety of these systems?

Safe Robust Invariance Verification

$$x(k+1) = f(x(k), d(k))$$

- $d(k) \in \mathcal{D}$ is the perturbation input

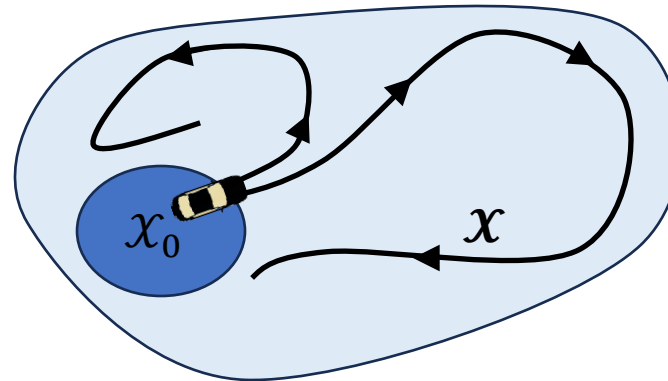
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Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0 \subseteq \mathcal{X}$, to verify that the system starting from $\mathcal{X}_0$ will **remain inside the safe set $\mathcal{X}$ for all time, regardless of disturbances**



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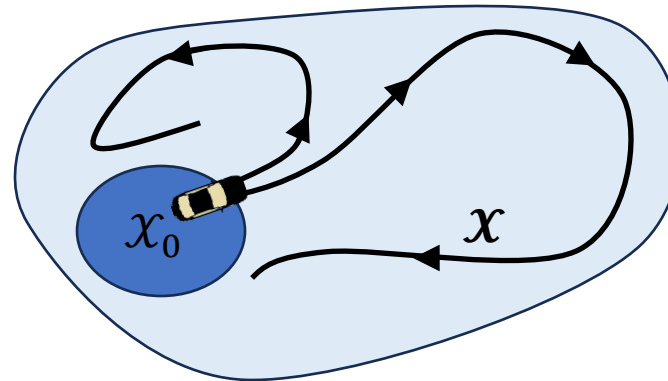
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Given $\lambda > 0$, finding Barrier Functions $B(x)$:

$$\begin{cases} B(x) \geq 0, \\ B(f(x, d)) \geq \lambda B(x), \\ B(x) < 0, \end{cases} \quad \begin{aligned} &\forall x \in \mathcal{X}_0, \\ &\forall x \in \mathcal{X}, \forall d \in \mathcal{D}, \\ &\forall x \in \mathbb{R}^n \setminus \mathcal{X}. \end{aligned}$$



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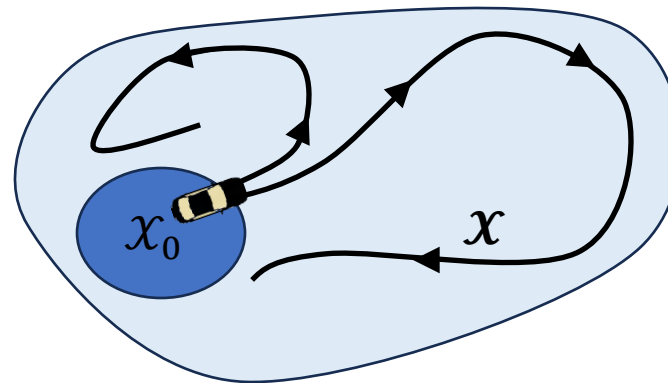
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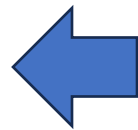
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Too Conservative!

Safe Probabilistic Invariance Verification

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Safe probabilistic invariance verification:

Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0 \subseteq \mathcal{X}$, to verify that the system starting from $\mathcal{X}_0$ will **remain inside the safe set $\mathcal{X}$ with a certain probability**

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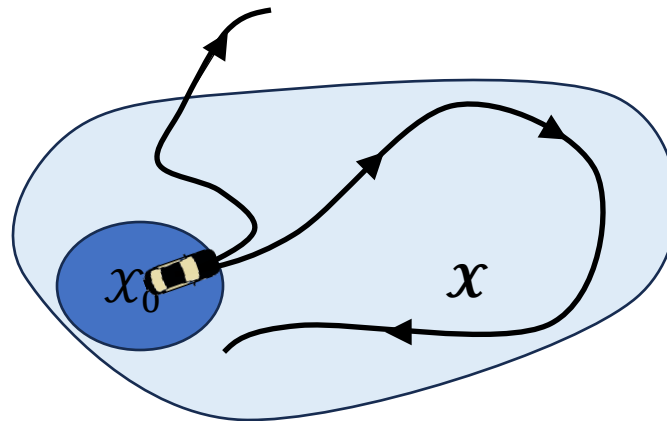
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This approach reduces conservatism by allowing probabilistic violations



Related Work

- Probabilistic Invariant Sets [e.g., E. Kofman, et.al., 2012 (Kofman); E. Kofman, et.al., 2016 (Kofman); L. Hewing, et.al., ECC 2018]: **Linear Systems**
- Finite-time Probabilistic Invariance Problem [e.g., A. Abate, et. al., 2008 (Automatica); A. Abate, et. Al., 2010 (European Journal of Control); C. Santoyo et.al, 2021 (Automatica)]
- Infinite-time Probabilistic Invariance Problem:
 - **Finite-time Probabilistic Invariance + Robust Invariant Sets** [e.g., I. Tkachev and A. Abate, et.al., CDC 2011; I. Tkachev and A. Abate, et.al., 2014 (Theoretical Computer Science)]
 - **Barrier Certificates Methods** [e.g., M. Anand, et. al., HSCC 2022; Peixin Wang, et.al., CAV 2024]
- ...

Problem Formulation

Stochastic Discrete-time Systems

$$x(k+1) = f(x(k), d(k)), \quad x(0) = x_0$$

- $d(k) \in \mathcal{D}$ is the stochastic perturbation input.
- $d(0), d(1), \dots$, are independent and identically distributed (i.i.d) on a probability space $(\mathcal{D}, \mathcal{F}, \mathbb{P})$, with support $\mathcal{D}$: for any measurable set $B \subseteq \mathcal{D}$, $\text{Prob}(d(l) \in B) = \mathbb{P}(B)$, $\forall l \in \mathbb{N}$. The expectation is denoted by $\mathbb{E}[\cdot]$.

A disturbance signal π is an ordered sequence $\{d(k), k \in \mathbb{N}\}$: a sample path of a stochastic process defined on the canonical sample space $\Omega^\infty = \mathcal{D} \times \mathcal{D} \times \dots$ with the probability measure $\mathbb{P}^\infty = \mathbb{P} \times \mathbb{P} \times \dots$.

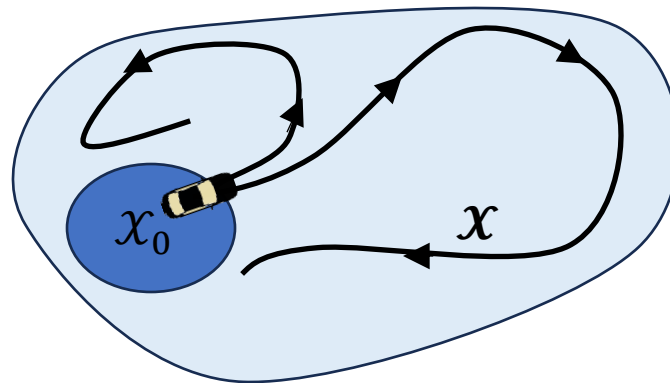
Trajectory $\phi_\pi^{x_0}(\cdot): \mathbb{N} \rightarrow \mathbb{R}^n$: $\phi_\pi^{x_0}(k+1) = f(\phi_\pi^{x_0}(k), \pi(k)), \phi_\pi^{x_0}(0) = x_0$

Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0 \subseteq \mathcal{X}$,

the safe probabilistic invariance verification is to compute lower and upper bounds, denoted by $\epsilon_1 \in [0,1]$ and $\epsilon_2 \in [0,1]$ respectively, for the liveness probability that the system, starting from any state in $\mathcal{X}_0$, will remain inside the safe set $\mathcal{X}$ for all time, i.e.,

to compute ϵ_1 and ϵ_2 such that

$$\epsilon_1 \leq \mathbb{P}^\infty \left(\forall k \in \mathbb{N}. \phi_\pi^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0 \right) \leq \epsilon_2.$$



Doob's Supermartingale Inequality Based Method

Doob's Supermartingale Inequality [J. Ville, 1939]

Let $(\Omega_1, \mathcal{F}, \mathbb{P}_1)$ be the probability space and $\{B_i\}_{i \in \mathbb{N}}$ be a non-negative supermartingale, then for $b > 0$,

$$\mathbb{P}_1 \left(\sup_{i \in \mathbb{N}} B_i \geq b \mid B_0 \right) \leq \frac{B_0}{b}.$$

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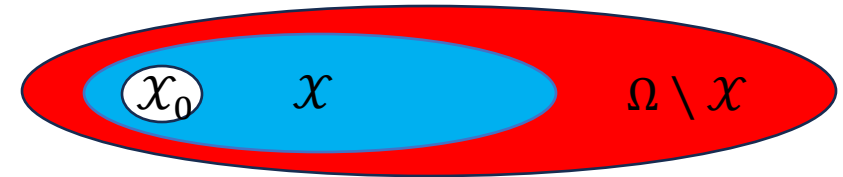
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Under the assumption that $\Omega \subset \mathbb{R}^n$ is a robust invariant set, i.e., $f(x, d): \Omega \times \mathcal{D} \rightarrow \Omega$, and $\mathcal{X}_0 \subseteq \Omega$, if there exists $v(x): \Omega \rightarrow \mathbb{R}$ such that

$$\begin{cases} v(x) \leq 1 - \epsilon_1, & \forall x \in \mathcal{X}_0, \\ v(x) \geq 0, & \forall x \in \Omega, \\ \mathbb{E}[v(f(x, d))] \leq v(x), & \forall x \in \Omega, \\ v(x) \geq 1, & \forall x \in \Omega \setminus \mathcal{X}, \end{cases}$$

where $\Omega \setminus \mathcal{X}$ is a set of unsafe states, then

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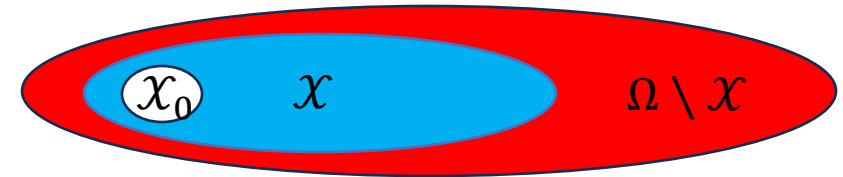
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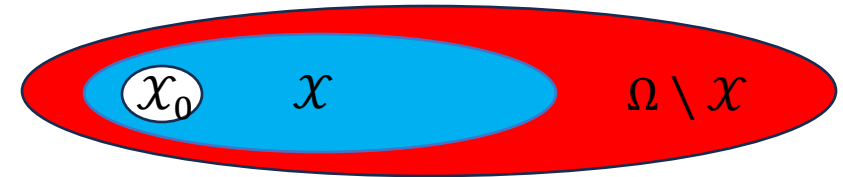
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- $\Omega \neq \mathbb{R}^n$: challenging to compute (if it exists)
- $\Omega = \mathbb{R}^n$: producing conservative lower bounds

Running Example

A computer-based model, which is modified from the reversed-time Van der Pol oscillator based on Euler's method with the time step 0.01:

$$\begin{cases} x(l+1) = x(l) - 0.02y(l), \\ y(l+1) = y(l) + 0.01 \left((0.8 + d(l))x(l) + 10(x^2(l) - 0.21)y(l) \right). \end{cases}$$

- $d(\cdot): \mathbb{N} \rightarrow \mathcal{D} = [-0.1, 0.1]$
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❖ Monto-carlo method: $\mathbb{P}^\infty(\forall k \in \mathbb{N}. \phi_\pi^{x_0}(\mathbf{k}) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \approx \mathbf{1}$

❖ Method in [M. Anand, et. al., HSCC 2022]($\Omega = \mathbb{R}^2$)+ semi-definite programming tool:
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Doob's Supermartingale Inequality Based Method

An auxiliary system

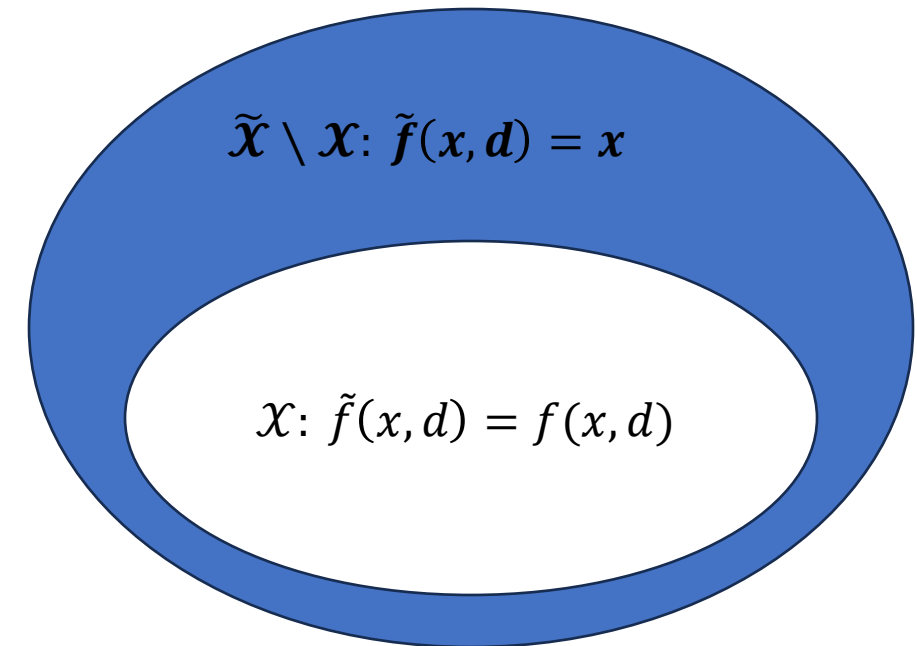
$$x(k+1) = \tilde{f}(x(k), d(k))$$

with

$$\tilde{f}(x, d) = f(x, d) \cdot 1_{\mathcal{X}}(x) + x \cdot 1_{\tilde{\mathcal{X}} \setminus \mathcal{X}}(x)$$

$\tilde{\mathcal{X}}$ is a set containing the union of the set $\mathcal{X}$ and all reachable states starting from $\mathcal{X}$ within one step:

$$\{x \mid x = f(x_0, d), x_0 \in \mathcal{X}, d \in \mathcal{D}\} \cup \mathcal{X} \subseteq \tilde{\mathcal{X}}$$



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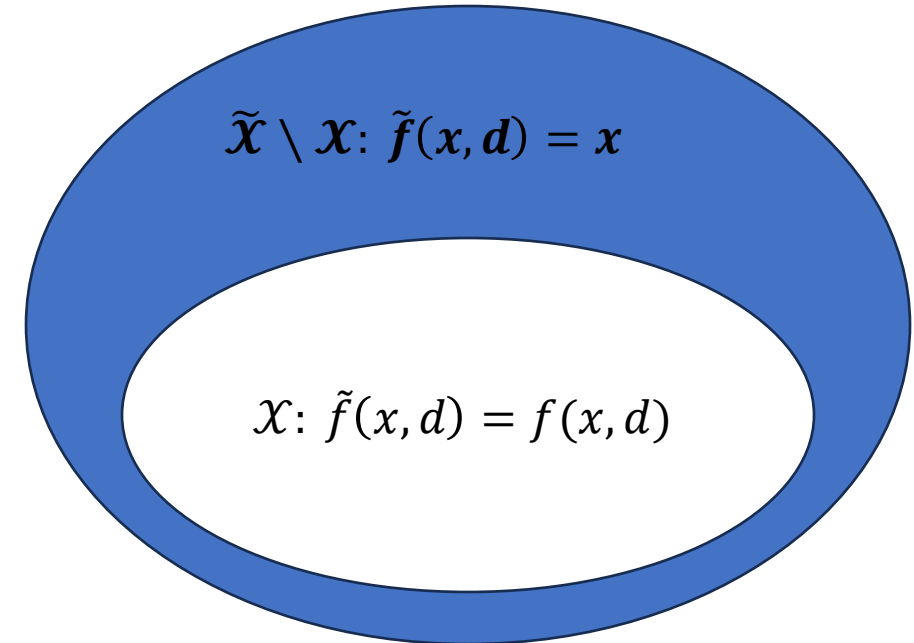
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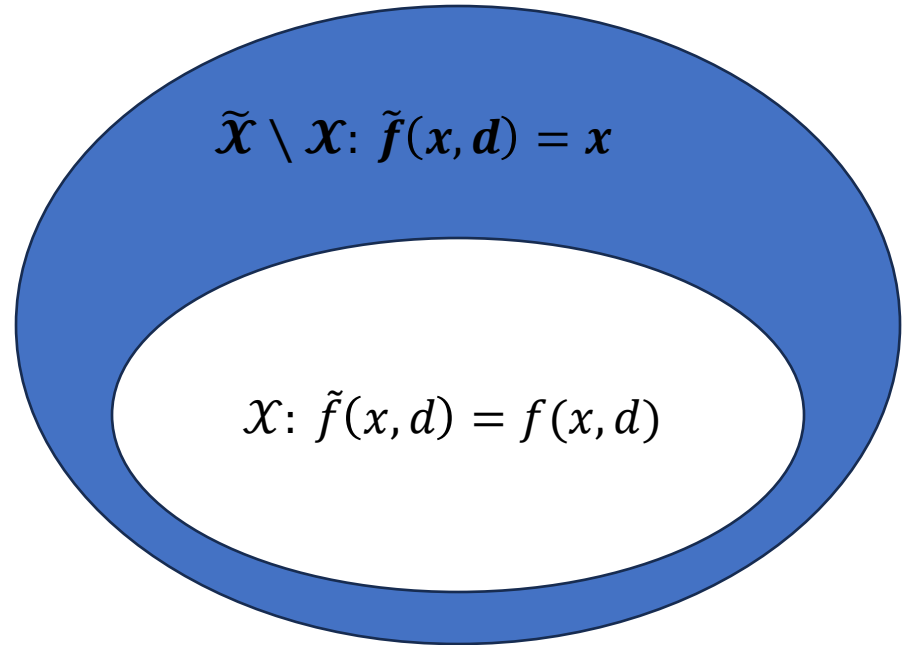
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Proposition 1 $\mathbb{P}^\infty(\forall k \in \mathbb{N}. \phi_\pi^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) = \mathbb{P}^\infty(\forall k \in \mathbb{N}. \tilde{\phi}_\pi^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0)$

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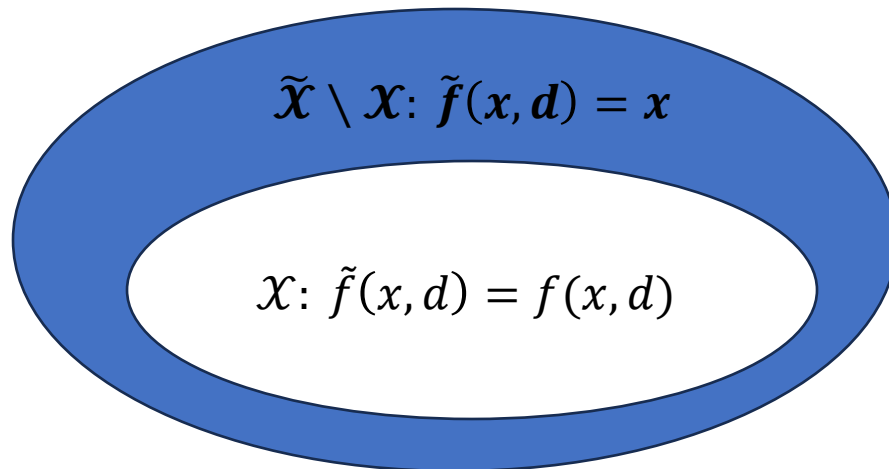
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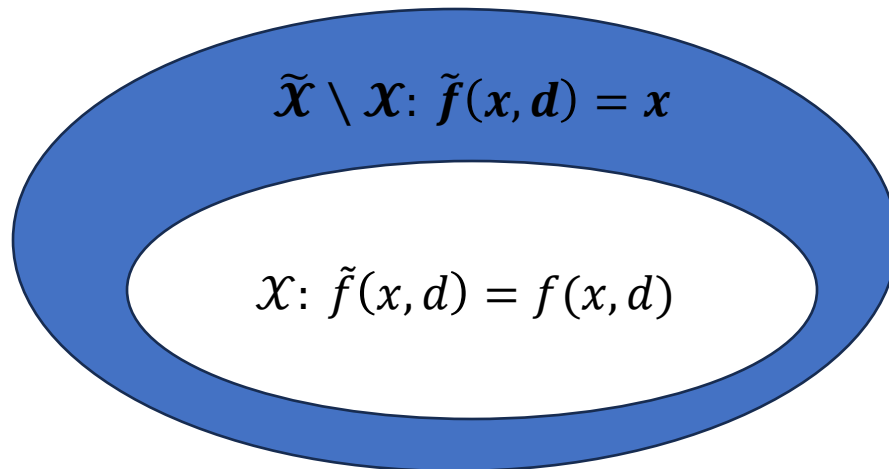


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Theorem 1 If there exists $v(x): \tilde{\mathcal{X}} \rightarrow \mathbb{R}$ such that

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$$\mathbb{P}^\infty(\forall k \in \mathbb{N}. \phi_\pi^{x_0}(\mathbf{k}) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \geq \mathbf{2.1368e - 07}$$

❖ **Our method** ($\tilde{\mathcal{X}} = \{(x, y) \mid x^2 + y^2 - \mathbf{2} \leq \mathbf{0}\}$) + semi-definite programming tool:

$$\mathbb{P}^\infty(\forall k \in \mathbb{N}. \phi_\pi^{x_0}(\mathbf{k}) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \geq \mathbf{0.9465}$$

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then $\mathbb{P}^\infty(\forall k \in \mathbb{N}. \phi_\pi^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \leq \epsilon_2$.

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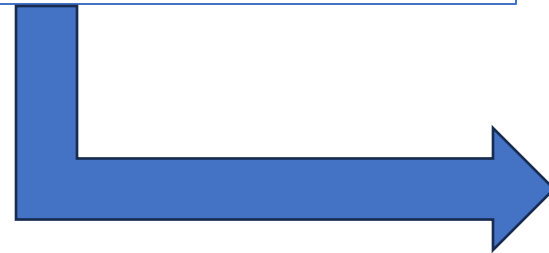
If $v(x)$ is bounded over $\tilde{\mathcal{X}}$, it provides strong guarantees of leaving the safe set $\mathcal{X}$ almost surely,
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Equation Relaxation Based Method

In [B. Xue, et. al., ACC 2021],

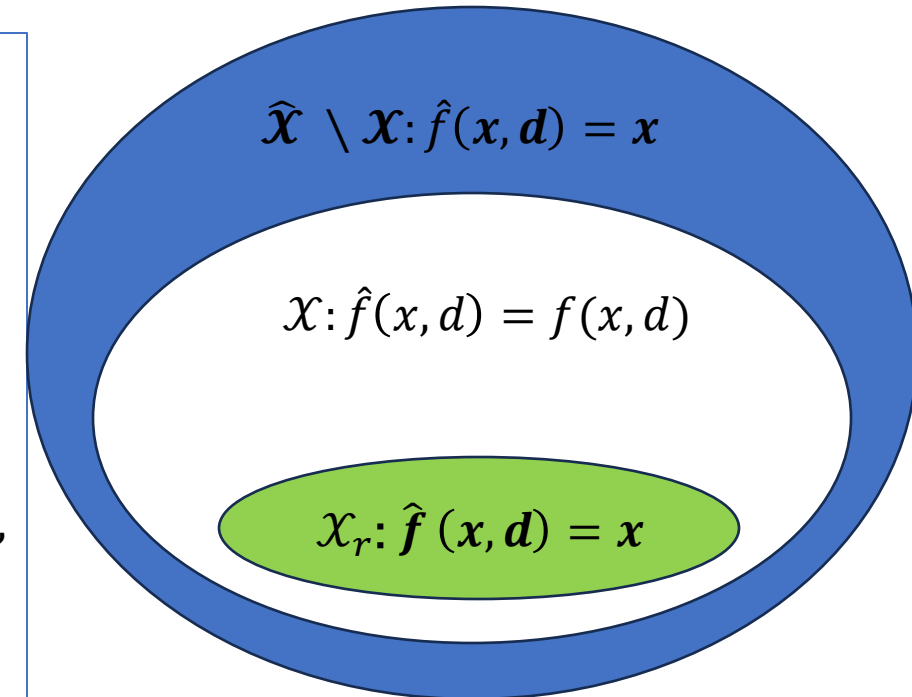
Given a safe set $\mathcal{X}$, a target set $\mathcal{X}_r$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_r, \mathcal{X}_0 \subseteq \mathcal{X}$, if there exist bounded functions $v(x): \hat{\mathcal{X}} \rightarrow \mathbb{R}$ and $w(x): \hat{\mathcal{X}} \rightarrow \mathbb{R}$ such that

$$\begin{cases} v(x) = \mathbb{E} \left[v \left(\hat{f}(x, d) \right) \right], \forall x \in \hat{\mathcal{X}}, \\ v(x) = 1_{\mathcal{X}_r}(x) + \mathbb{E} \left[w \left(\hat{f}(x, d) \right) \right] - w(x), \forall x \in \hat{\mathcal{X}}. \end{cases}$$

Then,

$$\mathbb{P}^\infty \left(\exists k \in \mathbb{N}. \phi_{\pi}^{x_0}(k) \in \mathcal{X}_r \wedge \forall l \in [0, k]. \phi_{\pi}^{x_0}(l) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0 \right) = v(x_0),$$

where $\hat{f}(x, d) = f(x, d) \cdot 1_{\mathcal{X}}(x) + x \cdot 1_{\hat{\mathcal{X}} \setminus \mathcal{X}}(x) + x \cdot 1_{\mathcal{X}_r}(x)$



B. Xue, R. Li, N. Zhan, and M. Fraenzle. Reach-avoid analysis for stochastic discrete-time systems. In 2021 American Control Conference (ACC), pages 4879–4885. IEEE, 2021

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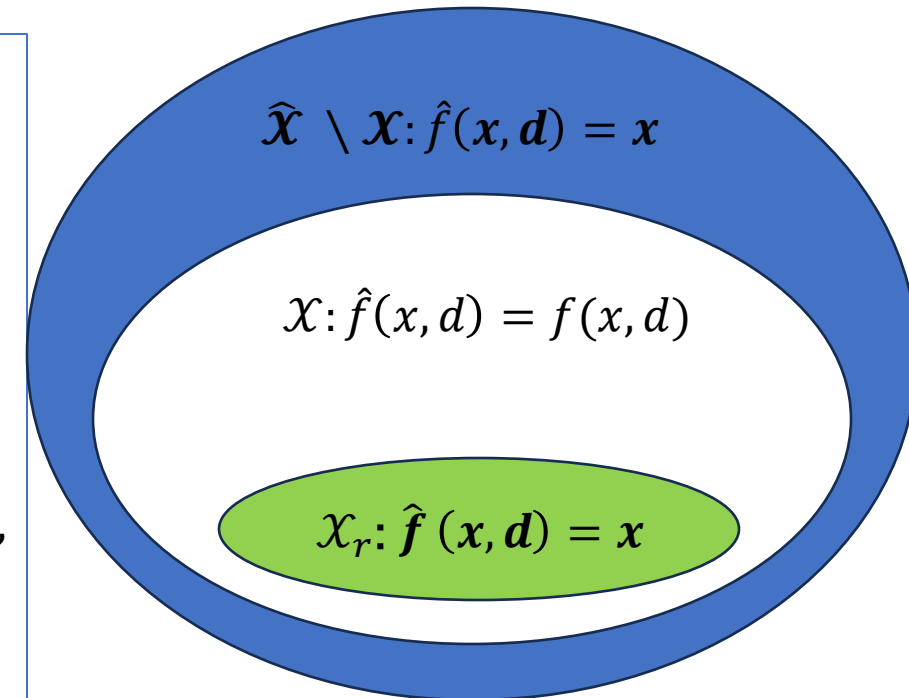
Then,

$$\mathbb{P}^\infty \left(\exists k \in \mathbb{N}. \phi_{\pi}^{x_0}(k) \in \mathcal{X}_r \wedge \forall l \in [0, k]. \phi_{\pi}^{x_0}(l) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0 \right) = v(x_0),$$

where $\hat{f}(x, d) = f(x, d) \cdot 1_{\mathcal{X}}(x) + x \cdot 1_{\hat{\mathcal{X}} \setminus \mathcal{X}}(x) + x \cdot 1_{\mathcal{X}_r}(x)$

$\hat{\mathcal{X}}$ is a set containing the union of the set $\mathcal{X}$ and all reachable states starting from $\mathcal{X}$ within one step

$$\{x \mid x = f(x_0, d), x_0 \in \mathcal{X}, d \in \mathcal{D}\} \cup \mathcal{X} \subseteq \hat{\mathcal{X}}$$



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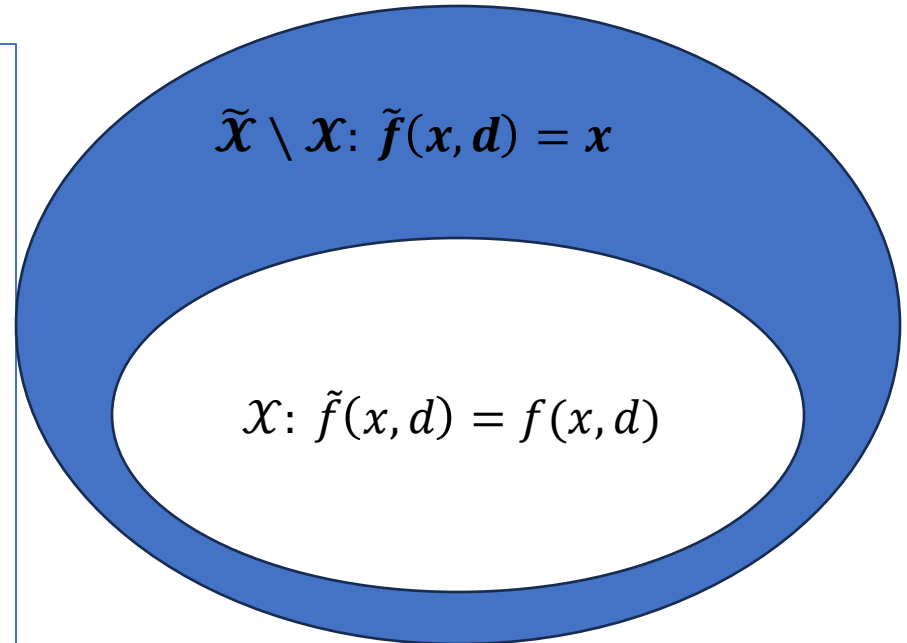
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Then,

$$\mathbb{P}^\infty \left(\exists k \in \mathbb{N}. \phi_\pi^{x_0}(k) \in \tilde{\mathcal{X}} \setminus \mathcal{X} \mid x_0 \in \mathcal{X}_0 \right) = v(x_0).$$

Thus,

$$\mathbb{P}^\infty \left(\forall k \in \mathbb{N}. \phi_\pi^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0 \right) = 1 - v(x_0).$$



Equation Relaxation Based Method

Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist bounded functions $v(x): \tilde{\mathcal{X}} \rightarrow \mathbb{R}$ and $w(x): \tilde{\mathcal{X}} \rightarrow \mathbb{R}$ such that

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Then,

$$\mathbb{P}^\infty \left(\forall k \in \mathbb{N}. \phi_\pi^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0 \right) \geq \epsilon_1.$$

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$$\tilde{f}(x, d) = f(x, d) \cdot 1_{\mathcal{X}}(x) + x \cdot 1_{\tilde{\mathcal{X}} \setminus \mathcal{X}}(x)$$

Theorem 3 Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist bounded functions $v(x): \tilde{\mathcal{X}} \rightarrow \mathbb{R}$ and $w(x): \tilde{\mathcal{X}} \rightarrow \mathbb{R}$ such that

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Then,

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Comparison

Doob's Supermartingale Inequality Based Method

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then

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Equation Relaxation Based Method

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Sufficient and Necessary Barrier-like Conditions

Comparison

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then

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Theorem 2 Let $\mathcal{X}$ be a closed set. If there exists $v(x): \tilde{\mathcal{X}} \rightarrow \mathbb{R}$ such that

$$\begin{cases} v(x) \leq \epsilon_2, & \forall x \in \mathcal{X}_0, \\ \mathbb{E}[v(f(x, d))] - v(x) \leq -\delta, & \forall x \in \mathcal{X}, \\ v(x) \geq 1, & \forall x \in \partial \tilde{\mathcal{X}} \setminus \partial(\tilde{\mathcal{X}} \setminus \mathcal{X}), \\ v(x) \geq 0, & \forall x \in \tilde{\mathcal{X}}, \end{cases}$$

then

$$\mathbb{P}^\infty(\forall k \in \mathbb{N}. \phi_\pi^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \leq \epsilon_2.$$

Equation Relaxation Based Method

Theorem 3 Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist bounded functions $v(x): \tilde{\mathcal{X}} \rightarrow \mathbb{R}$ and $w(x): \tilde{\mathcal{X}} \rightarrow \mathbb{R}$ such that

$$\begin{cases} v(x) \leq 1 - \epsilon_1, & \forall x \in \mathcal{X}_0, \\ v(x) \geq \mathbb{E}[v(f(x, d))], & \forall x \in \mathcal{X}, \\ v(x) \geq \mathbb{E}[w(f(x, d))] - w(x), & \forall x \in \mathcal{X}, \\ v(x) \geq 1, & \forall x \in \tilde{\mathcal{X}} \setminus \mathcal{X}, \end{cases}$$

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$$\mathbb{P}^\infty(\forall k \in \mathbb{N}. \phi_\pi^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \geq \epsilon_1.$$

Theorem 4 Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist bounded functions $v(x): \tilde{\mathcal{X}} \rightarrow \mathbb{R}$ and $w(x): \tilde{\mathcal{X}} \rightarrow \mathbb{R}$ such that

$$\begin{cases} v(x) \geq 1 - \epsilon_2, & \forall x \in \mathcal{X}_0, \\ v(x) \leq \mathbb{E}[v(f(x, d))], & \forall x \in \mathcal{X}, \\ v(x) \leq \mathbb{E}[w(f(x, d))] - w(x), & \forall x \in \mathcal{X}, \\ v(x) \leq 1, & \forall x \in \tilde{\mathcal{X}} \setminus \mathcal{X}, \end{cases}$$

then

$$\mathbb{P}^\infty(\forall k \in \mathbb{N}. \phi_\pi^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \leq \epsilon_2.$$

Comparison

Doob's Supermartingale Inequality Based Method

Theorem 1 Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exists $v(x): \tilde{\mathcal{X}} \rightarrow \mathbb{R}$ such that

$$\begin{cases} v(x) \leq 1 - \epsilon_1, & \forall x \in \mathcal{X}_0, \\ v(x) \geq 0, & \forall x \in \tilde{\mathcal{X}}, \\ \mathbb{E}[v(f(x, d))] \leq v(x), & \forall x \in \mathcal{X}, \\ v(x) \geq 1, & \forall x \in \tilde{\mathcal{X}} \setminus \mathcal{X}, \end{cases}$$

then

$$\mathbb{P}^\infty(\forall k \in \mathbb{N}. \phi_\pi^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \geq \epsilon_1.$$

Theorem 2 Let $\mathcal{X}$ be a closed set. If there exists $v(x): \tilde{\mathcal{X}} \rightarrow \mathbb{R}$ such that

$$\begin{cases} v(x) \leq \epsilon_2, & \forall x \in \mathcal{X}_0, \\ \mathbb{E}[v(f(x, d))] - v(x) \leq -\delta, & \forall x \in \mathcal{X}, \\ v(x) \geq 1, & \forall x \in \partial \tilde{\mathcal{X}} \setminus \partial(\tilde{\mathcal{X}} \setminus \mathcal{X}), \\ v(x) \geq 0, & \forall x \in \tilde{\mathcal{X}}, \end{cases}$$

then

$$\mathbb{P}^\infty(\forall k \in \mathbb{N}. \phi_\pi^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \leq \epsilon_2.$$

Equation Relaxation Based Method

Theorem 3 Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist bounded functions $v(x): \tilde{\mathcal{X}} \rightarrow \mathbb{R}$ and $w(x): \tilde{\mathcal{X}} \rightarrow \mathbb{R}$ such that

$$\begin{cases} v(x) \leq 1 - \epsilon_1, & \forall x \in \mathcal{X}_0, \\ v(x) \geq \mathbb{E}[v(f(x, d))], & \forall x \in \mathcal{X}, \\ v(x) \geq \mathbb{E}[w(f(x, d))] - w(x), & \forall x \in \mathcal{X}, \\ v(x) \geq 1, & \forall x \in \tilde{\mathcal{X}} \setminus \mathcal{X}, \end{cases}$$

then

$$\mathbb{P}^\infty(\forall k \in \mathbb{N}. \phi_\pi^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \geq \epsilon_1.$$

Theorem 4 Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist bounded functions $v(x): \tilde{\mathcal{X}} \rightarrow \mathbb{R}$ and $w(x): \tilde{\mathcal{X}} \rightarrow \mathbb{R}$ such that

$$\begin{cases} v(x) \geq 1 - \epsilon_2, & \forall x \in \mathcal{X}_0, \\ v(x) \leq \mathbb{E}[v(f(x, d))], & \forall x \in \mathcal{X}, \\ v(x) \leq \mathbb{E}[w(f(x, d))] - w(x), & \forall x \in \mathcal{X}, \\ v(x) \leq 1, & \forall x \in \tilde{\mathcal{X}} \setminus \mathcal{X}, \end{cases}$$

then

$$\mathbb{P}^\infty(\forall k \in \mathbb{N}. \phi_\pi^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \leq \epsilon_2.$$

$$1 - v(x) \text{ with } w(x) = M(1 - v(x)), \text{ where } M\delta \geq \sup_{x \in \tilde{\mathcal{X}}} 1 - v(x)$$

Optimization

Doob's Supermartingale Inequality Based Method

Op1

Max_v ϵ_1

$$\text{s.t.} \begin{cases} v(x) \leq 1 - \epsilon_1, & \forall x \in \mathcal{X}_0, \\ v(x) \geq 0, & \forall x \in \tilde{\mathcal{X}}, \\ \mathbb{E}[v(f(x, d))] \leq v(x), & \forall x \in \mathcal{X}, \\ v(x) \geq 1, & \forall x \in \tilde{\mathcal{X}} \setminus \mathcal{X}. \end{cases}$$

$$\mathbb{P}^\infty(\forall k \in \mathbb{N}. \phi_\pi^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \geq \epsilon_1.$$

Equation Relaxation Based Method

Op3

Max_{v,w} ϵ_1

$$\text{s.t.} \begin{cases} v(x) \leq 1 - \epsilon_1, & \forall x \in \mathcal{X}_0, \\ v(x) \geq \mathbb{E}[v(f(x, d))], & \forall x \in \mathcal{X}, \\ v(x) \geq \mathbb{E}[w(f(x, d))] - w(x), & \forall x \in \mathcal{X}, \\ v(x) \geq 1, & \forall x \in \tilde{\mathcal{X}} \setminus \mathcal{X}. \end{cases}$$

$$\mathbb{P}^\infty(\forall k \in \mathbb{N}. \phi_\pi^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \geq \epsilon_1.$$

Op2

Min_v ϵ_2

$$\text{s.t.} \begin{cases} v(x) \leq \epsilon_2, & \forall x \in \mathcal{X}_0, \\ \mathbb{E}[v(f(x, d))] - v(x) \leq -\delta, & \forall x \in \mathcal{X}, \\ v(x) \geq 1, & \forall x \in \partial\tilde{\mathcal{X}} \setminus \partial(\tilde{\mathcal{X}} \setminus \mathcal{X}), \\ v(x) \geq 0, & \forall x \in \tilde{\mathcal{X}}. \end{cases}$$

$$\mathbb{P}^\infty(\forall k \in \mathbb{N}. \phi_\pi^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \leq \epsilon_2.$$

Op4

Min_{v,w} ϵ_2

$$\text{s.t.} \begin{cases} v(x) \geq 1 - \epsilon_2, & \forall x \in \mathcal{X}_0, \\ v(x) \leq \mathbb{E}[v(f(x, d))], & \forall x \in \mathcal{X}, \\ v(x) \leq \mathbb{E}[w(f(x, d))] - w(x), & \forall x \in \mathcal{X}, \\ v(x) \leq 1, & \forall x \in \tilde{\mathcal{X}} \setminus \mathcal{X}. \end{cases}$$

$$\mathbb{P}^\infty(\forall k \in \mathbb{N}. \phi_\pi^{x_0}(k) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \leq \epsilon_2.$$

They were relaxed into semi-definite programming problems

Example

Consider

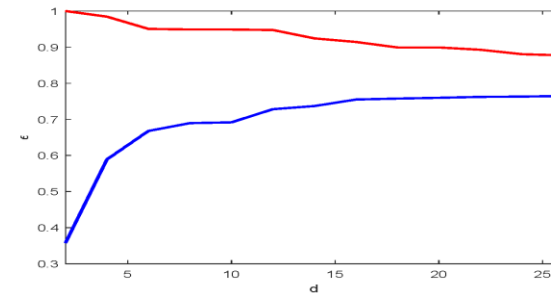
$$x(l+1) = (-0.5 + d(l))x(l)$$

- $d(\cdot): \mathbb{N} \rightarrow \mathcal{D} = [-1, 1]$ (**uniform distribution**)
- $\mathcal{X} = \{x \mid x^2 - 1 \leq 0\}$
- $\mathcal{X}_0 = \{x \mid (x + 0.8)^2 = 0\}$ (i.e., $x_0 = -0.8$)
- $\hat{\mathcal{X}} = \{x \mid x^2 - 2 \leq 0\}$

The lower and upper bounds of the liveness probability obtained by Monte Carlo are $\epsilon_1 = \epsilon_2 = 0.8312$

Op3 and Op4													
d	2	4	6	8	10	12	14	16	18	20	22	24	26
ϵ_1	0.3574	0.5890	0.6678	0.6895	0.6917	0.7281	0.7368	0.7549	0.7575	0.7597	0.7622	0.7630	0.7647
ϵ_2	1.0000	0.9844	0.9505	0.9489	0.9488	0.9474	0.9242	0.9143	0.8991	0.8991	0.8927	0.8804	0.8771

Op1													
d	2	4	6	8	10	12	14	16	18	20	22	24	26
ϵ_1	0.3574	0.5890	0.6678	0.6895	0.6917	0.7281	0.7368	0.7549	0.7575	0.7597	0.7622	0.7630	0.7647



Safe Probabilistic Invariance Verification of Stochastic Continuous-time Systems

Stochastic continuous-time systems modeled by time-homogeneous SDEs:

$$dX(t, w) = b(X(t, w))dt + \sigma(X(t, w))dW(t, w), t \geq 0$$

Its trajectory

$$X^{x_0}(\cdot, w): [0, T^{x_0}(w)) \times \Omega \rightarrow R^n$$

satisfies

$$X^{x_0}(t, w) = x_0 + \int_0^t b(X^{x_0}(s, w))ds + \int_0^t \sigma(X^{x_0}(s, w))dW(s, w)$$

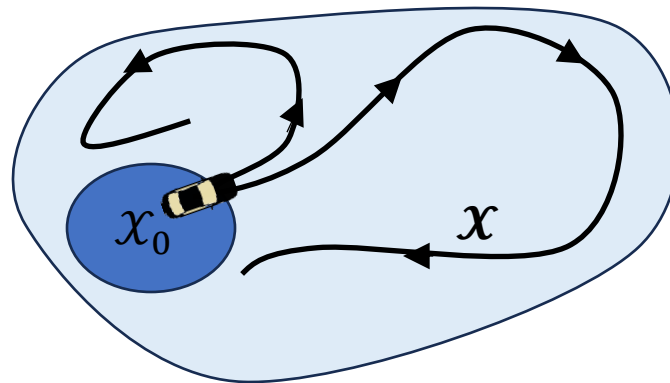
Bai Xue, Naijun Zhan and Martin Fränzle. Reach-Avoid Analysis for Polynomial Stochastic Differential Equations. IEEE Transactions on Automatic Control (IEEE TAC), 69(3): 1882--1889, 2024.

Given a safe set $\mathcal{X}$ (bounded and open) and an initial set $\mathcal{X}_0 \subseteq \mathcal{X}$,

the safe probabilistic invariance verification is to compute lower and upper bounds, denoted by $\epsilon_1 \in [0,1]$ and $\epsilon_2 \in [0,1]$ respectively, for the liveness probability that the system, starting from any state in $\mathcal{X}_0$, will remain inside the safe set $\mathcal{X}$ for all time, i.e.,

to compute ϵ_1 and ϵ_2 such that

$$\epsilon_1 \leq \mathbb{P}(\forall t \in \mathbb{R}_{\geq 0}. X^{x_0}(t, w) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \leq \epsilon_2.$$



Doob's Supermartingale Inequality Based Method

Doob's Supermartingale Inequality [J. Ville, 1939]

Let $(\Omega_1, \mathcal{F}, \mathbb{P}_1)$ be the probability space and $\{B_i\}_{i \in \mathbb{N}}$ be a non-negative supermartingale, then for $b > 0$,

$$\mathbb{P}_1 \left(\sup_{i \in \mathbb{N}} B_i \geq b \mid B_0 \right) \leq \frac{B_0}{b}$$

In [S. Prajna, et. al., 2007(IEEE TAC)],

Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist $v(x) \in \mathcal{C}^2(\bar{\mathcal{X}})$ and $u(x) \in \mathcal{C}^2(\bar{\mathcal{X}})$ such that

$$\begin{cases} v(x) \leq 1 - \epsilon_1, & \forall x \in \mathcal{X}_0, \\ \mathcal{A}v(x) \leq 0, & \forall x \in \bar{\mathcal{X}}, \\ v(x) \geq 1, & \forall x \in \partial\mathcal{X}, \\ v(x) \geq 0, & \forall x \in \bar{\mathcal{X}}. \end{cases}$$

Then,

$$\mathbb{P}(\forall t \in \mathbb{R}_{\geq 0}. X^{x_0}(t, w) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \geq \epsilon_1.$$

S. Prajna, A. Jadbabaie, and G. J. Pappas. A framework for worst-case and stochastic safety verification using barrier certificates. IEEE Transactions on Automatic Control, 52(8):1415–1428, 2007.

Doob's Supermartingale Inequality Based Method

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$$\begin{cases} v(x) \leq 1 - \epsilon_1, & \forall x \in \mathcal{X}_0, \\ \mathcal{A}v(x) \leq 0, & \forall x \in \bar{\mathcal{X}}, \\ v(x) \geq 1, & \forall x \in \partial\mathcal{X}, \\ v(x) \geq 0, & \forall x \in \bar{\mathcal{X}}. \end{cases}$$

Then,

$$\mathbb{P}(\forall t \in \mathbb{R}_{\geq 0}. X^{x_0}(t, w) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \geq \epsilon_1.$$

S. Prajna, A. Jadbabaie, and G. J. Pappas. A framework for worst-case and stochastic safety verification using barrier certificates. IEEE Transactions on Automatic Control, 52(8):1415–1428, 2007.

There are **no barrier-like conditions** based on the Doob's nonnegative supermartingale inequality that have been developed in previous studies to **examine upper bounds of the reachability probability**

Equation Relaxation Based Method

In [B. Xue, et. al., 2024(IEEE TAC)],

Given a safe set $\mathcal{X}$, a target set $\mathcal{X}_r$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_r, \mathcal{X}_0 \subseteq \mathcal{X}$, if there exist $v(x) \in \mathcal{C}^2(\bar{\mathcal{X}})$ and $u(x) \in \mathcal{C}^2(\bar{\mathcal{X}})$ such that

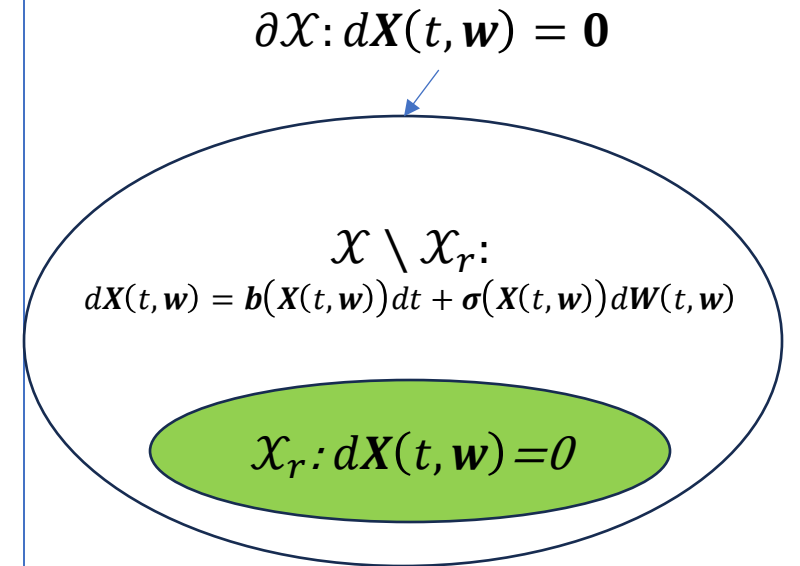
$$\begin{cases} \tilde{\mathcal{A}}v(x) = 0, \forall x \in \bar{\mathcal{X}}, \\ v(x) = 1_{\mathcal{X}_r}(x) + \tilde{\mathcal{A}}u(x), \forall x \in \bar{\mathcal{X}}. \end{cases}$$

Then,

$$\mathbb{P}(\exists \tau \in \mathbb{R}_{\geq 0}. X^{x_0}(\tau, w) \in \mathcal{X}_r \wedge \forall t \in [0, \tau]. X^{x_0}(t, w) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) = v(x_0),$$

where

$$\tilde{\mathcal{A}}v(x) = \begin{cases} \mathcal{A}v(x) \left(= \frac{\partial v(x)}{\partial x} b(x) + \frac{1}{2} \text{tr}(\sigma^\top(x) \frac{\partial^2 v(x)}{\partial x^2} \sigma(x)) \right), & \text{if } x \in \mathcal{X} \setminus \mathcal{X}_r, \\ 0, & \text{if } x \in \partial\mathcal{X} \cup \mathcal{X}_r. \end{cases}$$



B. Xue, N. Zhan, and M. Fraenzle. Reach-avoid analysis for polynomial stochastic differential equations. IEEE Transactions on Automatic Control, 69(3):1882–1889, 2024.

Equation Relaxation Based Method

Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist $v(x) \in \mathcal{C}^2(\bar{\mathcal{X}})$ and $u(x) \in \mathcal{C}^2(\bar{\mathcal{X}})$ such that

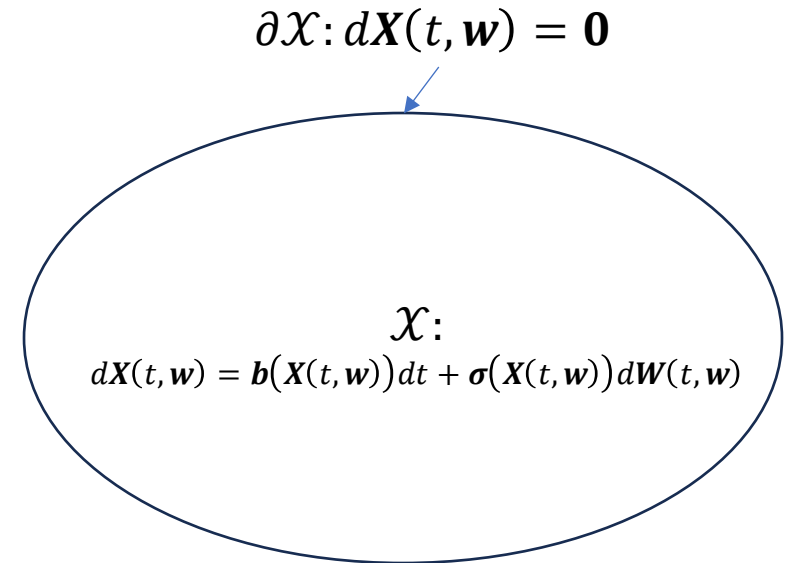
$$\begin{cases} \tilde{\mathcal{A}}v(x) = 0, \forall x \in \bar{\mathcal{X}}, \\ v(x) = 1_{\partial\mathcal{X}}(x) + \tilde{\mathcal{A}}u(x), \forall x \in \bar{\mathcal{X}}. \end{cases}$$

Then,

$$\mathbb{P}(\exists \tau \in \mathbb{R}_{\geq 0}. X^{x_0}(\tau, w) \in \partial\mathcal{X} \wedge \forall t \in [0, \tau). X^{x_0}(t, w) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) = v(x_0).$$

Thus,

$$\mathbb{P}(\forall t \in \mathbb{R}_{\geq 0}. X^{x_0}(t, w) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) = 1 - v(x_0).$$



Equation Relaxation Based Method

Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist $v(x) \in \mathcal{C}^2(\bar{\mathcal{X}})$ and $u(x) \in \mathcal{C}^2(\bar{\mathcal{X}})$ such that

$$\begin{cases} v(x) \leq 1 - \epsilon_1, & \forall x \in \mathcal{X}_0, \\ \tilde{\mathcal{A}}v(x) \leq 0, & \forall x \in \bar{\mathcal{X}}, \\ v(x) \geq 1_{\partial\mathcal{X}}(x) + \tilde{\mathcal{A}}u(x), & \forall x \in \bar{\mathcal{X}}. \end{cases}$$

Then,

$$\mathbb{P}(\forall t \in \mathbb{R}_{\geq 0}. X^{x_0}(\tau, w) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \geq \epsilon_1.$$

Equation Relaxation Based Method

Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist $v(x) \in \mathcal{C}^2(\bar{\mathcal{X}})$ and $u(x) \in \mathcal{C}^2(\bar{\mathcal{X}})$ such that

$$\begin{cases} v(x) \leq 1 - \epsilon_1, & \forall x \in \mathcal{X}_0, \\ \tilde{\mathcal{A}}v(x) \leq 0, & \forall x \in \bar{\mathcal{X}}, \\ v(x) \geq 1_{\partial\mathcal{X}}(x) + \tilde{\mathcal{A}}u(x), & \forall x \in \bar{\mathcal{X}}. \end{cases}$$

Then,

$$\mathbb{P}(\forall t \in \mathbb{R}_{\geq 0}. X^{x_0}(t, w) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \geq \epsilon_1.$$



$$\tilde{\mathcal{A}}u(x) = \tilde{\mathcal{A}}v(x) = 0 \quad \forall x \in \partial\mathcal{X}$$

Theorem 6 Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist $v(x) \in \mathcal{C}^2(\bar{\mathcal{X}})$ and $u(x) \in \mathcal{C}^2(\bar{\mathcal{X}})$ such that

$$\begin{cases} v(x) \leq 1 - \epsilon_1, & \forall x \in \mathcal{X}_0, \\ \mathcal{A}v(x) \leq 0, & \forall x \in \mathcal{X}, \\ v(x) \geq 1, & \forall x \in \partial\mathcal{X}, \\ v(x) \geq \mathcal{A}u(x), & \forall x \in \mathcal{X}. \end{cases}$$

Then,

$$\mathbb{P}(\forall t \in \mathbb{R}_{\geq 0}. X^{x_0}(t, w) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \geq \epsilon_1.$$

Equation Relaxation Based Method

Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist $v(x) \in \mathcal{C}^2(\bar{\mathcal{X}})$ and $u(x) \in \mathcal{C}^2(\bar{\mathcal{X}})$ such that

$$\begin{cases} v(x) \leq 1 - \epsilon_1, & \forall x \in \mathcal{X}_0, \\ \tilde{\mathcal{A}}v(x) \leq 0, & \forall x \in \bar{\mathcal{X}}, \\ v(x) \geq 1_{\partial\mathcal{X}}(x) + \tilde{\mathcal{A}}u(x), & \forall x \in \bar{\mathcal{X}}, \end{cases}$$

Then,

$$\mathbb{P}(\forall t \in \mathbb{R}_{\geq 0}. X^{x_0}(\tau, w) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \geq \epsilon_1.$$



$$\tilde{\mathcal{A}}u(x) = \tilde{\mathcal{A}}v(x) = 0 \quad \forall x \in \partial\mathcal{X}$$



Theorem 6 Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist $v(x) \in \mathcal{C}^2(\bar{\mathcal{X}})$ and $u(x) \in \mathcal{C}^2(\bar{\mathcal{X}})$ such that

$$\begin{cases} v(x) \leq 1 - \epsilon_1, & \forall x \in \mathcal{X}_0, \\ \mathcal{A}v(x) \leq 0, & \forall x \in \mathcal{X}, \\ v(x) \geq 1, & \forall x \in \partial\mathcal{X}, \\ v(x) \geq \mathcal{A}u(x), & \forall x \in \mathcal{X}. \end{cases}$$

Then,

$$\mathbb{P}(\forall t \in \mathbb{R}_{\geq 0}. X^{x_0}(\tau, w) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \geq \epsilon_1.$$

Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist $v(x) \in \mathcal{C}^2(\bar{\mathcal{X}})$ and $u(x) \in \mathcal{C}^2(\bar{\mathcal{X}})$ such that

$$\begin{cases} v(x) \geq 1 - \epsilon_2, & \forall x \in \mathcal{X}_0, \\ \tilde{\mathcal{A}}v(x) \geq 0, & \forall x \in \bar{\mathcal{X}}, \\ v(x) \leq 1_{\partial\mathcal{X}}(x) + \tilde{\mathcal{A}}u(x), & \forall x \in \bar{\mathcal{X}}. \end{cases}$$

Then,

$$\mathbb{P}(\forall t \in \mathbb{R}_{\geq 0}. X^{x_0}(\tau, w) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \leq \epsilon_2.$$

Theorem 7 Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist $v(x) \in \mathcal{C}^2(\bar{\mathcal{X}})$ and $u(x) \in \mathcal{C}^2(\bar{\mathcal{X}})$ such that

$$\begin{cases} v(x) \geq 1 - \epsilon_2, & \forall x \in \mathcal{X}_0, \\ \mathcal{A}v(x) \geq 0, & \forall x \in \mathcal{X}, \\ v(x) \leq 1, & \forall x \in \partial\mathcal{X}, \\ v(x) \leq \mathcal{A}u(x), & \forall x \in \mathcal{X}. \end{cases}$$

Then,

$$\mathbb{P}(\forall t \in \mathbb{R}_{\geq 0}. X^{x_0}(\tau, w) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \leq \epsilon_2.$$

Comparison

Doob's Supermartingale Inequality Based Method

Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist $v(x) \in \mathcal{C}^2(\bar{\mathcal{X}})$ and $u(x) \in \mathcal{C}^2(\bar{\mathcal{X}})$ such that

$$\begin{cases} v(x) \leq 1 - \epsilon_1, & \forall x \in \mathcal{X}_0, \\ \mathcal{A}v(x) \leq 0, & \forall x \in \bar{\mathcal{X}}, \\ v(x) \geq 1, & \forall x \in \partial\mathcal{X}, \\ v(x) \geq 0, & \forall x \in \bar{\mathcal{X}}. \end{cases}$$

Then,

$$\mathbb{P}(\forall t \in \mathbb{R}_{\geq 0}. X^{x_0}(t, w) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \geq \epsilon_1.$$



Equation Relaxation Based Method

Theorem 6 Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist $v(x) \in \mathcal{C}^2(\bar{\mathcal{X}})$ and $u(x) \in \mathcal{C}^2(\bar{\mathcal{X}})$ such that

$$\begin{cases} v(x) \leq 1 - \epsilon_1, & \forall x \in \mathcal{X}_0, \\ \mathcal{A}v(x) \leq 0, & \forall x \in \mathcal{X}, \\ v(x) \geq 1, & \forall x \in \partial\mathcal{X}, \\ v(x) \geq \mathcal{A}u(x), & \forall x \in \mathcal{X}. \end{cases}$$

Then,

$$\mathbb{P}(\forall t \in \mathbb{R}_{\geq 0}. X^{x_0}(t, w) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \geq \epsilon_1.$$

Theorem 7 Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist $v(x) \in \mathcal{C}^2(\bar{\mathcal{X}})$ and $u(x) \in \mathcal{C}^2(\bar{\mathcal{X}})$ such that

$$\begin{cases} v(x) \geq 1 - \epsilon_2, & \forall x \in \mathcal{X}_0, \\ \mathcal{A}v(x) \geq 0, & \forall x \in \mathcal{X}, \\ v(x) \leq 1, & \forall x \in \partial\mathcal{X}, \\ v(x) \leq \mathcal{A}u(x), & \forall x \in \mathcal{X}. \end{cases}$$

Then,

$$\mathbb{P}(\forall t \in \mathbb{R}_{\geq 0}. X^{x_0}(t, w) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \leq \epsilon_2.$$

Optimization

Doob's Supermartingale Inequality Based Method

Op5

Max_v ϵ_1

$$\text{s.t.} \begin{cases} v(x) \leq 1 - \epsilon_1, & \forall x \in \mathcal{X}_0, \\ \mathcal{A}v(x) \leq 0, & \forall x \in \bar{\mathcal{X}}, \\ v(x) \geq 1, & \forall x \in \partial\mathcal{X}, \\ v(x) \geq 0, & \forall x \in \bar{\mathcal{X}}. \end{cases}$$

$$\mathbb{P}(\forall t \in \mathbb{R}_{\geq 0}. X^{x_0}(t, w) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \geq \epsilon_1$$

Equation Relaxation Based Method

Op6

Max_{v,u} ϵ_1

$$\text{s.t.} \begin{cases} v(x) \leq 1 - \epsilon_1, & \forall x \in \mathcal{X}_0, \\ \mathcal{A}v(x) \leq 0, & \forall x \in \mathcal{X}, \\ v(x) \geq 1, & \forall x \in \partial\mathcal{X}, \\ v(x) \geq \mathcal{A}u(x), & \forall x \in \mathcal{X}. \end{cases}$$

$$\mathbb{P}(\forall t \in \mathbb{R}_{\geq 0}. X^{x_0}(t, w) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \geq \epsilon_1$$

Op7

Min_{v,u} ϵ_2

$$\text{s.t.} \begin{cases} v(x) \geq 1 - \epsilon_2, & \forall x \in \mathcal{X}_0, \\ \mathcal{A}v(x) \geq 0, & \forall x \in \mathcal{X}, \\ v(x) \leq 1, & \forall x \in \partial\mathcal{X}, \\ v(x) \leq \mathcal{A}u(x), & \forall x \in \mathcal{X}. \end{cases}$$

$$\mathbb{P}(\forall t \in \mathbb{R}_{\geq 0}. X^{x_0}(t, w) \in \mathcal{X} \mid x_0 \in \mathcal{X}_0) \leq \epsilon_2$$

They were relaxed into semi-definite programming problems

Example

Consider the stochastic differential equation:

$$\begin{cases} dX_1(t, w) = X_2(t, w)dt, \\ dX_2(t, w) = -\left(X_1(t, w) + X_2(t, w) + 0.5X_1^3(t, w)\right)dt + \left(X_1(t, w) + X_2(t, w)\right)dW(t, w). \end{cases}$$

- the safe set is $\mathcal{X} = \{(x_1, x_2) \mid x_1^2 + x_2^2 - 1 < 0\}$
- the initial set is $\mathcal{X}_0 = \{(x_1, x_2) \mid 100(x_1 + 0.4)^2 + 100(x_2 + 0.5)^2 - 1 < 0\}$

The lower and upper bounds of the liveness probability obtained by Monte Carlo are

$$\epsilon_1 = 0.5338$$

$$\epsilon_2 = 0.7101$$

Op6 and Op7							
d	4	6	8	10	12	14	16
ϵ_1	0.3957	0.4217	0.4590	0.4660	0.4675	0.4682	0.4686
ϵ_2	0.7313	0.7279	0.7233	0.7224	0.7216	0.7213	0.7208
Op5							
ϵ_1	0.3957	0.4217	0.4590	0.4660	0.4675	0.4682	0.4686

Conclusion

Two sets of optimizations for computing lower and upper bounds of the liveness probability are proposed.

1. The first one is based on Doob's supermartingale inequality.
2. The second one is based on relaxing an equation that characterizes the exact reachability probability.

The comparative results show that the second one is more powerful and can provide tighter lower and upper bounds.

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Extensions

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Thanks for Your Attention!