Under-Approximating Reach Sets for Polynomial Continuous Systems

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ABSTRACT

In this paper we suggest a method based on convex programming for computing semi-algebraic under-approximations of reach sets for polynomial continuous systems with initial sets being the zero sub-level set of a polynomial function. It is well-known that the reachable set can be formulated as the zero sub-level set of a value function to a Hamilton-Jacobi partial differential equation (HJE), and our approach in this paper consequently focuses on searching for approximate analytical polynomial solutions to associated HJEs, of which the zero sub-level sets converge to the exact reachable set from inside in measure, without discretizing the state space. Such approximate solutions can be computed via a classical hierarchy of convex programs consisting of linear matrix inequalities, which are constructed by sum-of-squares decomposition techniques. In contrast to traditional numerical methods approximately solving HJEs, such as level-set methods, our method reduces HJE solving to convex optimization, avoiding the complexity associated to gridding the state space. Compared to existing approaches computing under-approximations, the approach described in this paper is structurally simpler as the under-approximations are the outcome of a single semi-definite program. Furthermore, an over-approximation of the reach set, shedding light on the quality of the constructed under-approximation, can be constructed via solving the same semi-definite program. Several illustrative examples and comparisons with existing methods demonstrate the merits of our approach.

Keywords

Reachability Analysis, Semi-algebraic Sets, Convex Programming

1. INTRODUCTION

Reachability analysis, which involves computing reach sets of a given dynamical system, has emerged as a powerful formal method for addressing a broad range of important engineering problems such as the reach-avoid verification (e.g. [31, 37]), satisfiability checking of temporal logic formulae (e.g.,[7]), and safe design of embedded controllers for aircrafts, cars, medical devices, and other safety-critical applications. Consequently, attention from scientists across multiple disciplines has been devoted to the problem of performing reachability analysis.

Unfortunately, the exact computation of reachable sets of an initial-value problem for a nonlinear system modeled by an ordinary differential equation is generally impossible. Despite existence of solvable cases [10], general practice consequently is the computation of safe approximations of the reachable set. Such approximations can be classified into two categories: over-approximations, which are sets covering the exact reachable set, and under-approximations, which are subsets of the exact reachable set. Due to its relation to sufficient conditions in safety verification, the over-approximation problem has traditionally attracted more attention and significant advances have been reported based on various representations of sets in the $\mathbb{R}^n$ such as intervals [30], zonotopes [1], polyhedra and support functions for polyhedral sets [5, 8], ellipsoids [19], level sets [26], Taylor models [3] and semi-algebraic sets [36, 15]. Contrastingly, computational methods for under-approximations have been less explored. Existing methods mainly focus on linear systems (e.g., [18]), and methods addressing non-linear systems [17, 36, 39] have been proposed only recently. Consequently, the under-approximate reachability analysis for nonlinear systems is still in its infancy.

For nonlinear systems modelled by polynomial ordinary differential equations with initial sets defined by a zero sub-level set of a polynomial function, we propose a method based on semi-definite programming. It is able to construct semi-algebraic under-approximations of reachable sets over a finite time horizon. The idea behind our approach originates from the application of Hamilton-Jacobi partial differential equations (HJE) to reachability analysis [23, 26]. Note that the general Hamilton-Jacobi(-Bellman) equation is able to synthesize reachable sets for nonlinear systems with control/disturbance inputs, which are not the focus of this paper. An under-approximation of the reachable set in our approach is constructed via the approximate analytical polynomial solution to the HJE. However, our approach in dealing with HJEs is completely different from the traditional grid-based numerical methods such as level-set methods [26]. Without any discretization of the state space, we approximate the problem of addressing the solution to HJEs via a hierarchy of semi-definite programming problems, which provide a converging sequence of under-approximations to the reachable set by polynomial sub-level sets in measure. The contributions of the paper are summarized as follows.

1). We construct an optimization problem in which the constraints approximate the HJE associated with the nonlinear system under consideration. Any feasible solution satisfying these constraints can be used to define an under-approximation of the reachable set and the optimal solution represents the exact reachable set.

2). We show that the above optimization problem, when restricted to a locally compact state-space, can be uniformly approximated via a hierarchy of semi-definite programming problems as the order of the relaxation tends to infinity. The presence of the local approximation under appropriate requirements does not introduce any conservativeness in estimating reachable sets. Moreover, an under-approximation can be synthesized by computing an approximately optimal solution to a single semi-definite program. An over-approximation can be extracted from the same near-optimal solution, and an inspection of its difference to the under-approximation provides a rigorous measure of the approximation quality obtained.

3). Finally, we test and discuss our method on several illustrative examples of nonlinear dynamics based on the Matlab package YALMIP [22] interfacing with the semi-definite programming solver Mosek [27], and highlight the merits of our method by comparing with the method in [17].

Related Work

As mentioned above, less attention has been paid to the problem of under-approximating reachable sets of nonlinear systems, compared with the over-approximation problem. Most approaches are confined to linear systems (e.g., [18, 2, 11, 6]), although some methods for nonlinear systems have been proposed recently as presented below.

Polytopic under-approximations permit the analysis of some specified properties such as the falsification of safety properties using reasoning in the theory of linear arithmetic. [12, 13] proposed a method based on modal intervals with affine forms to under-approximate reachable sets using intervals for continuous nonlinear systems modelled by ordinary differential equations. By making use of the homeomorphism property of the solution mapping, a boundary-based reachability analsysis method was proposed to under-approximate reachable sets with general polytopes in [39], and it was extended to under-approximate reachability for a class of nonlinear systems modelled by delay differential equations in [38]. As the reachable sets of nonlinear systems tend to be non-convex, the above methods for under-approximating reachable sets using convex set representations may result in poor approximations. As accuracy is an important factor in performing reachability analysis (e.g.,[34, 25]), more complex shapes of representations such as semi-algebraic sets and Taylor models are desirable.

[36] proposed an iterative method, with each iteration relying on solving semi-definite programming problems, to compute semi-algebraic under-approximations of reachable sets for polynomial systems using the advection map of the given dynamical system. By decomposing the initial set's boundary into smaller interval boxes and performing boundary reachability analysis as in [39], [40] proposed an iterative method to compute semi-algebraic under-approximations of reachable sets for polynomial systems and beyond by solving semi-definite programming problems. Compared to these methods, our method proposed in this paper reduces the computation of the under-approximation to a single semi-definite programming problem (SDP). Safe approximate solutions to both the SDP and the reach set are thus easy to obtain and the quality of approximation of the reachable set can easily be scaled to desired precision. A Taylor model backward flowpipe method was presented to compute under-approximations in [4]. The algorithm in [4] attempted to find implicit Taylor models such that the semi-algebraic set formed by them is connected. However, the reachable set representation considered in our method is not limited to connected sets, thus avoiding the high computational complexity of verifying the connectedness of the obtained semi-algebraic set.

[17] provided an inner approximations of the region of attraction to an open target set subject to basic semi-algebraic state constraints by solving a single semi-definite program as well. The inner-approximation in [17] was constructed by computing over-approximation of the complement of the reachable set of interest. The under-approximation problem in this paper is free of state constraints. However, any information on the convergence rate and the approximation quality is inaccessible for the method in [17]. Contrastingly, such important information can be gained or measured in our approach since the approximation errors are calculated during our computations, and an over-approximation of the reachable set can be produced simultaneously by solving a related single semi-definite program.

The *structure of this paper* is organized as follows. In Section 2, we introduce our problem of interest as well as Hamilton-Jacobi equations and Putinar's Positivstellensatz used to construct semi-definite programs. Our method for computing under-approximations of reachable sets by solving semi-definite programs is elucidated in Section 3. After illustrating our method through several examples and comparisons with existing works in Section 4, we conclude our our method in Section 5.

2. PRELIMINARIES

In this section we present an introduction to reachable sets, Hamilton-Jacobi equations and Putinar's Positivstellensatz used to construct semi-definite programs. The notation will be used throughout this paper: For a set Δ, $\partial\Delta$ is its boundary. The symbol $\mathbb{R}[\cdot]$ denotes the ring of polynomials in variables given by the argument. The space of continuously differentiable functions on a set X is denoted by $C^1(X)$. The difference of two sets of A and B is denoted by $A \setminus B$. $\mu(A)$ denotes the Lebesgue measure on $A \subset \mathbb{R}^n$. Vectors are denoted by boldface letters.

We consider continuous dynamical systems specified by the following Ordinary Differential Equation (ODE):

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}),\ \boldsymbol{x}_0 \in \mathcal{X}_0,\ t \in [t_0, T], \tag{1}$$

where $\boldsymbol{x} = (x_1, x_2, \dots, x_n)' \in \mathbb{R}^n$, $\boldsymbol{f} = (f_1, f_2, \dots, f_n)'$ and $\mathcal{X}_0 = \{\boldsymbol{x} \mid V_0(\boldsymbol{x}) \leq 0\}$ is a compact set. Furthermore, $f_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, 2, \dots, n$, is a polynomial over $\boldsymbol{x}$, and $V_0 : \mathbb{R}^n \to \mathbb{R}$ is a polynomial over $\boldsymbol{x}$.

Since $f_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, 2, \dots, n$, is a polynomial over the state variable $\boldsymbol{x}$ and the time variable t, $\boldsymbol{f}(\boldsymbol{x})$ is locally Lipschitz continuous, assuring existence and uniqueness of the trajectories of System (1) over some time interval $[t_0 - \delta, t_0 + \delta]$ in a given compact state space $\mathcal{X}$. Further, the trajectory of System (1) over time interval $[t_0, T]$, where $T \leq t_0 + \delta$, is defined to be $\boldsymbol{\phi}(t; \boldsymbol{x}_0, t_0) = \boldsymbol{x}(t)$, where $\boldsymbol{x}(t)$ is the solution to System (1) with the initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$. Hence comes the definition of reachable sets of the initial set $\mathcal{X}_0$ at $t \in [t_0, T]$ and over time interval $[t_0, t]$.

DEFINITION 1. *Given a system (1) and a time interval* $[t_0, T]$*, the reachable set of the initial set* $\mathcal{X}_0$ *at time instant* $t \in [t_0, T]$ *is defined to be* $\Omega(t; \mathcal{X}_0, t_0) = \{\boldsymbol{x} \mid \boldsymbol{x}_0 \in \mathcal{X}_0 \bigwedge \boldsymbol{x} = \boldsymbol{\phi}(t; \boldsymbol{x}_0, t_0)\}$*; the reachable set of the initial set* $\mathcal{X}_0$ *over time interval* $[t_0, T]$ *is defined to be* $\Omega([t_0, t]; \mathcal{X}_0, t_0) = \cup_{\tau \in [t_0, t]} \{\boldsymbol{x} \mid \boldsymbol{x}_0 \in \mathcal{X}_0 \bigwedge \boldsymbol{x} = \boldsymbol{\phi}(\tau; \boldsymbol{x}_0, t_0)\}$.

In this paper, we put our attention on under-approximations, as defined below.

DEFINITION 2. *For* $t \in [t_0, T]$*, a set* $\mathcal{X}_t$ *is called an under-approximation of* $\Omega(t; \mathcal{X}_0, t_0)$ *if*

$$\mathcal{X}_t \subseteq \Omega(t; \mathcal{X}_0, t_0).$$

2.1 Hamilton-Jacobi Equation

The exact reachable set can be obtained by the solutions to Hamilton-Jacobi Equations associated with System (1). For details, please refer to [26]. Below we present a simple introduction to this concept.

Let $\mathcal{X}_0 = \{\boldsymbol{x} \mid V_0(\boldsymbol{x}) \leq 0\}$. We further assume that $\Phi(\boldsymbol{x}, t)$ with $\Phi(\boldsymbol{x}, t_0) = V_0(\boldsymbol{x})$ is the viscosity solution to Hamilton-Jacobi Equation

$$\frac{\partial \Phi(\boldsymbol{x}, t)}{\partial t} + \frac{\partial \Phi(\boldsymbol{x}, t)}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}) = 0, \tag{2}$$

solved forward in time from $t = t_0$ with the initial value $\Phi(\boldsymbol{x}, t_0) = V_0(\boldsymbol{x})$. Then

$$\Omega(t; \mathcal{X}_0, t_0) = \{\boldsymbol{x} \mid \Phi(\boldsymbol{x}, t) \leq 0\}$$

is the reachable set at time instant t for System (1).

THEOREM 1. *Assume that* $V_0(\boldsymbol{x})$ *is continuously differentiable over* $\boldsymbol{x} \in \mathbb{R}^n$*, then* $\Phi(\boldsymbol{x}, t)$ *is a classical solution to (2), i.e.* $\Phi(\boldsymbol{x}, t) \in C^1(\mathbb{R}^n \times [t_0, T])$*, and it is unique.*

PROOF. Apparently, $\Phi(\boldsymbol{x}, t) = V_0(\boldsymbol{\phi}^{-1}(t_0; \boldsymbol{x}, t))$, which is continuously differentiable over $(t, \boldsymbol{x}) \in [t_0, T] \times \mathbb{R}^n$, satisfies (2), where $\boldsymbol{\phi}^{-1}(t; \boldsymbol{x}, t_0)$ is the inverse of the solution mapping $\boldsymbol{x} = \boldsymbol{\phi}(t; \cdot, t_0) : \mathbb{R}^n \mapsto \mathbb{R}^n$.

Next, we prove the uniqueness. Assume that there exists another $\Phi_1(\boldsymbol{x}, t) \in C^1(\mathbb{R}^n \times [t_0, T])$ such that Hamilton-Jacobi Equation (2) holds. Substracting $\frac{d}{dt}\Phi_1(\boldsymbol{x}, t)$ from $\frac{d}{dt}\Phi(\boldsymbol{x}, t)$, we obtain

$$\frac{d}{dt}\Phi(\boldsymbol{x}, t) - \frac{d}{dt}\Phi_1(\boldsymbol{x}, t) = 0 \tag{3}$$

with the initial condition $\Phi(\boldsymbol{x}, t_0) - \Phi_0(\boldsymbol{x}, t_0) = 0$. For $\forall \tau \in [t_0, T]$, integrating (3) over t from t_0 to τ, we obtain that $\Phi(\boldsymbol{x}, \tau) = \Phi_1(\boldsymbol{x}, \tau)$ for $\forall \boldsymbol{x} \in \mathbb{R}^n$. Thus, there exists a unique solution $\Phi(\boldsymbol{x}, t) \in C^1(\mathbb{R}^n \times [t_0, T])$ to (2) $\square$

2.2 Putinar's Positivstellensatz

The problem of positivity of polynomials on a compact semi-algebraic set is a field of research attracting researchers from various areas including real algebra, semidefinite programming, and operator theory. Based on Putinar's Positivstellensatz [29], the above problem can be addressed in the semi-definite programming framework. In the following we briefly introduce such application of Putinar's Positivstellensatz.

Let $K \subset \mathbb{R}^m$ be a closed semi-algebraic set, i.e., there exist polynomials $g_1(\boldsymbol{y}), \ldots, g_l(\boldsymbol{y})$ such that

$$K = \{\boldsymbol{y} \in \mathbb{R}^m \mid g_1(\boldsymbol{y}) \geq 0, \ldots, g_l(\boldsymbol{y}) \geq 0\}.$$

Given a polynomial $p(\boldsymbol{y})$, we attempt to decide whether it is positive over K, i.e.

$$p(\boldsymbol{y}) > 0, \forall \boldsymbol{y} \in K.$$

Additionally, we define $\sum_m$ to be the set of sum of squares (SOS) polynomials over variables $\boldsymbol{y}$.

$$\sum_m := \{p \in \mathbb{R}[\boldsymbol{y}] \mid p = \sum_{i=1}^{k} q_i^2, q_i \in \mathbb{R}[\boldsymbol{y}], i = 1, \ldots, k\}.$$

Obviously if $h \in \sum_m$, then $h(\boldsymbol{y}) \geq 0$ for any $\boldsymbol{y} \in \mathbb{R}^m$.

Putinar's Positivstellensatz presented a linear formulation in constraints defining a given compact semi-algebraic set to characterize a polynomial that is positive on the compact semi-algebraic set, as stated in Theorem 2.

THEOREM 2. *[Putinar's Positivstellensatz [29]]Let* $K = \{\boldsymbol{y} \in \mathbb{R}^m \mid g_1(\boldsymbol{y}) \geq 0, \ldots, g_l(\boldsymbol{y}) \geq 0\}$ *be a compact set. Suppose there exists* $N > 0$ *such that*

$$N - \sum_{i=1}^{m} y_i^2 \in M(g_1, \ldots, g_l).$$

If $p(\boldsymbol{y})$ *is positive on* K*, then* $p(\boldsymbol{y}) \in M(g_1, \ldots, g_l)$*, where* $M(g_1, \ldots, g_l)$ *is the quadratic module of polynomials* $g_1, \ldots, g_l$*, i.e.*

$$M(g_1, \ldots, g_l) = \{\sigma_0(\boldsymbol{y}) + \sum_{i=1}^{l} \sigma_i(\boldsymbol{y}) g_i(\boldsymbol{y}) \mid \text{ each } \sigma_i \in \sum_m\}.$$

Therefore, the problem of verifying positivity of a given polynomial $p(\boldsymbol{y})$ on a compact set K satisfying Putinar's Positivstellensatz is reduced to finding appropriate sum-of-squares multipliers $\sigma_i(\boldsymbol{y}) \in \sum_m$ such that

$$p(\boldsymbol{y}) = \sigma_0(\boldsymbol{y}) + \sum_{i=1}^{l} \sigma_i(\boldsymbol{y}) g_i(\boldsymbol{y}). \tag{4}$$

When degrees of $\sigma_i(\boldsymbol{y})$'s are fixed, the problem of verifying whether (4) holds is a semi-definite programming problem.

3. REACHABLE SET COMPUTATIONS

In this section an approach to under-approximate reachability analysis for a given system (1) is presented. The approach begins with relaxing the associated Hamilton-Jacobi Equation (2) and ends with solving semi-definite programs.

An under-approximation $\mathcal{X}_t$ of the reachable set at time $t \in [t_0, T]$ can be constructed by the zero sub-level set of a continuously differentiable function solving a system of constraints derived by relaxing the HJE (2). This is formally stated in Theorem 3. For ease of exposition, we denote $\frac{\partial \Phi(\boldsymbol{x}, t)}{\partial t} + \frac{\partial \Phi(\boldsymbol{x}, t)}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x})$ by $\mathcal{L}\Phi(\boldsymbol{x}, t)$ for the rest of this paper.

THEOREM 3. *Assume a system (1) with a compact initial set* $\mathcal{X}_0 = \{\boldsymbol{x} \mid V_0(\boldsymbol{x}) \leq 0\}$ *and a time interval* $[t_0, T]$*, where* $V_0(\boldsymbol{x})$ *is continuously differentiable over* $\boldsymbol{x} \in \mathbb{R}^n$ *and* $t_0 < T$*. If a continuously differentiable function* $\Phi(\boldsymbol{x}, t)$ *and a non-negative value* ϵ *satisfy the following constraints* (5)*, namely*

$$\begin{cases} 0 \leq \mathcal{L}\Phi(\boldsymbol{x}) \leq \epsilon, \ \forall (\boldsymbol{x}, t) \in \mathbb{R}^n \times [t_0, T] \\ \Phi(\boldsymbol{x}, t_0) \geq V_0(\boldsymbol{x}) \\ \Phi(\boldsymbol{x}, t_0) - \epsilon \leq V_0(\boldsymbol{x}) \\ \epsilon \geq 0 \end{cases}, \tag{5}$$

then the set $\{\boldsymbol{x} \mid \Phi(\boldsymbol{x}, \tau) \leq 0\}$ *is an under-approximation of the reachable set for* $\tau \in [t_0, T]$.

PROOF. Assume that $\boldsymbol{y} = \boldsymbol{\phi}(\tau; \boldsymbol{x}_0, t_0)$ with $\tau \in [t_0, T]$ and $\boldsymbol{x}_0 \in \mathbb{R}^n$. Additionally, assume that $\boldsymbol{x}(t) = \boldsymbol{\phi}(t; \boldsymbol{x}_0, t_0)$ with $t \in [t_0, \tau]$. Obviously, $\boldsymbol{y} = \boldsymbol{x}(\tau)$. We can deduce that

$$\begin{aligned} \Phi(\boldsymbol{y}, \tau) =& \Phi(\boldsymbol{x}_0, t_0) + \int_{t_0}^{\tau} \frac{d}{dt}\Phi(\boldsymbol{x}(t), t) dt \\ =& \Phi(\boldsymbol{x}_0, t_0) + \int_{t_0}^{\tau} \mathcal{L}\Phi(\boldsymbol{x}, t) dt \end{aligned}. \tag{6}$$

Let $\Phi_1(\boldsymbol{x}, t) \in C^1(\mathbb{R}^n \times [t_0, T])$ satisfy HJE (2) with the initial condition $\Phi_1(\boldsymbol{x}, t_0) = V_0(\boldsymbol{x})$. Since $\mathcal{L}\Phi_1(\boldsymbol{x}, t) = 0$ over $\mathbb{R}^n \times [t_0, T]$, the following equality can be inferred:

$$\Phi(\boldsymbol{y}, \tau) - \Phi_1(\boldsymbol{y}, \tau) = \Phi(\boldsymbol{x}_0, t_0) - V_0(\boldsymbol{x}) + \int_{t_0}^{\tau} \mathcal{L}\Phi(\boldsymbol{x}, t) dt. \tag{7}$$

Also, since

$$0 \leq \Phi(\boldsymbol{x}_0, t_0) - V_0(\boldsymbol{x}) \leq \epsilon$$

and

$$0 \leq \mathcal{L}\Phi(\boldsymbol{x}, t) \leq \epsilon$$

over $\mathbb{R}^n \times [t_0, T]$, it is deduced that

$$0 \leq \Phi(\boldsymbol{y}, \tau) - \Phi_1(\boldsymbol{y}, \tau) \leq \epsilon(1 + \tau - t_0) \tag{8}$$

holds over the $\mathbb{R}^n$. It is evident that thus $\boldsymbol{y} \in \{\boldsymbol{x} \mid \Phi(\boldsymbol{y}, \tau) \leq 0\}$ implies $\boldsymbol{y} \in \{\boldsymbol{x} \mid \Phi_1(\boldsymbol{y}, \tau) \leq 0\}$. Moreover, from Subsection 2.1, we have the conclusion that $\boldsymbol{x}_0 \in \mathcal{X}_0$ if $\boldsymbol{y} \in \{\boldsymbol{x} \mid \Phi_1(\boldsymbol{y}, \tau) \leq 0\}$ holds. That is, the state $\boldsymbol{y} \in \{\boldsymbol{x} \mid \Phi(\boldsymbol{x}, \tau) \leq 0\}$ is reached definitely by a trajectory originating from $\mathcal{X}_0$ at time instant t_0. Therefore, $\{\boldsymbol{x} \mid \Phi(\boldsymbol{x}, \tau) \leq 0\}$ is an under-approximation of the reachable set for $\tau \in [t_0, T]$. □

From Theorem 3, we have the following observations: Let the pair $(\Phi(\boldsymbol{x}, t), \epsilon)$ be in (5) from Theorem 3.

1). The non-negative value ϵ can be regarded as a characterization of the discrepancy between the approximate analytical solution $\Phi(\boldsymbol{x}, t)$ to HJE (2) and its exact solution according to (8) in the proof of Theorem 3: the smaller ϵ is, the closer the analytical solution $\Phi(\boldsymbol{x}, t)$ approximates the exact solution to HJE (2);

2). An over-approximation at time $\tau \in [t_0, T]$, which covers the reachable set $\Omega(\tau; \mathcal{X}_0, t_0)$, can be constructed using the function $\Phi(\boldsymbol{x}, \tau)$ and ϵ satisfying the constraint (5), as formulated in Corollary 1. The over-approximation could be employed to quantitatively characterize how close the computed under-approximation is to the exact reachable set.

COROLLARY 1. *Assume a system (1) with a compact initial set* $\mathcal{X}_0 = \{\boldsymbol{x} \mid V_0(\boldsymbol{x}) \leq 0\}$ *and a time interval* $[t_0, T]$*, where* $V_0(\boldsymbol{x})$ *is continuously differentiable over* $\boldsymbol{x} \in \mathbb{R}^n$ *and* $t_0 < T$*. If a continuously differentiable function* $\Phi(\boldsymbol{x}, t)$ *and a non-negative value* ϵ *satisfy the constraint* (5)*, the set* $\{\boldsymbol{x} \mid \Phi(\boldsymbol{x}, \tau) \leq \epsilon(\tau - t_0 + 1)\}$ *is an over-approximation of the reachable set for* $\tau \in [t_0, T]$.

PROOF. According to (8) in the proof of Theorem 3, the conclusion can be assured. □

Also, when optimizing the feasible pair $(\Phi(\boldsymbol{x}, t), \epsilon)$ in the constraint (5) in Theorem 3, as formulated in (9), we are able to gain the least conservative under-approximation of the reachable set $\Omega(t; \mathcal{X}_0, t_0)$ at time $t \in [t_0, T]$.

$$\begin{aligned} & \epsilon^* = \min_{\Phi(\boldsymbol{x}, t), \epsilon} \epsilon \\ & s.t. \\ & \epsilon \geq 0, \\ & \mathcal{L}\Phi(\boldsymbol{x}, t) \geq 0, \quad \forall (\boldsymbol{x}, t) \in \mathbb{R}^n \times [t_0, T], \\ & \epsilon - \mathcal{L}\Phi(\boldsymbol{x}, t) \geq 0, \quad \forall (\boldsymbol{x}, t) \in \mathbb{R}^n \times [t_0, T], \\ & \Phi(\boldsymbol{x}, t_0) - V_0(\boldsymbol{x}) \geq 0, \quad \forall \boldsymbol{x} \in \mathbb{R}^n, \\ & V_0(\boldsymbol{x}) + \epsilon - \Phi(\boldsymbol{x}, t_0) \geq 0, \forall \boldsymbol{x} \in \mathbb{R}^n, \\ & \Phi(\boldsymbol{x}, t) \in C^1(\mathbb{R}^n \times [t_0, T]) \end{aligned} \tag{9}$$

Let $(\Phi^*(\boldsymbol{x}, t), \epsilon^*)$ be the optimal pair solving the optimization (9). We obtain the following conclusion.

LEMMA 1. *The optimal value* ϵ^* *of the optimization problem (9) is equal to zero and the zero sub-level set of the function* $\Phi^*(\boldsymbol{x}, t)$*, i.e.* $\{\boldsymbol{x} \mid \Phi^*(\boldsymbol{x}, t) \leq 0\}$*, is equal to the exact reachable set* $\Omega(t; \mathcal{X}_0, t_0)$ *at time* $t \in [t_0, T]$.

PROOF. As stated in Theorem 1, there exists a unique continuously differentiable function over $\mathbb{R}^n \times [t_0, T]$ such that the HJE (2) holds. Thus, the optimal value ϵ^* of problem (9) is equal to zero and the zero sub-level set of its corresponding continuously differentiable function $\Phi^*(\boldsymbol{x}, t)$ is equal to the reachable set at time $t \in [t_0, T]$, i.e. $\Omega(t; \mathcal{X}_0, t_0) = \{\boldsymbol{x} \mid \Phi^*(\boldsymbol{x}, t) \leq 0\}$. □

Through Theorem 3 and Lemma 1, an under-approximation $\mathcal{X}_t$ at $t \in [t_0, T]$ can be constructed via the zero sub-level set of $\Phi(\boldsymbol{x}, t)$ in the pair $(\Phi(\boldsymbol{x}, t), \epsilon)$ feasible in (9), and the reachable set $\Omega(t; \mathcal{X}_0, t_0)$ equals the zero sub-level set of the function $\Phi^*(\boldsymbol{x}, t)$ in the optimal pair $(\Phi^*(\boldsymbol{x}, t), \epsilon^*)$ solving the optimization (9), i.e. $\Omega(t; \mathcal{X}_0, t_0) = \{\boldsymbol{x} \mid \Phi^*(\boldsymbol{x}, t) \leq 0\}$. However, solving the optimization problem (9) is intractable in general.

3.1 Under-Approximating Reachable Sets

In this subsection we show how a hierarchy of under-approximations of the reachable set represented by polynomial sub-level sets can be computed via solving semi-definite programs, with a guarantee of convergence to the exact reachable set in measure. Such semi-definite programs approximate the optimization (9) in an appropriate compact state space $\mathcal{Y} \subset \mathbb{R}^n$ rather than $\mathbb{R}^n$, i.e. the continuously differentiable function $\Phi(\boldsymbol{x}, t)$ defined in $\mathbb{R}^n \times [t_0, T]$ in the optimization (9) is restricted to the compact set $\mathcal{Y} \times [t_0, T]$ in the constructed semi-definite programs.

Firstly, we introduce the compact set $\mathcal{Y} \subset \mathbb{R}^n$ required in our computations.

DEFINITION 3. *Assume* $\Omega([t_0, T]; \mathcal{X}_0, t_0) \subseteq \mathcal{X}$*, the set* $\mathcal{Y}$ *of the form* $\{\boldsymbol{x} \mid g(\boldsymbol{x}) \geq 0\}$ *is a set such that all trajectories starting from* $\mathcal{X}$ *at time instant* $t = t_0$ *do not leave it within time interval* $[t_0, T]$*, where* $g(\boldsymbol{x}) = R - \|\boldsymbol{x}\|_2^2$ *with* $R \geq 0$.

The compact set $\mathcal{Y}$ in Definition 3 always exists since the reachable set $\Omega([t_0,T];\mathcal{X},t_0)$ is bounded if $\mathcal{X}$ is bounded, and there definitely exists some constant $R \geq 0$ such that $\Omega([t_0,T];\mathcal{X},t_0) \subseteq \mathcal{Y} := \{\boldsymbol{x} \mid g(\boldsymbol{x}) \geq 0\}$ holds. By searching for the continuously differentiable function $\Phi(\boldsymbol{x},t)$ restricted in the compact domain $\mathcal{Y} \times [t_0,T]$ rather than $\mathbb{R}^n \times [t_0,T]$, the optimization (9) is relaxed into the optimization (10), as formulated below. However, such relaxation does not introduce conservativeness in estimating the reachable set.

$$\begin{aligned}
&\epsilon^* = \min_{\Phi(\boldsymbol{x},t),\epsilon} \epsilon \\
&s.t. \\
&\epsilon \geq 0, \\
&\mathcal{L}\Phi(\boldsymbol{x},t) \geq 0, \qquad \forall(\boldsymbol{x},t) \in \mathcal{Y} \times [t_0,T], \\
&\epsilon - \mathcal{L}\Phi(\boldsymbol{x},t) \geq 0, \qquad \forall(\boldsymbol{x},t) \in \mathcal{Y} \times [t_0,T], \\
&\Phi(\boldsymbol{x},t_0) - V_0(\boldsymbol{x}) \geq 0, \quad \forall \boldsymbol{x} \in \mathcal{Y}, \\
&V_0(\boldsymbol{x}) + \epsilon - \Phi(\boldsymbol{x},t_0) \geq 0, \forall \boldsymbol{x} \in \mathcal{Y},
\end{aligned} \tag{10}$$

where the minimum is over $\Phi(\boldsymbol{x},t) \in C^1(\mathcal{Y} \times [t_0,T])$.

LEMMA 2. *The set $\{\boldsymbol{x} \in \mathcal{X} \mid \Phi(\boldsymbol{x},t) \leq 0\}$ is an under-approximation of the reachable set at time $t \in [t_0,T]$, where $\Phi(\boldsymbol{x},t)$ satisfies the constraints in* (10).

PROOF. By restricting the state space $\mathbb{R}^n$ in the constraint (5) to the compact set $\mathcal{Y}$ and due to the fact that every trajectory starting from $\mathcal{X}$ at time $t = t_0$ stays within the $\mathcal{Y}$ over the time interval $[t_0,T]$, the proof follows from the proof of Theorem 3. □

LEMMA 3. *Let $(\Phi^*(\boldsymbol{x},t),\epsilon^*)$ be the optimal pair solving the optimization problem* (10)*, then $\epsilon^* = 0$ and the restriction of the continuous differentiable function $\Phi^*(\boldsymbol{x},t)$ to the compact set $\mathcal{X} \times [t_0,T]$, i.e. $\Phi^*(\boldsymbol{x},t)|_{\mathcal{X}\times[t_0,T]}$, is unique.*

PROOF. According to Kirszbraun's theorem (e.g., [9]) on extending Lipschitz mappings in Hilbert space, the existence of a globally Lipschitz function $\boldsymbol{f}'(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^n$ such that $\boldsymbol{f}'(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x})$ over $\mathcal{Y}$ is guaranteed. Thus the solution to $\dot{\boldsymbol{x}} = \boldsymbol{f}'(\boldsymbol{x})$ exists for $\forall \boldsymbol{x} \in \mathbb{R}^n$ and $\forall t \in \mathbb{R}$ and coincides with the solution to (1) for $\forall \boldsymbol{x} \in \mathcal{Y}$. Thus, a feasible pair $(\Phi(\boldsymbol{x},t),\epsilon)$ to (9) associated with $\dot{\boldsymbol{x}} = \boldsymbol{f}'(\boldsymbol{x})$ is a feasible pair to (10). Furthermore, the optimal value ϵ^* to the optimization (10) is equal to zero, implying that the optimal pair to the optimization (9) is also an optimal pair to the optimization (10). Moreover, $\Phi^*(\boldsymbol{x},t)|_{\mathcal{X}\times[t_0,T]}$ is equal to the restriction of the $\Phi-$component to the set $\mathcal{X} \times [t_0,T]$ in the optimal pair to (9). Since $\Omega([t_0,T];\mathcal{X},t_0) \subseteq \mathcal{Y}$, the uniqueness of $\Phi^*(\boldsymbol{x},t)|_{\mathcal{X}\times[t_0,T]}$ can be assured by following the corresponding proof in Theorem 2. □

Consequently, we have the following theorem assuring computation of the exact reachable set via solving (10).

THEOREM 4. *Let $(\Phi^*(\boldsymbol{x},t),\epsilon^*)$ be the optimal solution to* (10)*, then the reachable set $\Omega(t;\mathcal{X}_0,t_0)$ is equal to the set $\{\boldsymbol{x} \in \mathcal{X} \mid \Phi^*(\boldsymbol{x},t) \leq 0\}$.*

In the following, we show that the optimization problem (10) can be approximated by a hierarchy of semi-definite programming problems with the approximation error ϵ vanishing as the relaxation order tends to infinity.

Let $\mathbb{R}_k[\boldsymbol{x},t]$ denote the vector space of real multivariate polynomials of total degree $\leq k$. Each polynomial $p(\boldsymbol{x},t) \in \mathbb{R}_k[\boldsymbol{x},t]$ can be expressed in the monomial basis as

$$p(\boldsymbol{x},t) = \sum_{|\boldsymbol{\alpha}|\leq k} p_{\boldsymbol{\alpha}} \boldsymbol{y}^{\alpha} = \sum_{|\boldsymbol{\alpha}|\leq k} p_{\boldsymbol{\alpha}} (x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} \cdot t^{\alpha_{n+1}}),$$

where $\boldsymbol{y} = (\boldsymbol{x},t)$, the $p_{\boldsymbol{\alpha}}$'s are the coefficients of $p(\boldsymbol{x},t)$, and $\boldsymbol{\alpha}$ ranges over the multi-indices (vectors of non-negative integers) such that $|\boldsymbol{\alpha}| = \sum_{i=1}^{n+1} \alpha_i \leq k$.

Assuming $\Phi_k(\boldsymbol{x},t) \in \mathbb{R}_k[\boldsymbol{x},t]$, the optimization (10) is relaxed to the following semi-definite program,

$$\begin{aligned}
&\epsilon_k^* = \min_{\epsilon,\Phi_k(\boldsymbol{x},t),s_i,i=0,\ldots,9} \epsilon \\
&s.t. \\
&\epsilon \geq 0 \\
&\mathcal{L}\Phi_k(\boldsymbol{x},t) = s_0 + s_1(t-t_0)(T-t) + s_2 g(\boldsymbol{x}) \\
&\epsilon - \mathcal{L}\Phi_k(\boldsymbol{x},t) = s_3 + s_4(t-t_0)(T-t) + s_5 g(\boldsymbol{x}) \\
&\Phi_k(\boldsymbol{x},t_0) - V_0(\boldsymbol{x}) = s_6 + s_7 g(\boldsymbol{x}) \\
&\epsilon + V_0(\boldsymbol{x}) - \Phi_k(\boldsymbol{x}_0,t_0) = s_8 + s_9 g(\boldsymbol{x})
\end{aligned} , \tag{11}$$

where the minimum is over polynomials $\Phi_k(\boldsymbol{x},t) \in \mathbb{R}_k[\boldsymbol{x},t]$, ϵ and polynomial sum-of-squares $s_i \in \mathbb{R}[\boldsymbol{x},t], i = 0,\ldots,5$, $s_j \in \mathbb{R}[\boldsymbol{x}]$, $j = 6,\ldots,9$, of appropriate degrees. The constraints that polynomials are sum-of-squares can be written explicitly as linear matrix inequalities, and the objective is linear in ϵ; therefore problem (11) can be formulated as a semi-definite program, which falls within the convex optimization framework and is solvable in polynomial time up to any desired accuracy by interior point methods (e.g., [35]).

3.2 Convergence Analysis

In this subsection we show how (11) gives rise to a converging sequence of inner-approximations to the reachable set at time $t \in [t_0,T]$ in measure.

Let $(\Phi_k^*(\boldsymbol{x},t),\epsilon_k^*)$ be the optimal solution to the semi-definite program (11). Our first convergence result proves the convergence of the restriction of the function Φ_k^* to the set $\mathcal{X} \times [t_0,T]$ to the restriction of the function $\Phi^*(\boldsymbol{x},t)$ to the set $\mathcal{X} \times [t_0,T]$, i.e. $\Phi_k^*(\boldsymbol{x},t)|_{\mathcal{X}\times[t_0,T]} \to \Phi^*(\boldsymbol{x},t)|_{\mathcal{X}\times[t_0,T]}$ as $k \to \infty$, where Φ^* is the $\Phi-$component of the optimal solution (Φ^*,ϵ^*) to the optimization problem (10).

THEOREM 5. *Let $(\Phi_k^*(\boldsymbol{x},t),\epsilon_k^*)$ and $(\Phi^*(\boldsymbol{x},t),\epsilon^*)$ be the optimal solutions to the optimizations* (11) *and* (10) *respectively, where $\mathcal{L}\Phi^* \equiv 0$ for $\forall(\boldsymbol{x},t) \in \mathcal{Y} \times [t_0,T]$. Then as the relaxation order k approaches infinity, $\Phi_k^*(\boldsymbol{x},\boldsymbol{t})$ converges to $\Phi^*(\boldsymbol{x},t)$ with $\Phi_k(\boldsymbol{x},t) \geq \Phi^*(\boldsymbol{x},t)$ over $\mathcal{X} \times [t_0,T]$ uniformly, and ϵ_k converges monotonically from above to ϵ^*.*

PROOF. Since $\mathcal{Y} \times [t_0,T]$ is compact, for $\forall \epsilon > 0$, there exists polynomials $\tilde{\Phi}$ of a sufficiently high degree such that

$$\sup_{\mathcal{Y}\times[t_0,T]} |\tilde{\Phi} - \Phi^*| < \epsilon$$

and

$$\sup_{\mathcal{Y}\times[t_0,T]} |\mathcal{L}\tilde{\Phi} - \mathcal{L}\Phi^*| < \epsilon.$$

Also, since $\mathcal{L}\Phi^* \equiv 0$ for $\forall(\boldsymbol{x},t) \in \mathcal{Y} \times [t_0,T]$, we obtain $\sup_{\mathcal{Y}\times[t_0,T]} |\mathcal{L}\tilde{\Phi}| < \epsilon$. Set

$$\hat{\Phi}(\boldsymbol{x},t) := \tilde{\Phi}(\boldsymbol{x},t) + \epsilon(t - t_0 + 1).$$

We derive that

$$0 < \hat{\Phi}(\boldsymbol{x},t) - \Phi^*(\boldsymbol{x},t) < \delta, \forall(\boldsymbol{x},t) \in \mathcal{Y} \times [t_0,T]$$

and

$$0 < \sup_{\mathcal{Y}\times[t_0,T]} \mathcal{L}\hat{\Phi} < \delta,$$

where $\delta = \epsilon(T - t_0 + 2)$. Thus the polynomial function $\hat{\Phi}(\boldsymbol{x}, t)$ is strictly feasible in (9) and as a result, under Definition 3, feasible for a sufficiently large relaxation order k, which follows Theorem 2. Thus, ϵ_k^* converges to $\epsilon*$, i.e. 0 (according to Lemma 3), since ϵ is arbitrary. Monotone convergence of ϵ_k^* to $\epsilon*$ follows from that the higher the relaxation order k, the looser the constraint set of the minimization problem (11) is.

Since $\Omega([t_0,T];\mathcal{X},t_0) \subseteq \mathcal{Y}$ and $\Phi^*(\boldsymbol{x},t)$ is unique over $\mathcal{X}\times[t_0,T]$ according to Lemma 3, the claim that $\Phi_k(\boldsymbol{x},t) \geq \Phi^*(\boldsymbol{x},t)$ over $\mathcal{X}\times[t_0,T]$ follows from (8) and $\Phi_k^*(\boldsymbol{x},\boldsymbol{t})$ converges to $\Phi^*(\boldsymbol{x},t)$ over $\mathcal{X}\times[t_0,T]$ as k tends to infinity. □

Theorem 5 establishes a convergence of $\Phi_k^*(\boldsymbol{x},t)$ to $\Phi^*(\boldsymbol{x},t)$ with $\Phi_k^*(\boldsymbol{x},t) \geq \Phi^*(\boldsymbol{x},t)$ over $(\boldsymbol{x},t)\in\mathcal{X}\times[t_0,T]$ in function space. Finally, the following theorem establishes a set-wise convergence of the sets

$$\mathcal{X}_{t,k} := \{\boldsymbol{x}\in\mathcal{X} \mid \Phi_k^*(\boldsymbol{x},t) \leq 0\} \tag{12}$$

and

$$\overline{\mathcal{X}}_{t,k} := \{\boldsymbol{x}\in\mathcal{X} \mid \overline{\Phi}_k^*(\boldsymbol{x},t) \leq 0\} \tag{13}$$

to $\Omega(t;\mathcal{X}_0,t_0)$ at $t\in[t_0,T]$, where $\overline{\Phi}_k^* := \min_{i\leq k}\Phi_i^*$.

THEOREM 6. *Let $(\Phi_k^*(\boldsymbol{x},t),\epsilon_k^*)$ be the optimal pair solving the optimization (11) and $\mu(\{\boldsymbol{x} \mid V_0(\boldsymbol{x})=0\})=0$. Then the sets $\mathcal{X}_{t,k}$ and $\overline{\mathcal{X}}_{t,k}$ defined in (12) and (13) converge to the reachable set $\Omega(t;\mathcal{X}_0,t_0)$ from inside such that $\Omega(t;\mathcal{X}_0,t_0)\supseteq\overline{\mathcal{X}}_{t,k}\supseteq\mathcal{X}_{t,k}$ and*

$$\texttt{lim}_{k\to\infty}\mu(\Omega(t;\mathcal{X}_0,t_0)\setminus\mathcal{X}_{t,k}) = 0$$

and

$$\texttt{lim}_{k\to\infty}\mu(\Omega(t;\mathcal{X}_0,t_0)\setminus\overline{\mathcal{X}}_{t,k}) = 0.$$

Moreover the convergence of $\overline{\mathcal{X}}_{t,k}$ is monotonous, i.e., $\overline{\mathcal{X}}_{t,i}\subseteq\overline{\mathcal{X}}_{t,j}$ whenever $i\leq j$.

PROOF. The inclusion $\Omega(t;\mathcal{X}_0,t_0)\supset\overline{\mathcal{X}}_{t,k}\supset\mathcal{X}_{t,k}$ follows from the facts that $\Phi_k^*(\boldsymbol{x},t)\geq\Phi^*(\boldsymbol{x},t)$ over $(\boldsymbol{x},t)\in\mathcal{X}\times[t_0,T]$ in Theorem 3, $\Omega(t;\mathcal{X}_0,t_0)=\{\boldsymbol{x}\in\mathcal{X}\mid\Phi^*(\boldsymbol{x},t)\leq 0\}$ in Theorem 4 and $\overline{\mathcal{X}}_{t,k}=\cup_{i=1}^k\mathcal{X}_{t,i}$. The last fact also proves the monotonicity of the sequence $\overline{\mathcal{X}}_{t,k}$. Next, from Theorem 5 we have that $\Phi_k^*(\boldsymbol{x},t)$ converges to $\Phi^*(\boldsymbol{x},t)$ over $\mathcal{X}\times[t_0,T]$ uniformly as the order k tends to infinity, thus we have

$$\texttt{lim}_{k\to\infty}\int_{\mathcal{X}}|\Phi_k^*(\boldsymbol{x},t)-\Phi^*(\boldsymbol{x},t)|\,d\boldsymbol{x}=0;$$

also, since $\mu(\{\boldsymbol{x}|V_0(\boldsymbol{x})=0\})=0$ and the trajectory mapping $\phi(t;\cdot,t_0):\mathcal{X}_0\mapsto\Omega(t;\mathcal{X}_0,t_0)$ is absolutely continuous for $t\in[t_0,T]$ and thus is measure preserving, $\mu(\{\boldsymbol{x}|\Phi^*(\boldsymbol{x},t)=0\})=0$ for $t\in[t_0,T]$ holds; additionally, the fact that $\Phi_k^*(\boldsymbol{x},t)\geq\Phi^*(\boldsymbol{x},t)$ over $(\boldsymbol{x},t)\in\mathcal{X}\times[t_0,T]$ is ensured by Theorem 5. Accordind to Theorem 4 stating the fact that $\Omega(t;\mathcal{X}_0,t_0)=\{\boldsymbol{x}\in\mathcal{X}|\Phi^*(\boldsymbol{x},t)\leq 0\}$ and Theorem 3 in [21], we have $\texttt{lim}_{k\to\infty}\mu(\Omega(t;\mathcal{X}_0,t_0)\setminus\mathcal{X}_{t,k})=0$ and $\texttt{lim}_{k\to\infty}\mu(\Omega(t;\mathcal{X}_0,t_0)\setminus\overline{\mathcal{X}}_{t,k})=0$, completing the proof. □

REMARK 1. *In analogy to Corollary 1, a converging sequence of approximations to the reachable set $\Omega(t;\mathcal{X}_0,t_0)$ at time $t\in[t_0,T]$ from outside can be constructed by solving the semi-definite programming problem (11), as stated below. Let*

$$\mathcal{Y}_{t,k} := \{\boldsymbol{x}\in\mathcal{X}\mid\Phi_k^*(\boldsymbol{x},t)\leq\epsilon(1+t-t_0)\} \tag{14}$$

and

$$\overline{\mathcal{Y}}_{t,k} := \{\boldsymbol{x}\in\mathcal{X}\mid\widehat{\Phi}_k^*(\boldsymbol{x},t)\leq\epsilon(1+t-t_0)\} \tag{15}$$

where $t\in[t_0,T]$ and $\widehat{\Phi}_k^* := max_{i\leq k}\Phi_i^*$.

THEOREM 7. *Let $(\Phi_k^*(\boldsymbol{x},t),\epsilon_k^*)$ be the optimal solution to (11). Then the sets $\mathcal{Y}_{t,k}$ and $\overline{\mathcal{Y}}_{t,k}$ defined in (14) and (15) converge to the reachable set $\Omega(t;\mathcal{X}_0,t_0)$ from outside such that $\Omega(t;\mathcal{X}_0,t_0)\subseteq\overline{\mathcal{Y}}_{t,k}\subseteq\mathcal{Y}_{t,k}$ and*

$$\texttt{lim}_{k\to\infty}\mu(\mathcal{Y}_{t,k}\setminus\Omega(t;\mathcal{X}_0,t_0)) = 0$$

and

$$\texttt{lim}_{k\to\infty}\mu(\overline{\mathcal{Y}}_{t,k}\setminus\Omega(t;\mathcal{X}_0,t_0)) = 0.$$

Moreover, the convergence of $\overline{\mathcal{Y}}_{t,k}$ is monotonous, i.e., $\overline{\mathcal{Y}}_{t,i}\supseteq\overline{\mathcal{Y}}_{t,j}$ whenever $i\leq j$.

PROOF. *The statement follows from corresponding statements in Corollary 1, Theorem 4, and Theorem 5 and Theorem 3.2 in [24].* □

4. EXAMPLES AND COMPARISONS

In this section we present three examples to illustrate our approach and discuss the benefits of our method by comparing with the method in [17]. All computations were performed on an i7-T450s 2.6GHz CPU with 4GB RAM running Ubuntu Linux 17.04. For numerical implementation, we formulate the sum-of-square problem (11) using the MATLAB package YALMIP [22] and use Mosek[1] as a semi-definite programming solver.

4.1 Examples

In this subsection we test our method on three illustrative examples. The parameters that control the performance of our approach are presented in Table 1.

EXAMPLE 1. *Consider a two-dimensional example constructed to illustrate the barrier certificate in [28],*

$$\begin{aligned}\dot{x}_1 &= x_1 - 2x_2\\ \dot{x}_2 &= x_1x_2 + 0.5x_2^2\end{aligned}\ ,$$

where $\mathcal{X}_0=\{\boldsymbol{x}\mid 100x_1^2+100x_2^2\leq 1\}$, and 1a) $\mathcal{Y}=\{\boldsymbol{x}|1-x^2-y^2\geq 0\}$; 1b) $\mathcal{Y}=\{\boldsymbol{x}|0.25-x^2-y^2\geq 0\}$.

The computed under-approximations are illustrated in Fig. 1 and 2. The obtained under-approximations with $k=20$ for 1a) and $k=18$ for 1b) almost match the corresponding exact reachable sets by comparing with the respective gained over-approximations.

EXAMPLE 2. *Consider the following van-der-Pol system*

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -0.2x_1 + x_2 - 0.2x_1^2x_2\end{aligned}\ ,$$

where $\mathcal{X}_0=\{\boldsymbol{x}\mid x_1^2+x_2^2-0.25\leq 0\}$, and 2a) $\mathcal{Y}=\{\boldsymbol{x}|15-x^2-y^2\geq 0\}$; 2b) $\mathcal{Y}=\{\boldsymbol{x}|16-x^2-y^2\geq 0\}$.

The computation results are illustrated in Fig. 3 and 4. When $k=12$, the obtained under-approximations and the corresponding over-approximations at $t=2.0$ for 2a) and 2b) are too close to be shown so that just the under-approximation for 2a) is presented in Fig. 4.

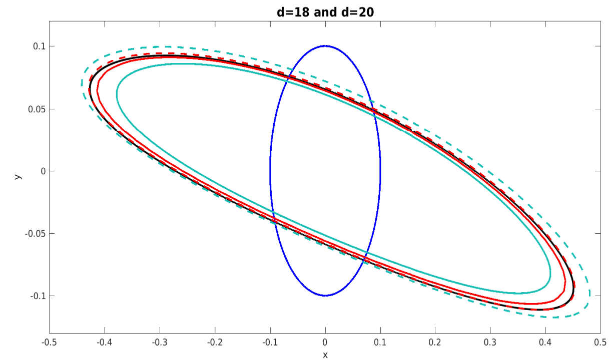

Figure 1: Reachable sets for Example 1 in 1a). (Solid red and green curves: $\partial\mathcal{X}_{1,20}$ and $\partial\mathcal{X}_{1,18}$, respectively. Dashed red and green curves: $\partial\mathcal{Y}_{1,20}$ and $\partial\mathcal{Y}_{1,18}$, respectively. Blue curve: $\partial\mathcal{X}_0$. Black curve: $\partial\Omega(1.0, \mathcal{X}_0, 0)$ from Runge-Kutta methods.)

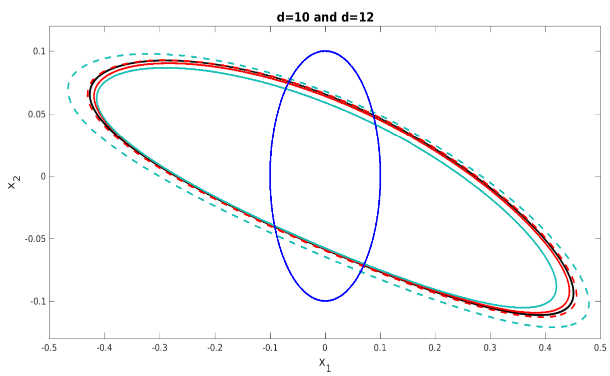

Figure 2: Reachable sets for Example 1 in 1b). (Solid red and green curves: $\partial\mathcal{X}_{1,12}$ and $\partial\mathcal{X}_{1,10}$, respectively. Dashed red and green curves: $\partial\mathcal{Y}_{1,12}$ and $\partial\mathcal{Y}_{1,10}$, respectively. Blue curve: $\partial\mathcal{X}_0$. Black curve: $\partial\Omega(1, \mathcal{X}_0, 0)$ from Runge-Kutta methods.)

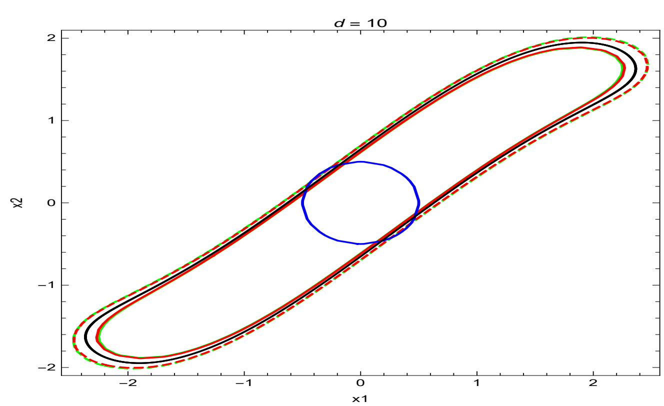

Figure 3: Reachable sets for Example 2. (Solid green and red curves: $\partial\mathcal{X}_{2,10}$ for 2a) and 2b), respectively. Dashed green and red curves: $\partial\mathcal{Y}_{2,10}$ for 2a) and 2b), respectively. Blue curve: $\partial\mathcal{X}_0$. Black curve: $\partial\Omega(2.0, \mathcal{X}_0, 0)$ from Runge-Kutta methods.)

Ex.	$[t_0, T]$	k	d_i	ϵ	*Time*
1a	[0,1.0]	18	18	1.50×10^{-3}	109.41
1a	[0,1.0]	20	20	7.07×10^{-4}	238.55
1b	[0,1.0]	10	10	1.30×10^{-3}	12.36
1b	[0,1.0]	12	12	2.75×10^{-4}	7.82
2a	[0,2.0]	10	10	2.85×10^{-2}	6.07
2a	[0,2.0]	12	12	2.9×10^{-3}	10.63
2b	[0,2.0]	10	10	2.71×10^{-2}	6.25
2b	[0,2.0]	12	12	1.4×10^{-3}	10.69
3a	[0,3.0]	12	12	1.75×10^{-2}	153.32
3a	[0,3.0]	14	14	1.21×10^{-2}	665.57
3b	[0,3.0]	12	12	4.35×10^{-2}	191.19
3b	[0,3.0]	14	14	2.59×10^{-2}	742.96

Table 1: *Parameters and performance of our implementations on the examples presented in this section.* $[t_0, T]$*: the reference interval;* k*: the relaxation order in* (11)*, i.e. degree for* $\Phi_k(\boldsymbol{x}, t)$*;* d_i*: degree of the sum-of-squares multiplier* s_i *in the optimization* (11)*;* ϵ*: tolerance of the near-optimal solution to* (11)*; Time: seconds.*

EXAMPLE 3. *Consider the reversed-time 3D-Lotka-Volterra system*

$$\begin{aligned} \dot{x}_1 &= -x_1x_2 + x_1x_3 \\ \dot{x}_2 &= -x_2x_3 + x_2x_1 \\ \dot{x}_3 &= -x_3x_1 + x_3x_2 \ , \end{aligned}$$

where $\mathcal{X}_0 = \{\boldsymbol{x} \mid \sum_{i=1}^{3} 100x_i^2 \leq 1\}$ *and 3a)* $\mathcal{Y} = \{\boldsymbol{x} \mid 0.16 - \sum_{i=1}^{3} x_i^2 \geq 0\}$*; 3b)* $\mathcal{Y} = \{\boldsymbol{x} \mid 0.25 - \sum_{i=1}^{3} x_i^2 \geq 0\}$*. The computation results are illustrated in Fig. 5 and 6.*

From Example 1, we observe by comparing the results in Fig. 1 and Fig. 2 that polynomials of degree 10 (or 12) in the case of 1b) can under-approximate the reachable set

[1]For academic use, the software Mosek can be obtained for free from `https://www.mosek.com/`.

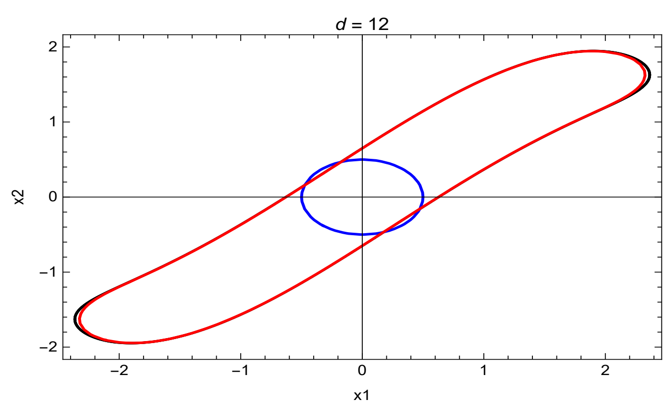

Figure 4: Reachable sets for Example 2. (Red curve: $\partial\mathcal{X}_{2,12}$. Blue curve: $\partial\mathcal{X}_0$. Black curve: $\partial\Omega(2.0, \mathcal{X}_0, 0)$ from Runge-Kutta methods.)

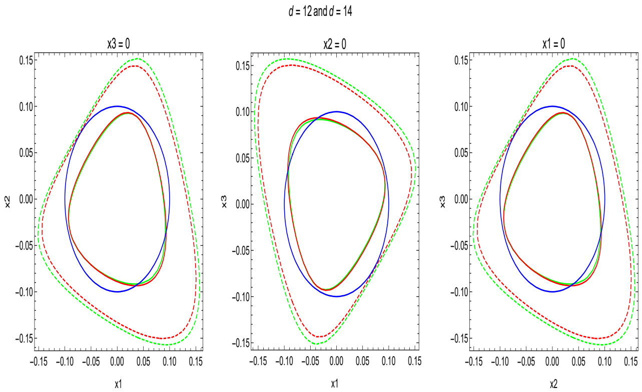

Figure 5: Reachable sets for Example 3 in 3a). (Solid and dashed red curves: $\partial\mathcal{X}_{3,14}$ and $\partial\mathcal{Y}_{3,14}$, respectively. Solid and dashed green curves: $\partial\mathcal{X}_{3,12}$ and $\partial\mathcal{Y}_{3,12}$, respectively. Blue curve: $\partial\mathcal{X}_0$.)

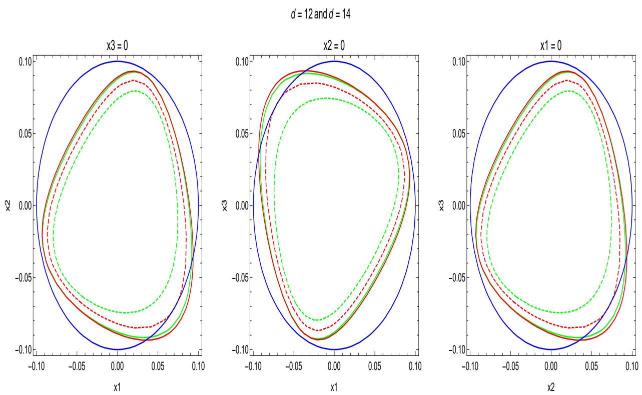

Figure 6: Reachable sets for Example 3 with 3a) and 3b). (Solid red and green curves : $\partial\mathcal{X}_{3,14}$ and $\partial\mathcal{X}_{3,12}$ for 3a), respectively. Dashed red and green curves: $\partial\mathcal{X}_{3,14}$ and $\partial\mathcal{X}_{3,12}$ for 3b), respectively. Blue curve: $\partial\mathcal{X}_0$.)

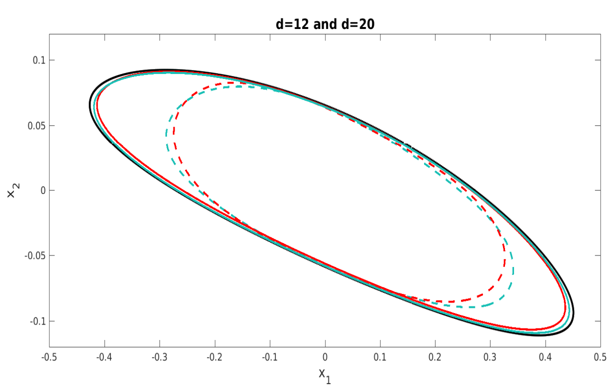

Figure 7: Reachable sets for Example 1. (Solid red and green curves: $\partial\mathcal{X}_{1,20}$ and $\partial\mathcal{X}_{1,12}$ from our method in 1a) and 1b), respectively. Dashed red and green curves: $\partial\mathcal{X}_{1,20}$ and $\partial\mathcal{X}_{1,12}$ from the method in [17] in 1a) and 1b), respectively. Black curve: $\partial\Omega(1.0, \mathcal{X}_0, 0)$ obtained by Runge-Kutta methods.)

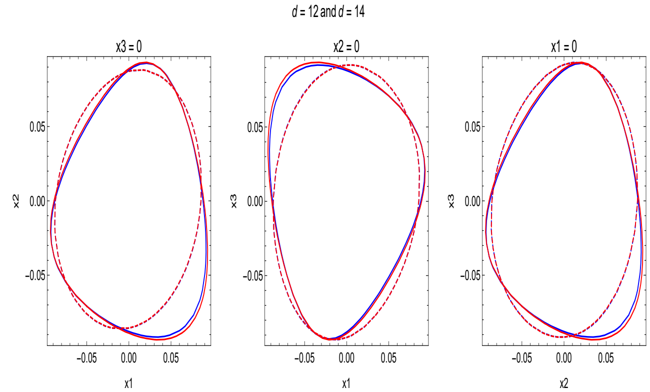

Figure 8: Reachable sets for Example 3 in 3a). (Solid red and blue curves: $\partial\mathcal{X}_{3,14}$ and $\partial\mathcal{X}_{3,12}$ obtained from our method, respectively. Dashed red and blue curves: $\partial\mathcal{X}_{3,14}$ and $\partial\mathcal{X}_{3,12}$ obtained from the method in [17], respectively.)

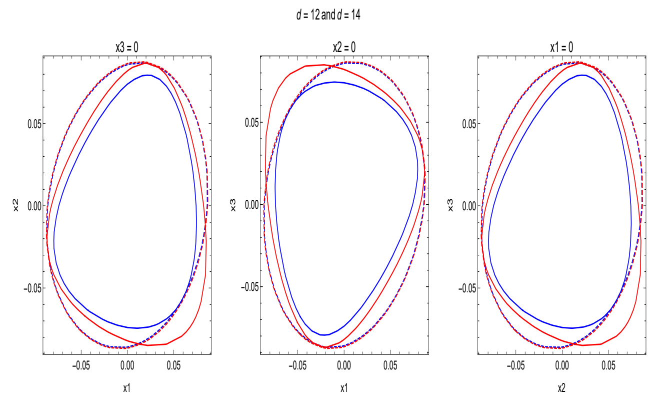

Figure 9: Reachable sets for Example 3 in 3b). (Solid red and blue curves: $\partial\mathcal{X}_{3,14}$ and $\partial\mathcal{X}_{3,12}$ obtained from our method, respectively. Dashed red and blue curves: $\partial\mathcal{X}_{3,14}$ and $\partial\mathcal{X}_{3,12}$ obtained from the method in [17], respectively.)

less conservatively than polynomials of degree 18 (or 20) in the case of 1a). This observation intuitively implies that a less conservatively a-priori enclosure $\mathcal{Y}$ would help compute a tighter approximation of the reachable set with polynomials of lower degree. This claim also applies to Example 3 by scrutinizing the results in Fig. 6. In contrast, such statement does not seem to apply to Example 2, for which no significant differences between state set estimates, as illustrated in Fig. 3, are found for cases 2a) and 2b). In real applications, we would suggest the use of a less conservatively set $\mathcal{Y}$ to perform computations on (11). Another observation, from all the above three examples, is that the semi-definite program (11) can provide a sequence of inner-approximations with guarantee of convergence to the exact reachable set as well as a monotonically decreasing sequence of infima ϵ characterizing the discrepancy between the approximate solution to the HJE and its exact solution over the specified compact set $\mathcal{X} \times [t_0, T]$ with the relaxation order increasing, as claimed in Theorem 5 and Theorem 6 respectively.

4.2 Comparisons

As discussed previously, analogous to our method in this paper, an under-approximation of the rechable set can be gained by solving a single semi-defnite program for the method in [17] as well. Except the differences between our method and the method in [17] as presented in the related work section, we in this subsection further compare the performances of these two methods based on Examples 1 $\sim$ 3.

From the visualized results in Fig. 7, it is evident that the under-approximations computed by our method, when the relaxation orders are 14 and 12 for the cases $1b$) and $1a$) respectively, are tighter than those synthesized by the method in [17]. Compared with the results illustared in Fig. 3 (or, Fig. 4), the results obtained by applying the method in [17] to both cases in Example 2 are too conservative. Therefore, we do not show them in this paper. Contrasting with Examples 1 and 2, the comparison as to Example 3 becomes slightly complicated. For the case of 3a), according to the results illustrated in Fig. 8, the claim that our method outperforms the method in [17] is still valid when relaxation orders 14 and 12 are adopted in both methods. From Fig. 9, it is difficult to make a conslusion that one outperforms the other in case that the relxation order for both methods is 14. However, when applying both methods with the relaxation order 12 to 3b) and observing the results in Fig. 9, we gain less conservative under-approxiamtions using the method in [17]. Note that in 3a) (and 3b)), the corresponding under-approximations obtained by [17] with relaxation orders 14 and 12 respectively are almost the same and therefore it is hard to distinguish the visualized results presented in Fig. 8 (and Fig. 9).

Based on the comparison results above, we conclude that our method can provide tighter under-approximations than the method in [17] for some cases when the same relaxation order for both methods is used. Besides, apart from the development of more advanced semi-definite programming solvers, which are capable of dealing with numerical issues in the numerical solving of semi-definite programming problems more maturely, the investigation of structure differences of the semi-definite programs constructed in this paper and in [17] helps shed more light on the merits of one over the other for these two methods.

5. CONCLUSION

An approach based on convex programming was proposed for computing semi-algebraic under-approximations of reachable sets for polynomial nonlinear systems. The construction of reachable sets depends on computing approximate analytical solutions of HJEs. The exact solution to the HJE of interest can be uniformly approximated in a compact set by solving a hierarchy of semi-definite programs, providing a converging sequence of under-approximations to the reachable set. There are two notable benefits of our approach in comparison to related approaches: one is that the under-approximation is obtained by approximately solving just a single semi-definite programming problem; and the other is that the very same formulation also provides a description of an over-approximation such that the accuracy of the reach-set approximation can easily be inspected. We tested and discussed our approach on several illustrative examples.

In order to make our method more practical, additonal work remains to be done: numerical issues (e.g., [32]) and computational efficiency in dealing with large-scale semi-definite programming problems are pressing problems. For alleviating these, and beyond developing more powerful semi-definite programming solvers being capable of more maturely and efficiently dealing with numerical issues ubiquitous in the current solvers, an exploitation of the sparsity of large-scale semi-definite programms [16] seems attractive. An alternative to the semi-definite programming formulation employed in this paper could be linear programs manipulatting Handelman representations [33, 14]. While coming with the drawback that semi-definite programming representation are more general than the linear programming representation in algebraic geometry [20], the solvers for linear programs are obviously more stable and much more scalable. We will investigate the trade-offs in our future work.

Instead of considering initial sets defined by sub-level sets of a continuously differentiable function, one could also generalize to sub-level sets of Lipschitz-continuous functions. If the initial set is defined by the zero sub-level sets of a system of Lipschitz-continuous functions and the system is subject to state constraints, we can still under-approximate the reachable set tightly by the use of a sequence of smooth viscosity super-solutions to Hamilton-Jacobi equations, which can again be found by solving semi-definite programming problems. Moreover, we will extend our method in this paper to under-approximate reachability analysis for systems subject to disturbances and state-constraints, which is an open problem. These extensions are ongoing work.

Acknowledgements. This research from Bai Xue is partly funded by CAS Pioneer Hundred Talents Program under grant No. Y8YC235015, and from Martin Fränzle is funded by Deutsche Forschungsgemeinschaft within the Research Training Group "SCARE - System Correctness under Adverse Conditions" (DFG GRK 1765), and from Naijun Zhan is supported partly by NSFC under grant No. 61625206 and 61732001, by "973 Program" under grant No. 2014CB340701 and by the CAS/SAFEA International Partnership Program for Creative Research Teams.

6. REFERENCES

[1] M. Althoff, O. Stursberg, and M. Buss. Reachability analysis of nonlinear systems with uncertain parameters using conservative linearization. In *CDC'08*, pages 4042–4048. IEEE, 2008.

[2] E. Asarin, O. Bournez, T. Dang, and O. Maler. Approximate reachability analysis of piecewise-linear dynamical systems. In *HSCC'00*, pages 20–31. Springer, 2000.

[3] X. Chen, E. Ábrahám, and S. Sankaranarayanan. Taylor model flowpipe construction for non-linear hybrid systems. In *RTSS'12*, pages 183–192. IEEE, 2012.

[4] X. Chen, S. Sankaranarayanan, and E. Ábrahám. Under-approximate flowpipes for non-linear continuous systems. In *FMCAD'14*, pages 59–66. FMCAD Inc, 2014.

[5] T. Dang, O. Maler, and R. Testylier. Accurate hybridization of nonlinear systems. In *HSCC'10*, pages 11–20. ACM, 2010.

[6] A. N. Daryin and A. B. Kurzhanski. Estimation of reachability sets for large-scale uncertain systems: from theory to computation. In *CDC'12*, pages 7401–7406. IEEE, 2012.

[7] A. Eggers, N. Ramdani, N. S. Nedialkov, and M. Fränzle. Improving the SAT modulo ODE

approach to hybrid systems analysis by combining different enclosure methods. *Software & Systems Modeling*, 14(1):121–148, 2015.

[8] G. Frehse, S. Bogomolov, M. Greitschus, T. Strump, and A. Podelski. Eliminating spurious transitions in reachability with support functions. In *HSCC'15*, pages 149–158. ACM, 2015.

[9] D. H. Fremlin. Kirszbraun's theorem. *Preprint*, 2011.

[10] T. Gan, M. Chen, Y. Li, B. Xia, and N. Zhan. Reachability Analysis for Solvable Dynamical Systems. *To appear in IEEE Trans. Autom. Control.*, 2017.

[11] A. Girard, C. Le Guernic, and O. Maler. Efficient computation of reachable sets of linear time-invariant systems with inputs. In *HSCC'06*, pages 257–271. Springer, 2006.

[12] E. Goubault, O. Mullier, S. Putot, and M. Kieffer. Inner approximated reachability analysis. In *HSCC'14*, pages 163–172. ACM, 2014.

[13] E. Goubault and S. Putot. Forward Inner-Approximated Reachability of Non-Linear Continuous Systems. In *HSCC'17*, pages 1–10. ACM, 2017.

[14] D. Handelman. Representing polynomials by positive linear functions on compact convex polyhedra. *Pacific Journal of Mathematics*, 132(1):35–62, 1988.

[15] D. Henrion and M. Korda. Convex computation of the region of attraction of polynomial control systems. *IEEE Trans. Autom. Control.*, 59(2):297–312, 2014.

[16] M. Kojima. Exploiting Structured Sparsity in Large Scale Semidefinite Programming Problems. In *ICMS'10*, pages 4–9. Springer, 2010.

[17] M. Korda, D. Henrion, and C. N. Jones. Inner approximations of the region of attraction for polynomial dynamical systems. *IFAC Proceedings Volumes*, 46(23):534–539, 2013.

[18] A. B. Kurzhanski and P. Varaiya. Ellipsoidal techniques for reachability analysis: internal approximation. *Systems & Control Letters*, 41(3):201–211, 2000.

[19] A. B. Kurzhanski and P. Varaiya. Ellipsoidal Techniques for Reachability Analysis. In *HSCC'00*, pages 202–214. Springer, 2000.

[20] J. B. Lasserre. Semidefinite programming vs. LP relaxations for polynomial programming. *Mathematics of Operations Research*, 27(2):347–360, 2002.

[21] J. B. Lasserre. Tractable approximations of sets defined with quantifiers. *Mathematical Programming*, 151(2):507–527, 2015.

[22] J. Lofberg. Yalmip: A toolbox for modeling and optimization in MATLAB. In *CACSD'04*, pages 284–289. IEEE, 2004.

[23] J. Lygeros. On reachability and minimum cost optimal control. *Automatica*, 40(6):917–927, 2004.

[24] V. Magron, D. Henrion, and J.-B. Lasserre. Semidefinite approximations of projections and polynomial images of semialgebraic sets. *SIAM Journal on Optimization*, 25(4):2143–2164, 2015.

[25] O. Maler. Algorithmic verification of continuous and hybrid systems. *arXiv preprint arXiv:1403.0952*, 2014.

[26] I. M. Mitchell, A. M. Bayen, and C. J. Tomlin. A time-dependent Hamilton-Jacobi formulation of reachable sets for continuous dynamic games. *IEEE Trans. Autom. Control.*, 50(7):947–957, 2005.

[27] A. Mosek. The MOSEK optimization toolbox for MATLAB manual. *Version 7.1 (Revision 28)*, page 17, 2015.

[28] S. Prajna. Barrier certificates for nonlinear model validation. *Automatica*, 42(1):117–126, 2006.

[29] M. Putinar. Positive polynomials on compact semi-algebraic sets. *Indiana University Mathematics Journal*, 42(3):969–984, 1993.

[30] N. Ramdani, N. Meslem, and Y. Candau. A hybrid bounding method for computing an over-approximation for the reachable set of uncertain nonlinear systems. *IEEE Trans. Autom. Control.*, 54(10):2352–2364, 2009.

[31] S. Ratschan and Z. She. Safety verification of hybrid systems by constraint propagation based abstraction refinement. In *HSCC'05*, volume 5, pages 573–589. Springer, 2005.

[32] P. Roux, Y.-L. Voronin, and S. Sankaranarayanan. Validating Numerical Semidefinite Programming Solvers for Polynomial Invariants. In *SAS'16*, pages 424–446. Springer, 2016.

[33] S. Sankaranarayanan, X. Chen, et al. Lyapunov Function Synthesis using Handelman Representations. *IFAC Proceedings Volumes*, 46(23):576–581, 2013.

[34] S. Schupp, E. Ábrahám, X. Chen, I. B. Makhlouf, G. Frehse, S. Sankaranarayanan, and S. Kowalewski. Current challenges in the verification of hybrid systems. In *CyPhy'15*, pages 8–24. Springer, 2015.

[35] L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Review*, 38(1):49–95, 1996.

[36] T.-C. Wang, S. Lall, and M. West. Polynomial level-set method for polynomial system reachable set estimation. *IEEE Trans. Autom. Control.*, 58(10):2508–2521, 2013.

[37] B. Xue, A. Easwaran, N.-J. Cho, and M. Fränzle. Reach-avoid verification for nonlinear systems based on boundary analysis. *IEEE Trans. Autom. Control.*, 62(7):3518–3523, 2017.

[38] B. Xue, P. N. Mosaad, M. Fränzle, M. Chen, Y. Li, and N. Zhan. Safe Over-and Under-Approximation of Reachable Sets for Delay Differential Equations. In *FORMATS'17*, pages 281–299. Springer, 2017.

[39] B. Xue, Z. She, and A. Easwaran. Under-approximating backward reachable sets by polytopes. In *CAV'16*, pages 457–476. Springer, 2016.

[40] B. Xue, Z. She, and A. Easwaran. Under-Approximating Backward Reachable Sets by Semialgebraic Sets. *IEEE Trans. Autom. Control.*, 62:5185–5197, 2017.