

# Robust Regions of Attraction Generation for State-Constrained Perturbed Discrete-Time Polynomial Systems

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**Abstract:** In this paper we propose a convex programming based method for computing robust regions of attraction for state-constrained perturbed discrete-time polynomial systems. The robust region of attraction of interest is a set of states such that every possible trajectory initialized in it will approach an equilibrium state while never violating the specified state constraint, regardless of the actual perturbation. Based on a Bellman equation which characterizes the interior of the maximal robust region of attraction as the strict one sub-level set of its unique bounded and continuous solution, we construct a semi-definite program for computing robust regions of attraction. Under appropriate assumptions, the existence of solutions to the constructed semi-definite program is guaranteed and there exists a sequence of solutions such that their strict one sub-level sets inner-approximate and converge to the interior of the maximal robust region of attraction in measure. Finally, we demonstrate the method by two examples.

*Keywords:* Robust Regions of Attraction; State-Constrained Perturbed Discrete-Time Polynomial Systems; Convex Programming.

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## 1. INTRODUCTION

Discrete-time systems, which are governed by difference equations or iterative processes, may result from discretizing continuous systems or modeling evolution systems for which the time scale is discrete. They are prevalent in signal processing, population dynamics, scientific computation and so forth, e.g., (Kot and Schaffer, 1986). The polynomial discrete-time systems are the type of systems whose dynamics are described in polynomial forms. This system is classified as an important class of nonlinear systems due to the fact that many nonlinear systems can be modelled as, transformed into, or approximated by polynomial systems, e.g., (Halanay and Rasvan, 2000).

A fundamental problem in control engineering consists of determining the robust region of attraction of an equilibrium (Slotine et al., 1999), which is a set of states such that every trajectory starting from it will move towards this equilibrium while never leaving a specified state-constraint set irrespective of the actual perturbation. Its applications include biology systems (Merola et al., 2008) and ecology systems (Ludwig et al., 1997), and among others. Computing robust regions of attraction has been the subject of extensive research over the past several decades, resulting in the emergence of many computational approaches, e.g., Lyapunov function-based methods (Zubov, 1964; Salle and Lefschetz, 1961; Coutinho and de Souza, 2013; Genesio et al., 1985; Giesl and Hafstein, 2014), trajectory reversing methods (Genesio et al., 1985), moment-based optimization methods (Henrion and Korda, 2013; Korda et al., 2013) and so on.

Lyapunov-based methods are still dominant in estimating robust regions of attraction (Khalil, 2002). Generally, the search for Lyapunov functions is non-trivial for nonlinear systems due to the non-constructive nature of the Lyapunov theory, apart from some cases where the Jacobian matrix of the linearized system associated with the nonlinear system of interest is Hurwitz. However, with the advance of real algebraic geometry and polynomial optimization in the last decades, especially the sum-of-squares (SOS) decomposition technique (Parrilo, 2000), finding a Lyapunov function which is decreasing over a given state constraint set can be reduced to a convex programming problem for polynomial systems (Papachristodoulou and Prajna, 2002). This results in a large amount of findings which adopt convex optimization based approaches to the search for polynomial Lyapunov functions, e.g., (Anderson and Papachristodoulou, 2015). However, if we return to the problem of estimating robust domains of attraction, it resorts to addressing a bilinear semi-definite program, e.g., (Jarvis-Wloszek, 2003; Tan and Packard, 2008), which falls within the non-convex programming framework and is notoriously hard to solve. Also, the existence of polynomial solutions to (bilinear) semi-definite programs is not explored in the literature, especially for perturbed systems.

In this paper we propose a novel semi-definite programming based method for computing robust regions of attraction for state-constrained perturbed discrete-time polynomial systems with an equilibrium state, which is uniformly locally exponentially stable. It is worth remarking here that the method proposed in this paper can also be applied to the computation of robust regions of attraction for polynomial systems with an asymptotically stable equilibrium

state, as highlighted in Remark 1. The semi-definite program is constructed by relaxing a modified Bellman equation which characterizes the interior of the maximal robust region of attraction as the strict one sub-level set of its unique bounded and continuous solution Xue et al. (2020). It falls within the convex programming framework and can be solved efficiently in polynomial time via interior-point methods. Moreover, the existence of solutions to the constructed semi-definite program is guaranteed and there exists a sequence of polynomial solutions such that their strict one sub-level sets inner-approximate and converge to the interior of the maximal robust region of attraction in measure under appropriate assumptions. Finally, we demonstrate our method by two examples.

The closely related works to the present work in spirit are (Summers et al., 2013; Xue et al., 2018, 2019a,b; Henrion and Korda, 2013; Korda et al., 2013). The work in (Summers et al., 2013) employed semi-definite programming to solve discrete-time stochastic optimal control problems by relaxing the Bellman equation. Semi-definite programming based methods were proposed in (Xue et al., 2018, 2019a,b) for computing reachable sets for continuous-time systems by relaxing Hamilton-Jacobi equations. Reachable sets computation over finite time horizons was studied in (Xue et al., 2018, 2019a) and robust invariant sets computation over the infinite time horizon was studied in (Xue et al., 2019b). The trajectories starting from the robust invariant set in (Xue et al., 2019b) are not required to approach an equilibrium. The existence of solutions to the constructed semi-definite program in (Xue et al., 2019b) is not guaranteed. In contrast, the present work considers the computation of robust regions of attraction over the infinite time horizon for discrete-time systems by relaxing Bellman equations. The trajectories starting from the robust region of attraction are required to approach an equilibrium. Moreover, the existence of solutions to the constructed semi-definite program in the present work is guaranteed. Different from the works in (Henrion and Korda, 2013) and (Korda et al., 2013), which respectively considered outer and inner approximations of the maximal region of attraction over finite-time horizons based on moment-based optimization methods, the present work focuses on inner-approximations of the maximal region of attraction over the infinite-time horizon.

This paper is structured as follows. In Section 2 basic notions and the problem of interest are introduced. After presenting our method for synthesizing robust regions of attraction in Section 3, we evaluate it on two examples in Section 4. Finally, we conclude this paper in Section 5.

## 2. PRELIMINARIES

In this section we describe the system of interest and the concept of robust regions of attraction.

The notions will be used in this paper:  $\mathbb{R}^n$  denotes the set of  $n$ -dimensional real vectors.  $\mathbb{R}[\cdot]$  denotes the ring of polynomials with real coefficients in variables given by the argument.  $\mathbb{R}_k[\cdot]$  denotes the vector space of real multivariate polynomials of total degree  $\leq k$ .  $\Delta^\circ$ ,  $\partial\Delta$ ,  $\overline{\Delta}$  and  $\Delta^c$  denote the interior, boundary, closure and complement of a set  $\Delta$ , respectively. The difference of two sets  $A$  and  $B$  is denoted by  $A \setminus B$ .  $\mu(A)$  denotes

the Lebesgue measure on  $A \subset \mathbb{R}^n$ .  $\mathbb{N}$  denotes the set of non-negative integers.  $\|\mathbf{x}\|$  denotes the 2-norm, i.e.,  $\|\mathbf{x}\| = \sqrt{\sum_{i=1}^n x_i^2}$ , where  $\mathbf{x} = (x_1, \dots, x_n)'$ .  $B(\mathbf{0}, r)$  denotes a ball of radius  $r > 0$  and center  $\mathbf{0}$ , i.e.,  $B(\mathbf{0}, r) = \{\mathbf{x} \mid \|\mathbf{x}\|^2 \leq r\}$ . Vectors are denoted by boldface letters.

The perturbed discrete-time system of interest in this paper is of the following form

$$\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{d}(k)), k \in \mathbb{N}, \quad (1)$$

where  $\mathbf{x}(\cdot) : \mathbb{N} \rightarrow \mathbb{R}^n$ ,  $\mathbf{d}(\cdot) : \mathbb{N} \rightarrow D$ ,

$$D = \{\mathbf{d} \in \mathbb{R}^m \mid \bigwedge_{i=1}^{m_d} [h_i^D(\mathbf{d}) \leq 0]\}$$

is a compact semi-algebraic subset in  $\mathbb{R}^m$  with  $h_i^D \in \mathbb{R}[\mathbf{d}]$ ,  $\mathbf{f} \in \mathbb{R}[\mathbf{x}, \mathbf{d}]$  with  $\mathbf{f}(\mathbf{0}, \mathbf{d}) = \mathbf{0}$  for  $\mathbf{d} \in D$ .

In order to define our problem succinctly, we present the definition of a perturbation input policy  $\pi$ .

*Definition 1.* A perturbation input policy, denoted by  $\pi$ , refers to a function  $\pi(k) : \mathbb{N} \rightarrow D$ . In addition, we denote the set of all perturbation policies by  $\mathcal{D}$ .

Given a perturbation input policy  $\pi$ , a trajectory to system (1) is presented in Definition 2.

*Definition 2.* Given a perturbation input policy  $\pi \in \mathcal{D}$ , a trajectory of system (1) initialized in  $\mathbf{x}_0 \in \mathbb{R}^n$  is defined as  $\phi_{\mathbf{x}_0}^\pi(\cdot) : \mathbb{N} \rightarrow \mathbb{R}^n$ , where  $\phi_{\mathbf{x}_0}^\pi(0) = \mathbf{x}_0$  and

$$\phi_{\mathbf{x}_0}^\pi(k+1) = \mathbf{f}(\phi_{\mathbf{x}_0}^\pi(k), \pi(k)), \forall k \in \mathbb{N}. \quad (2)$$

We assume that  $\mathbf{0}$  is uniformly locally exponentially stable.

*Assumption 1.* The equilibrium state  $\mathbf{0}$  is uniformly locally exponentially stable for (1), i.e., there exist positive constants  $M > 0$ ,  $r > 0$  and  $0 < \lambda < 1$  such that

$$\|\phi_{\mathbf{x}_0}^\pi(k)\| \leq \lambda^k M \|\mathbf{x}_0\|, \forall \mathbf{x}_0 \in B(\mathbf{0}, r), \forall \pi \in \mathcal{D}, \forall k \in \mathbb{N},$$

where  $B(\mathbf{0}, r) \subset X$ .

Assumption 1 implies the existence of a positive constant  $\bar{\epsilon}$  such that  $B(\mathbf{0}, \bar{\epsilon}) \subseteq X$  and

$$\phi_{\mathbf{x}_0}^\pi(k) \in B(\mathbf{0}, \frac{r}{2}), \forall \mathbf{x}_0 \in B(\mathbf{0}, \bar{\epsilon}), \forall k \in \mathbb{N}, \forall \pi \in \mathcal{D}. \quad (3)$$

Since  $0 < \lambda < 1$  in Assumption 1,  $\bar{\epsilon}$  in (3) exists and can take the value of  $r$ .

Suppose that the state constraint set

$$X = \{\mathbf{x} \in \mathbb{R}^n \mid \bigwedge_{i=1}^{n_X} [h_i^X(\mathbf{x}) < 1]\}$$

is a bounded open set with  $h_i^X(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ . Also,  $h_i^X(\mathbf{x}) > 0$  for  $\mathbf{x} \neq \mathbf{0}$  and  $h_i^X(\mathbf{0}) = 0$ ,  $i = 1, \dots, n_X$ . We define robust regions of attraction.

*Definition 3.* (Robust Regions of Attraction). The maximal robust region of attraction  $\mathcal{R}$  is the set of states such that every possible trajectory of system (1) starting from it will approach the equilibrium state  $\mathbf{0}$  while never leaving the state constraint set  $X$ , i.e.

$$\mathcal{R} = \left\{ \mathbf{x}_0 \mid \left. \begin{array}{l} \phi_{\mathbf{x}_0}^\pi(k) \in X, \forall k \in \mathbb{N}, \forall \pi \in \mathcal{D}, \\ \text{and } \lim_{k \rightarrow \infty} \phi_{\mathbf{x}_0}^\pi(k) = \mathbf{0}, \forall \pi \in \mathcal{D} \end{array} \right\}. \quad (4)$$

Correspondingly, a robust region of attraction is a subset of the maximal robust region of attraction  $\mathcal{R}$ .

## 3. ROBUST REGIONS OF ATTRACTION GENERATION

In this section we present our semi-definite programming based method for computing robust regions of attraction

by relaxing a Bellman equation. Then we show that there exists a sequence of solutions to the semi-definite program such that their strict one sub-level sets can inner-approximate the interior of the maximal robust region of attraction in measure under appropriate assumptions.

### 3.1 Bellman Equations

In this subsection we introduce a modified Bellman equation, to which the strict one sub-level set of the unique bounded and continuous solution is equal to the interior of the maximal robust region of attraction.

*Theorem 1.* (Xue et al., 2020) The interior of the maximal robust region of attraction  $\mathcal{R}$  is equal to the strict one sub-level set of the unique bounded and continuous solution  $v(\mathbf{x}) : \mathbb{R}^n \rightarrow [0, 1]$  to the Bellman equation

$$\begin{cases} \min \left\{ \inf_{\mathbf{d} \in D} \{v - v(\mathbf{f}) - g \cdot (1 - v)\}, \right. \\ \quad \left. v - 1 + \min_{j \in \{1, \dots, n_x\}} l(1 - h_j^X) \right\} = 0, \forall \mathbf{x} \in \mathbb{R}^n, \\ v(\mathbf{0}) = 0, \end{cases} \quad (5)$$

where  $g(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$  is a non-negative polynomial satisfying that  $g(\mathbf{x}) = 0$  iff  $\mathbf{x} = \mathbf{0}$ , and  $l(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$  with

$$l(x) = \begin{cases} x, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$

That is,  $\mathcal{R}^\circ = \{\mathbf{x} \in \mathbb{R}^n \mid v(\mathbf{x}) < 1\}$ .

A direct consequence of Theorem 1 is that if a continuous function  $u(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfies (5), then  $u(\mathbf{x})$  satisfies the constraints:

$$\begin{cases} u - u(\mathbf{f}) - g \cdot (1 - u) \geq 0, \quad \forall \mathbf{x} \in \mathbb{R}^n, \forall \mathbf{d} \in D, \\ u - 1 + l(1 - h_j^X) \geq 0, \quad \forall \mathbf{x} \in \mathbb{R}^n, \\ j = 1, \dots, n_x. \end{cases} \quad (7)$$

*Corollary 1.* Suppose a continuous function  $u(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$  is a solution to (7), then  $u(\mathbf{x}) \geq v(\mathbf{x})$ , where  $v(\mathbf{x})$  is the unique bounded and continuous solution to (5). Consequently,  $\{\mathbf{x} \in \mathbb{R}^n \mid u(\mathbf{x}) < 1\} \subset \mathcal{R}^\circ$  and thus  $\{\mathbf{x} \in \mathbb{R}^n \mid u(\mathbf{x}) < 1\}$  is a robust region of attraction.

**Proof.** The second constraint in (7) implies that  $u(\mathbf{x}) \geq 0$  for  $\mathbf{x} \in \mathbb{R}^n$ .

Assume that there exists  $\mathbf{y}_0 \in \mathbb{R}^n$  such that  $u(\mathbf{y}_0) < v(\mathbf{y}_0)$ .

First let's assume  $v(\mathbf{y}_0) \geq 1$ . Obviously,  $\mathbf{y}_0 \neq \mathbf{0}$  and consequently  $g(\mathbf{y}_0) > 0$ . Since  $u$  satisfies (7) and  $v(\mathbf{y}_0) > u(\mathbf{y}_0)$ , we have that  $v(\mathbf{y}_0) - 1 + \min_{j \in \{1, \dots, n_x\}} l(1 - h_j^X(\mathbf{y}_0)) > 0$ . Also, since  $v$  satisfies (5), we have that

$$\inf_{\mathbf{d} \in D} \{v(\mathbf{y}_0) - v(\mathbf{f}(\mathbf{y}_0, \mathbf{d})) - g(\mathbf{y}_0)(1 - v(\mathbf{y}_0))\} = 0.$$

Since  $v$  is continuous over  $\mathbb{R}^n$  and  $\mathbf{f}$  is continuous over  $\mathbb{R}^n \times D$ , there exists  $\mathbf{d}'_1 \in D$  such that  $v(\mathbf{y}_0) - v(\mathbf{f}(\mathbf{y}_0, \mathbf{d}'_1)) - g(\mathbf{y}_0)(1 - v(\mathbf{y}_0)) = 0$ . Since  $u(\mathbf{y}_0) - u(\mathbf{f}(\mathbf{y}_0, \mathbf{d}'_1)) - g(\mathbf{y}_0)(1 - u(\mathbf{y}_0)) \geq 0$ , we obtain that

$$v(\mathbf{f}(\mathbf{y}_0, \mathbf{d}'_1)) - u(\mathbf{f}(\mathbf{y}_0, \mathbf{d}'_1)) \geq (v(\mathbf{y}_0) - u(\mathbf{y}_0))(1 + g(\mathbf{y}_0)).$$

Let  $\mathbf{y}_1 = \phi_{\mathbf{y}_0}^{\pi_1}(1)$ , where  $\pi_1(0) = \mathbf{d}'_1$ , then  $v(\mathbf{y}_1) > u(\mathbf{y}_1)$ . Also, we have  $v(\mathbf{y}_0) \leq v(\mathbf{y}_1)$ . Moreover,  $\mathbf{y}_1 \neq \mathbf{0}$ ,  $g(\mathbf{y}_1) > 0$ . We continue the above deduction for  $\mathbf{y}_0$  to  $\mathbf{y}_1$  and obtain that there exists  $\mathbf{d}'_2 \in D$  such that

$$v(\mathbf{f}(\mathbf{y}_1, \mathbf{d}'_2)) - u(\mathbf{f}(\mathbf{y}_1, \mathbf{d}'_2)) \geq (v(\mathbf{y}_1) - u(\mathbf{y}_1))(1 + g(\mathbf{y}_1)).$$

Thus, we have

$$\begin{aligned} v(\mathbf{f}(\mathbf{y}_1, \mathbf{d}'_2)) - u(\mathbf{f}(\mathbf{y}_1, \mathbf{d}'_2)) &\geq \\ (v(\mathbf{y}_0) - u(\mathbf{y}_0))(1 + g(\mathbf{y}_1))(1 + g(\mathbf{y}_0)). \end{aligned}$$

Let  $\mathbf{y}_2 = \phi_{\mathbf{y}_1}^{\pi_2}(1)$ , where  $\pi_2(0) = \mathbf{d}'_2$ , then  $v(\mathbf{y}_2) > u(\mathbf{y}_2)$ . Also,  $v(\mathbf{y}_1) \leq v(\mathbf{y}_2)$ .

Analogously, we deduce that for  $k \in \mathbb{N}$ ,

$$\begin{aligned} v(\mathbf{f}(\mathbf{y}_k, \mathbf{d}'_{k+1})) - u(\mathbf{f}(\mathbf{y}_k, \mathbf{d}'_{k+1})) &\geq \\ (v(\mathbf{y}_0) - u(\mathbf{y}_0))(1 + g(\mathbf{y}_k)) \cdots (1 + g(\mathbf{y}_0)). \end{aligned}$$

Moreover, let  $\mathbf{y}_{k+1} = \phi_{\mathbf{y}_k}^{\pi_{k+1}}(1)$ , then  $v(\mathbf{y}_k) \leq v(\mathbf{y}_{k+1})$ , where  $\pi_{k+1}(0) = \mathbf{d}'_{k+1}$ . This implies that  $\lim_{k \rightarrow \infty} \mathbf{y}_k \neq \mathbf{0}$  and thus  $\mathbf{y}_k \notin B(\mathbf{0}, \bar{\epsilon})$  for  $k \in \mathbb{N}$ , where  $B(\mathbf{0}, \bar{\epsilon})$  is defined in (3). Assume  $c_0 = \inf\{g(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^n \setminus B(\mathbf{0}, \bar{\epsilon})\}$ . Clearly,  $c_0 > 0$ . Such  $c_0$  exists since  $g(\mathbf{x})$  is a non-negative polynomial over  $\mathbb{R}^n$  and  $g(\mathbf{x}) = 0$  iff  $\mathbf{x} = \mathbf{0}$ . Therefore,

$$\begin{aligned} v(\mathbf{f}(\mathbf{y}_k, \mathbf{d}'_{k+1})) - u(\mathbf{f}(\mathbf{y}_k, \mathbf{d}'_{k+1})) &\geq \\ &\geq (v(\mathbf{y}_0) - u(\mathbf{y}_0))(1 + c_0)^{k+1}, \end{aligned}$$

implying that  $\lim_{k \rightarrow \infty} v(\mathbf{y}_k) = \infty$ , which contradicts the fact that  $v$  is bounded over  $\mathbb{R}^n$ . Thus,  $v(\mathbf{y}_0) \leq u(\mathbf{y}_0)$ .

Next, assume  $v(\mathbf{y}_0) > u(\mathbf{y}_0)$  and  $v(\mathbf{y}_0) < 1$ . According to Theorem 1, every possible trajectory starting from  $\mathbf{y}_0$  will eventually approach  $\mathbf{0}$ . Also, we have

$$\inf_{\mathbf{d} \in D} \{v(\mathbf{y}_0, \mathbf{d}) - v(\mathbf{f}(\mathbf{y}_0, \mathbf{d})) - g(\mathbf{y}_0)(1 - v(\mathbf{y}_0))\} = 0.$$

Following the deduction mentioned above, we have

$$v(\mathbf{y}_k) - u(\mathbf{y}_k) \geq v(\mathbf{y}_0) - u(\mathbf{y}_0), \forall k \in \mathbb{N}.$$

Therefore, we have that  $\lim_{k \rightarrow \infty} v(\mathbf{y}_k) \geq v(\mathbf{y}_0) - u(\mathbf{y}_0)$ , contradicting  $\lim_{k \rightarrow \infty} v(\mathbf{y}_k) = 0$ . Thus,  $v(\mathbf{y}_0) \leq u(\mathbf{y}_0)$ .

Therefore,  $v(\mathbf{x}) \leq u(\mathbf{x})$  for  $\mathbf{x} \in \mathbb{R}^n$ . Also, since  $\mathcal{R}^\circ = \{\mathbf{x} \in \mathbb{R}^n \mid v(\mathbf{x}) < 1\}$ ,  $\{\mathbf{x} \in \mathbb{R}^n \mid u(\mathbf{x}) < 1\} \subset \mathcal{R}^\circ$  holds.  $\square$

From Corollary 1 we observe that a robust region of attraction can be found by solving (7) instead of (5).

### 3.2 Semi-definite Programming Relaxation

In this subsection we construct a semi-definite program to compute robust regions of attraction based on (7). We observe that  $u(\mathbf{x})$  is required to satisfy (7) over  $\mathbb{R}^n$ , which is a strong condition. Regarding this issue, we further consider (7) on the set  $B(\mathbf{0}, R)$ , where  $B(\mathbf{0}, R)$  is defined in Assumption 2. In addition, we introduce another set  $X_\infty$ , which is also defined in Assumption 2.

*Assumption 2.* (a)  $X_\infty = \{\mathbf{x} \in \mathbb{R}^n \mid h_\infty(\mathbf{x}) < 1\}$  is a robust region of attraction, where  $h_\infty \in \mathbb{R}[\mathbf{x}]$ . Besides, we assume that  $\mathbf{0} \in X_\infty^\circ$ . It could be regarded as an initial conservative estimate of the maximal robust region of attraction.

(b)  $B(\mathbf{0}, R) = \{\mathbf{x} \in \mathbb{R}^n \mid h_0(\mathbf{x}) \leq R\}$ , where  $h_0(\mathbf{x}) = \sum_{i=1}^n x_i^2$ ,  $R$  is a positive constant such that  $\Omega(X) \subset B(\mathbf{0}, R)$ .  $\Omega(X)$  is the set of states being reachable from the set  $X$  within one step for system (1), i.e.,  $\Omega(X) = \{\mathbf{x} \mid \mathbf{x} = \mathbf{f}(\mathbf{x}_0, \mathbf{d}), \mathbf{x}_0 \in X, \mathbf{d} \in D\} \cup X$ .

$X_\infty \subset X$  from Assumption 2.  $X_\infty$  satisfies Assumption 2 if  $h_\infty$  is a (local) Lyapunov function for system (1). There are many efficient methods for computing  $h_\infty$ , e.g., semi-definite programming based methods (Giesl and Hafstein, 2015) or linear programming based methods (Giesl and

Hafstein, 2014).  $B(\mathbf{0}, R)$  can be efficiently computed by solving a semi-definite programming as in (Magron et al., 2019). In this paper, we assume  $X_\infty$  and  $B(\mathbf{0}, R)$  were given. Their computations are not the focus of this paper.

Based on  $B(\mathbf{0}, R)$  and  $X_\infty$  in Assumption 2, we further relax constraint (7) and restrict the search for a continuous function  $u(\mathbf{x})$  in the compact set  $B(\mathbf{0}, R)$ . We obtain the following constraints:

$$\begin{aligned} u - u(\mathbf{f}) - g \cdot (1 - u) &\geq 0, \forall \mathbf{x} \in \overline{B(\mathbf{0}, R)} \setminus X_\infty, \forall \mathbf{d} \in D, \\ u - 1 &\geq 0, \forall \mathbf{x} \in B(\mathbf{0}, R) \setminus X, \\ u - h_j^X &\geq 0, \forall \mathbf{x} \in \overline{X}, \\ j &= 1, \dots, n_X. \end{aligned} \quad (8)$$

Obviously,  $v(\mathbf{x})$  in (5) satisfies (8).

When the solution to (8) is restricted to a polynomial, based on sum-of-squares decomposition for multivariate polynomials, (7) could be reduced as the following sum-of-squares program, which is formulated in (9).

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$$\begin{aligned} p_k^* &= \inf \mathbf{w} \cdot \mathbf{l} \\ \text{s.t.} \\ u_k - u_k(\mathbf{f}) - g \cdot (1 - u_k) &= s_0 + s_1 \cdot (R - h_0) \\ &\quad + s_2 \cdot (h_\infty - 1) - \sum_{i=1}^{m_D} s_{3,i} \cdot h_i^D, \\ u_k - 1 &= s_{4,j} + s_{5,j} \cdot (R - h_0) + s_{6,j} \cdot (h_j^X - 1), \\ u_k - h_j^X &= s_{7,j} + s_{8,j} \cdot (R - h_0) + \sum_{l=1}^{n_X} s_{9,l,j} \cdot (1 - h_l^X), \\ j &= 1, \dots, n_X, \end{aligned} \quad (9)$$

where  $\mathbf{w} \cdot \mathbf{l} = \int_{B(\mathbf{0}, R)} u_k(\mathbf{x}) d\mathbf{x} - \int_{X_\infty} u_k(\mathbf{x}) d\mathbf{x}$ ,  $\mathbf{l}$  is the vector of the moments of the Lebesgue measure over  $\overline{B(\mathbf{0}, R)} \setminus X_\infty$  indexed in the same basis in which the polynomial  $u_k$  with coefficients  $\mathbf{w}$  is expressed. The minimum is over polynomial  $u_k(\mathbf{x}) \in R_k[\mathbf{x}]$  and sum-of-squares polynomial  $s_i(\mathbf{x}, \mathbf{d})$ ,  $i = 0, \dots, 2$ ,  $s_{3,i}(\mathbf{x}, \mathbf{d})$ ,  $i = 1, \dots, m_D$ ,  $s_{i,j}(\mathbf{x})$ ,  $s_{9,l,j}(\mathbf{x})$ ,  $i = 4, \dots, 8$ ,  $j, l = 1, \dots, n_X$ .

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*Theorem 2.* Under Assumption 2, if  $u(\mathbf{x}) \in \mathbb{R}_k[\mathbf{x}]$  is a solution to (9), then  $\{\mathbf{x} \in B(\mathbf{0}, R) \mid u(\mathbf{x}) < 1\}$  is a robust region of attraction.

**Proof.** According to the second constraint in (9), we have  $u(\mathbf{x}) \geq 1$  for  $\mathbf{x} \in B(\mathbf{0}, R) \setminus X$ . Therefore,  $\{\mathbf{x} \in B(\mathbf{0}, R) \mid u(\mathbf{x}) < 1\} \subset X$ . Next we prove that every possible trajectory initialized in the set  $\{\mathbf{x} \in B(\mathbf{0}, R) \mid u(\mathbf{x}) < 1\}$  will approach the equilibrium state  $\mathbf{0}$  eventually while never leaving the state constraint set  $X$ .

Assume that there exists  $\mathbf{y}_0 \in \{\mathbf{x} \in B(\mathbf{0}, R) \mid u(\mathbf{x}) < 1\}$  and a perturbation input policy  $\pi'$  such that  $\phi_{\mathbf{y}_0}^{\pi'}(k) \in X$  for  $k = 0, \dots, k_0$  and  $\phi_{\mathbf{y}_0}^{\pi'}(k_0 + 1) \notin X$ . It is obvious that  $\phi_{\mathbf{y}_0}^{\pi'}(k) \in X \setminus X_\infty$  for  $k = 0, \dots, k_0$  since  $X_\infty$  is a robust region of attraction. Since  $\Omega(X) \subseteq B(\mathbf{0}, R)$ , where  $\Omega(X)$  is defined in Assumption 2,  $\phi_{\mathbf{y}_0}^{\pi'}(k_0 + 1) \in B(\mathbf{0}, R) \setminus X$ , thus we obtain that

$$u(\phi_{\mathbf{y}_0}^{\pi'}(k_0 + 1)) \geq 1. \quad (10)$$

However, since  $\phi_{\mathbf{y}_0}^{\pi'}(k) \in B(\mathbf{0}, R) \setminus X_\infty$  for  $k = 0, \dots, k_0 + 1$  and  $u(\mathbf{y}_0) < 1$ , from the first constraint in (9), we have

$$u(\phi_{\mathbf{y}_0}^{\pi'}(k_0 + 1)) < 1,$$

contradicting (10). Thus, every possible trajectory initialized in  $\{\mathbf{x}_0 \in B(\mathbf{0}, R) \mid u(\mathbf{x}_0) < 1\}$  never leaves  $X$ .

Lastly, we prove that every possible trajectory initialized in  $\{\mathbf{x} \in B(\mathbf{0}, R) \mid u(\mathbf{x}) < 1\}$  will approach the equilibrium state  $\mathbf{0}$  eventually. Since every possible trajectory initialized in the set  $X_\infty$  will approach the equilibrium state  $\mathbf{0}$  eventually, it is enough to prove that every possible trajectory initialized in the set  $\{\mathbf{x} \in B(\mathbf{0}, R) \mid u(\mathbf{x}) < 1\} \setminus X_\infty$  will enter the set  $X_\infty$  over finite time horizons. Assume that there exist  $\mathbf{y}_0 \in \{\mathbf{x} \in B(\mathbf{0}, R) \mid u(\mathbf{x}) < 1\}$  and a perturbation input policy  $\pi'$  such that  $\phi_{\mathbf{y}_0}^{\pi'}(k) \notin X_\infty, \forall k \in \mathbb{N}$ . Since  $\phi_{\mathbf{y}_0}^{\pi'}(k) \in X$  for  $k \in \mathbb{N}$  and  $u(\mathbf{x}) \geq 0$  for  $\mathbf{x} \in X$  (The fact that  $u(\mathbf{x}) \geq 0$  for  $\mathbf{x} \in X$  can be obtained from the third constraint in (9).),

$$u(\phi_{\mathbf{y}_0}^{\pi'}(k)) \geq 0, \forall k \in \mathbb{N}.$$

Moreover,  $u(\phi_{\mathbf{y}_0}^{\pi'}(k)) < 1$  holds for  $k \in \mathbb{N}$ . According to the first constraint in (9), we have

$$u(\phi_{\mathbf{y}_0}^{\pi'}(k)) - u(\phi_{\mathbf{y}_0}^{\pi'}(k+1)) \geq g(\phi_{\mathbf{y}_0}^{\pi'}(k)) \cdot (1 - u(\phi_{\mathbf{y}_0}^{\pi'}(k)))$$

for  $k \in \mathbb{N}$ . Therefore,

$$u(\phi_{\mathbf{y}_0}^{\pi'}(k+1)) \leq u(\phi_{\mathbf{y}_0}^{\pi'}(k)) - g(\phi_{\mathbf{y}_0}^{\pi'}(k))(1 - u(\phi_{\mathbf{y}_0}^{\pi'}(k)))$$

and thus

$$u(\phi_{\mathbf{y}_0}^{\pi'}(k)) \geq u(\phi_{\mathbf{y}_0}^{\pi'}(k+1))$$

for  $k \in \mathbb{N}$ . Since  $g(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$  is positive over  $\mathbf{x} \neq \mathbf{0}$ , we obtain that  $g(\mathbf{x})$  can attain a minimum over the compact set  $\overline{X \setminus X_\infty}$ . Let

$$\epsilon' = \min_{\mathbf{x} \in \overline{X \setminus X_\infty}} g(\mathbf{x}),$$

it is obvious that  $\epsilon' > 0$ . Therefore, we have

$$u(\phi_{\mathbf{y}_0}^{\pi'}(k+1)) \leq u(\phi_{\mathbf{y}_0}^{\pi'}(k)) - \epsilon'(1 - u(\mathbf{y}_0)), \forall k \in \mathbb{N}.$$

Therefore,

$$u(\phi_{\mathbf{y}_0}^{\pi'}(k+1)) \leq u(\mathbf{y}_0) - (k+1)\epsilon'(1 - u(\mathbf{y}_0)), \forall k \in \mathbb{N}.$$

Thus, we obtain that there exists  $k'_0 \in \mathbb{N}$  such that

$$u(\phi_{\mathbf{y}_0}^{\pi'}(k'_0)) < 0,$$

contradicting the fact that  $u(\phi_{\mathbf{y}_0}^{\pi'}(k)) \geq 0, \forall k \in \mathbb{N}$ . Therefore, every possible trajectory initialized in the set  $\{\mathbf{x} \in B(\mathbf{0}, R) \mid u(\mathbf{x}) < 1\} \setminus X_\infty$  will enter the set  $X_\infty$  over finite time horizons. Consequently, every possible trajectory initialized in the set  $\{\mathbf{x} \in B(\mathbf{0}, R) \mid u(\mathbf{x}) < 1\}$  will approach the equilibrium state  $\mathbf{0}$ .

Combining above arguments, we conclude that  $\{\mathbf{x} \in B(\mathbf{0}, R) \mid u(\mathbf{x}) < 1\}$  is a robust region of attraction.  $\square$

*Remark 1.* Note that Theorem 2 still holds if the equilibrium  $\mathbf{0}$  is asymptotically stable rather than uniformly locally exponentially stable. The proof of Theorem 2 did not require Assumption 1.

### 3.3 Theoretical Analysis

This section shows that there exists a sequence of solutions to (9) such that their strict one sub-level sets inner-approximate the interior of the maximal robust region of attraction in measure under appropriate assumptions.

*Assumption 3.* One of the polynomials defining the set  $D$  is equal to  $h_i^D := \|\mathbf{d}\|^2 - R_D$  for some constant  $R_D \geq 0$ .

Assumption 3 is without loss of generality since  $D$  is compact, and thus redundant constraint of the form  $R_D - \|\mathbf{d}\|^2 \geq 0$  can always be added to the description of  $D$  for sufficiently large  $R_D$ .

*Lemma 1.* Under Assumptions 1, 2 and 3, there exists a sequence  $(u_{k_i}(\mathbf{x}))_{i=0}^\infty$  such that  $u_{k_i}(\mathbf{x})$  converges from above to  $v$  uniformly over  $B(\mathbf{0}, R)$ , where  $u_{k_i}(\mathbf{x}) \in \mathbb{R}_{k_i}[\mathbf{x}]$  denotes the  $u$ -component of a solution to (9) and  $v$  is the continuous and bounded solution to (5).

**Proof.** Let

$$\Omega(B(\mathbf{0}, R)) = \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y} = \phi_{\mathbf{x}_0}^\pi(i), i \in [0, 1], \mathbf{x}_0 \in B(\mathbf{0}, R), \pi \in \mathcal{D}\}. \quad (11)$$

Since  $\mathbf{f} \in \mathbb{R}[\mathbf{x}, \mathbf{d}]$ , and  $D$  and  $B(\mathbf{0}, R)$  are compact,  $\Omega(B(\mathbf{0}, R))$  is bounded and consequently  $\overline{\Omega(B(\mathbf{0}, R))}$  is compact. Moreover,  $B(\mathbf{0}, R) \subset \Omega(B(\mathbf{0}, R))$ . Let  $B = \overline{B(\mathbf{0}, R) \setminus X_\infty}$ . We infer that for every  $\epsilon > 0$ , there exists a continuous function  $v_\epsilon$  satisfying (8) and  $|v_\epsilon - v| \leq \epsilon$ . Obviously,  $v_\epsilon = v + \epsilon$  satisfies such requirement since

$$\begin{aligned} v_\epsilon - v_\epsilon(\mathbf{f}) - g \cdot (1 - v_\epsilon) &\geq \epsilon c_0, \forall \mathbf{x} \in B, \forall \mathbf{d} \in D, \\ v_\epsilon - 1 &\geq \epsilon, \forall \mathbf{x} \in B(\mathbf{0}, R) \setminus X, \\ v_\epsilon - h_j^X &\geq \epsilon, \forall \mathbf{x} \in \overline{X}, j = 1, \dots, n_X, \end{aligned} \quad (12)$$

where  $c_0 = \inf\{g(\mathbf{x}) \mid \mathbf{x} \in \overline{B(\mathbf{0}, R) \setminus X_\infty}\}$ . Since  $\overline{\Omega(B(\mathbf{0}, R))}$  is compact, according to Stone-Weierstrass theorem (Cotter, 1990), there exists a polynomial  $u_{k_i}$  of a sufficiently high degree  $k_i$  such that

$$0 < u_{k_i} - v_\epsilon < \frac{\epsilon}{2} c_0, \forall \mathbf{x} \in \overline{\Omega(B(\mathbf{0}, R))}.$$

Thus, we have

$$\epsilon < u_{k_i} - v < \epsilon + \frac{\epsilon}{2} c_0, \forall \mathbf{x} \in \overline{\Omega(B(\mathbf{0}, R))}. \quad (13)$$

According to the definition of  $\Omega(B(\mathbf{0}, R))$ , i.e., (11), we have that  $\mathbf{f}(\mathbf{x}, \mathbf{d}) \in \Omega(B(\mathbf{0}, R))$  holds for  $\mathbf{x} \in B(\mathbf{0}, R)$  and  $\mathbf{d} \in D$ . Therefore,

$$\epsilon < u_{k_i}(\mathbf{f}(\mathbf{x}, \mathbf{d})) - v(\mathbf{f}(\mathbf{x}, \mathbf{d})) < \epsilon + \frac{\epsilon}{2} c_0$$

holds for  $\mathbf{x} \in B(\mathbf{0}, R)$  and  $\mathbf{d} \in D$ . It is easy to check that  $u_{k_i}$  satisfies

$$\begin{aligned} u_{k_i} - u_{k_i}(\mathbf{f}) - g \cdot (1 - u_{k_i}) &> 0, \forall \mathbf{x} \in B, \forall \mathbf{d} \in D, \\ u_{k_i} - 1 &> 0, \forall \mathbf{x} \in B(\mathbf{0}, R) \setminus X, \\ u_{k_i} - h_j^X &> 0, \forall \mathbf{x} \in \overline{X}, j = 1, \dots, n_X. \end{aligned} \quad (14)$$

From Putinar's Positivstellensatz (Putinar, 1993) and arbitrariness of  $\epsilon$ , we obtain  $u_{k_i}(\mathbf{x})$  converges from above to  $v$  uniformly over  $B(\mathbf{0}, R)$  with  $i$  approaching infinity.  $\square$

Finally, we conclude that  $\{\mathbf{x} \in B(\mathbf{0}, R) \mid u_{k_i}(\mathbf{x}) < 1\}$  converges to the interior of the maximal robust region of attraction with  $i$  approaching infinity.

*Theorem 3.* Let  $u_{k_i}(\mathbf{x})$  satisfy the condition in Lemma 1. Then the set  $\mathcal{R}_{k_i} = \{\mathbf{x} \in B(\mathbf{0}, R) \mid u_{k_i}(\mathbf{x}) < 1\}$  satisfies

$$\begin{aligned} \mathcal{R}_{k_i} &\subset \mathcal{R}^\circ \text{ and} \\ \lim_{i \rightarrow \infty} \mu(\mathcal{R}^\circ \setminus \mathcal{R}_{k_i}) &= 0. \end{aligned}$$

**Proof.**  $\mathcal{R}_{k_i} \subset \mathcal{R}^\circ$  is an immediate consequence of Lemma 1 since  $u_{k_i} \geq v$  over  $B(\mathbf{0}, R)$  according to (13).

SDP						
Ex.	$k$	$d_{s_i}$	$d_{s_{3,i_1}}$	$d_{s_{i_2,j}}$	$d_{s_{9,l,j}}$	$T$
1	6	8	8	8	8	2.10
	10	12	12	12	12	17.50
2	4	4	4	4	4	5.45
	6	6	6	6	6	24.67
	8	8	8	8	8	316.12

Table 1. Parameters of our implementations on Example 1.  $k, d_{s_i}, d_{s_{3,i_1}}, d_{s_{i_2,j}}, d_{s_{9,l,j}}$ : degree of polynomials  $u, s_i, s_{3,i_1}, s_{i_2,j}, s_{9,l,j}$  in (9), respectively,  $i_1 = 1, \dots, m_D$ ,  $i = 0, \dots, 2$ ,  $i_2 = 4, \dots, 8$ ,  $j = 1, \dots, n_X$ ,  $l = 1, \dots, n_X$ ;  $T$ : computation times (seconds).

According to Theorem 1 as well as Theorem 3 in (Lasserre, 2015) and Lemma 1, we have  $\lim_{i \rightarrow \infty} \mu(\mathcal{R}^\circ \setminus \mathcal{R}_{k_i}) = 0$ .  $\square$

#### 4. ILLUSTRATIVE EXAMPLES

In this section we evaluate the semi-definite programming based method on two examples. The computations were performed on an i7-P51s 2.6GHz CPU with 4GB RAM running Windows 10. YALMIP (Lofberg, 2004) and Mosek (Mosek, 2015) were used to implement (9).

*Example 1.* Consider the discrete-generation predator-prey model from (Halanay and Rasvan, 2000),

$$\begin{cases} x(j+1) = 0.5x(j) - x(j)y(j), \\ y(j+1) = -0.5y(j) + (d(j)+1)x(j)y(j), \end{cases}$$

where  $j \in \mathbb{N}$ .

In this example we consider  $D = \{d \in \mathbb{R} \mid d^2 - 0.01 \leq 0\}$  and  $X = \{(x, y) \mid x^2 + y^2 - 1 \leq 0\}$ . The origin  $\mathbf{0}$  for this example is uniformly locally exponentially stable.  $g(x, y) = x^2 + y^2$ ,  $R = 1.6$ ,  $h_0(x, y) = x^2 + y^2$  and  $h_\infty(x, y) = 100(x^2 + y^2)$  are used to perform computations on the semi-definite program (9).

The function  $h_\infty(\mathbf{x}) = 100x^2 + 100y^2$  defining  $X_\infty$  is a Lyapunov function such that  $X_\infty \subset X$  is a robust region of attraction. This argument can be justified by first encoding the following constraint

$$h_\infty(\mathbf{x}) - h_\infty(\mathbf{f}(\mathbf{x}, \mathbf{d})) > 0, \forall \mathbf{x} \in X_\infty \setminus \{\mathbf{0}\}, \forall \mathbf{d} \in D$$

in the form of sum-of-squares constraints and then verifying the feasibility of the constructed sum-of-squares constraints, where  $\mathbf{f}(\mathbf{x}, \mathbf{d}) = (0.5x - xy, -0.5y + (d+1)xy)$ . Assumption 2(a) is satisfied.  $B(\mathbf{0}, R) = \{\mathbf{x} \mid h_0(x, y) \leq 1.6\}$ , which is a set satisfying Assumption 2(b). Since  $X \subseteq B(\mathbf{0}, R)$ , we just need to verify  $\{\mathbf{x} \mid \mathbf{x} = \mathbf{f}(\mathbf{x}_0, \mathbf{d}), \mathbf{x}_0 \in X, \mathbf{d} \in D\}$ . This argument is justified by first encoding the following constraint

$$\begin{aligned} 1.6 - (0.5x - xy)^2 \\ - (-0.5y + (d+1)xy)^2 \geq 0, \forall (x, y) \in X, \forall d \in D \end{aligned}$$

in the form of sum-of-squares constraints and then verifying its feasibility. Moreover, the function  $d^2 - 0.01$  defining  $D$  satisfies Assumption 3. Therefore, Lemma 1 holds, implying that the existence of solutions to the semi-definite program (9) is guaranteed.

Robust regions of attraction, which are computed via solving the semi-definite program (9) with approximating polynomials of degree 6 and 10 respectively, are illustrated in Fig. 1. We observe from Fig. 1 that the robust region of attraction computed when  $k = 10$  approximates the maximal robust region of attraction tightly by comparing

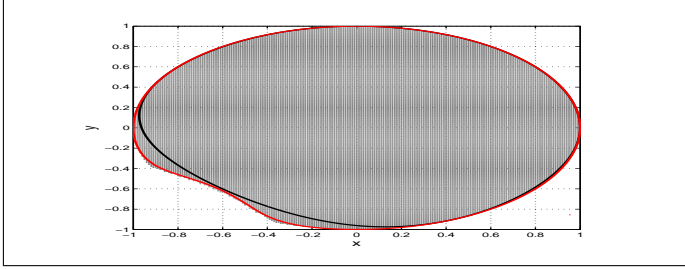


Fig. 1: Estimations of  $\mathcal{R}$  for Example 1. Black and red curves denote the boundaries of robust regions of attraction computed when  $k = 6$  and  $k = 10$ , respectively. Gray region denotes the maximal robust region of attraction estimated via simulation techniques.

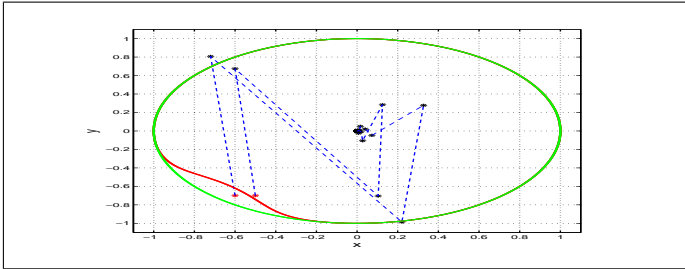


Fig. 2: An illustration of two trajectories for Example 1. Green and red curves denote the boundaries of the state constraint set  $X$  and the robust region of attraction computed when  $k = 10$ . Red stars and black stars denote the initial states and subsequent states, respectively. The dash blue line denotes the transition between states.

with the maximal one estimated via simulation methods. Here the simulation method requires gridding the state space and the disturbance space, and the check whether grid states will enter the region  $X_\infty$  while remaining inside the set  $X$  within a certain time interval. Two trajectories, one respecting the state constraint and one violating the state constraint, are illustrated in Fig. 2. They are generated by extracting the perturbation input  $d(j)$  from  $D$  randomly for  $j \in \mathbb{N}$ .

*Example 2.* Consider a three-dimensional perturbed discrete-time Lotka-Volterra model adopted from (Bischi and Tramtontana, 2010),

$$\begin{cases} x(j+1) = x(j)(e_1 + a_1x(j) + a_2y(j) + a_3z(j)), \\ y(j+1) = y(j)(e_2 + a_4x(j) + a_5y(j) + a_6z(j)), \\ z(j+1) = z(j)(e_3 + a_7x(j) + a_8y(j) + a_9z(j)), \end{cases}$$

where  $e_1 = e_2 = e_3 = 0.5$ ,  $a_1 = 0.5 + d$ ,  $a_2 = a_6 = -0.5$ ,  $a_3 = a_4 = a_5 = a_7 = a_8 = a_9 = 0.5$ ,  $D = \{d \in \mathbb{R} \mid d^2 - 0.01 \leq 0\}$  and  $X = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 - 1 \leq 0\}$ . The origin  $\mathbf{0}$  for this example is uniformly locally exponentially stable.

For this example,  $X_\infty = \{\mathbf{x} \in \mathbb{R}^3 \mid 100(x^2 + y^2 + z^2) < 1\}$  and  $B(\mathbf{0}, R) = \{\mathbf{x} \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1.6\}$  satisfy Assumption 2 and are used for perform computations. Moreover, the function  $d^2 - 0.01$  defining  $D$  satisfies Assumption 3. Therefore, Lemma 1 holds, implying that the existence of solutions to the semi-definite program (9) with  $g(\mathbf{x}) = x^2 + y^2 + z^2$  is guaranteed.

Plots of computed robust regions of attraction for approximating polynomials of degree  $k = 4, 6, 8$  on planes  $y - z$  with  $x = 0$ ,  $x - z$  with  $y = 0$ , and  $x - y$  with  $z = 0$  are shown

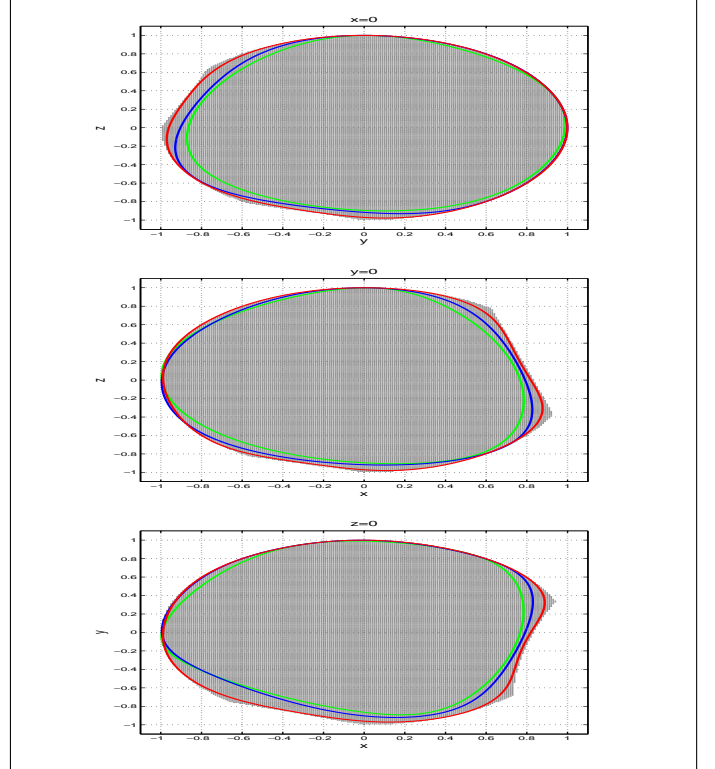


Fig. 3: Estimations of  $\mathcal{R}$  for Example 2. Green, blue and red curves denote the boundaries of robust regions of attraction computed when  $k = 4, 6$  and  $8$ , respectively. Gray region denotes the maximal robust region of attraction estimated via simulation techniques.

in Fig. 3. In order to shed light on the accuracy of the computed regions of attraction, we also use the simulation technique to synthesize estimations of the maximal robust region of attraction on planes  $y - z$  with  $x = 0$ ,  $x - z$  with  $y = 0$ , and  $x - y$  with  $z = 0$  by taking initial states in the state spaces  $\{\mathbf{x} \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1 \wedge x = 0\}$ ,  $\{\mathbf{x} \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1 \wedge y = 0\}$  and  $\{\mathbf{x} \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1 \wedge z = 0\}$ , respectively. They are the gray regions in Fig. 5. We observe from Fig. 3 that the robust region of attraction computed when  $k = 8$  could approximate the maximal robust region of attraction well.

## 5. CONCLUSION

In this paper we proposed a semi-definite programming based method for computing robust regions of attraction for state-constrained perturbed discrete-time polynomial systems. The semi-definite program was constructed based on a Bellman equation. There exists a sequence of solutions to the semi-definite program such that their strict one sub-level sets inner-approximate the interior of the maximal robust region of attraction in measure under appropriate assumptions. Two examples demonstrated the performance of our approach.

In near future we would like to compare the proposed method in this paper with existing methods on estimating robust domains of attraction for discrete-time systems. Also, we would extend the proposed method for computing robust regions of attraction of state-constrained perturbed continuous-time polynomial systems.

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