Reach-avoid Verification Based on Convex Optimization

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***Abstract*— In this paper we propose novel sufficient conditions for verifying reach-avoid properties of continuous-time systems modelled by ordinary differential equations (ODEs). Given a system, an initial set, a safe set and a target set of states, we say that the reach-avoid property holds, if for all initial conditions in the initial set, any trajectory of the system starting at them will eventually, i.e. in unbounded yet finite time, enter the target set while remaining inside the safe set until that first target hit (that is, if the system starting from the initial set can reach the target set safely). Based on a discount value function, two sets of quantified constraints are derived for verifying the reach-avoid property via the computation of exponential/asymptotic guidance-barrier functions (they form a barrier escorting the system to the target set safely at an exponential or asymptotic rate). It is interesting to find that one set of constraints whose solution is termed exponential guidance-barrier functions is just a simplified version of the existing one derived from the moment based method, while the other one whose solution is termed asymptotic guidance-barrier functions is completely new. Furthermore, built upon this new set of constraints, we derive a set of more expressive constraints, which includes the aforementioned two sets of constraints as special instances, providing more chances for verifying the reach-avoid property successfully. Finally, several examples demonstrate the theoretical developments and performance of proposed sufficient conditions using semi-definite programming methods.**

***Index Terms*— Ordinary Differential Equations; Reach-avoid Verification; Quantified Constraints**

I. Introduction

Cyber-physical technology is integrated into an ever-growing range of physical devices and increasingly pervades our daily life [17]. Examples of such systems range from intelligent highway systems, to air traffic management systems, to computer and communication networks, to smart houses and smart supplies, etc. [8], [25]. Many of the above-mentioned applications are safety-critical and require a rigorous guarantee of safe operation.

Among the many possible rigorous guarantees, reach-avoid verification, i.e., verifying whether the system's dynamics (generally modelled by ODEs) satisfy reach-avoid properties, is definitely in demand. One of the popular methods for reach-avoid verification is computational reachability analysis, which involves the explicit computation of reachable states [3], [12]. In general, the exact computation of reach sets is impossible for dynamical and hybrid systems [13]. Over-approximate reachability analysis, which computes an over-approximation (i.e., super-set) of the reach set based on set propagation techniques, is therefore studied in existing literature for verification purposes (e.g., [5]). Overly pessimistic over-approximations, however, render many properties unverifiable in practice, especially for large initial sets and/or large time horizons. This pessimism mainly arises due to the wrapping effect [16], which is the propagation and accumulation of over-approximation errors through the iterative computation of reach sets. There are many techniques developed in existing literature for controlling the wrapping effect. One way is to use complex sets such as Taylor models [7], [10] and polynomial zonotopes [2] to over-approximate the reach set. On the other hand, as the extent of the wrapping effect correlates strongly with the size of the initial set, another way is to exploit subsets of the initial set for performing over-approximate reachability analysis via exploiting the (topological) structure of the system. For instance, appropriate corner points of reach sets, called bracketing systems, are used in [11], [23] to bound the complete reach sets when the systems under consideration are monotonic; [26] proposed the set-boundary reachability method for for continuous-time systems featuring a locally Lipschitz-continuous vector field.

Another popular method is the optimization based method, which transforms the verification problem into a problem of determining the existence of solutions to a set of quantified constraints. This method avoids the explicit computation of reach sets and thus can handle verification with unbounded time horizons. A well-known method is the barrier certificate method, which was originally proposed in [20], [21] for safety verification of continuous and hybrid systems. The barrier certificate method was inspired by Lyapunov functions in control theory and relies on the computation of barrier certificates, which are a function of state satisfying a set of quantified inequalities on both the function itself and its Lie derivative along the flow of the system. In the state space, the zero level set of a barrier certificate separates an unsafe region from all system trajectories starting from a set of legally initial states and thus the existence of such a function provides an exact certificate/proof of system safety. Afterwards, a number of different kinds of barrier certificates were developed such as exponential barrier certificates and vector barrier certificates in the literature [6], [14], which mainly differ in their expressiveness. This method was also extended to reachability verification of continuous and hybrid systems. For instance, it was extended to reach-avoid verification in [22]. The set of constraints in [21] requires the Lie derivative of barrier certificates to be strictly decreasing along the trajectories of the dynamics. It is strong, limiting its applications as discussed in Subsection II-B. Recently, a set of new constraints based on moment theory was presented in [15] for inner-approximating the set of all initial states guaranteeing the satisfaction of the reach-avoid property. It can be straightforwardly extended to reach-avoid verification via supplementing a constraint that the designated initial set is included in the computed inner-approximation. The obtained set of constraints overcomes the strong requirement of the one in [21], but it is a special case of the proposed ones in this paper.

In this paper we study the reach-avoid verification problem of continuous-time systems modelled by ODEs in the framework of the optimization based method. The reach-avoid verification problem of

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interest is that given an initial set, a safe set and a target set, we verify whether any trajectory starting from the initial set will eventually enter the target set while remaining inside the safe set until the first target hit. The reach-avoid verification problem in our method is transformed into a problem of searching for so called guidance-barrier functions. Based on a discount value function, whose certain (sub) level set equals the set of all initial states enabling the satisfaction of reach-avoid properties, with the discount factor being larger than and equal to zero we first respectively derive two sets of quantified constraints whose solutions are termed exponential and asymptotic guidance barrier functions. If a solution to any of these two sets of constraints is found, the reach-avoid property is guaranteed. Based on the set of constraints associated with asymptotic guidance-barrier functions, we further construct a set of more expressive constraints, which admits more solutions and formulates the aforementioned two sets of constraints as its special instances, and thus offers more possibilities of verifying the reach-avoid property successfully. When the datum are polynomials, i.e., the system has polynomial dynamics, the initial set, target set and safe set are semi-algebraic sets, the problem of solving these constraints can be reduced to a semi-definite programming problem. Finally, several examples demonstrate the performance of the proposed constraints in reach-avoid verification with semi-definite programming.

The main contributions of this work are summarized below.

1) Based on a discount value function with a nonnegative discount factor, a novel unified framework is proposed for the reach-avoid verification of systems modelled by ODEs. In this framework, two sets of quantified inequalities are derived when the discount factor is zero and positive respectively. The one, which is obtained when the discount factor is positive, is a simplified version of the existing one from the moment based method in [15] however, the other one (i.e., the one obtained when the discount factor is zero) is completely new.
2) The differences and respective benefits of the aforementioned two sets of constraints are discussed in detail. Based on these discussions, an enhanced set of constraints, which is more expressive, is further developed such that the aforementioned two sets of constraints are its special cases. Performance comparisons with state-of-art ones are made based on several examples using semi-definite programming. Numerical results show that the enhanced set of constraints outperforms all others.

II. Preliminaries

In this section we formally present the concepts of continuous-time systems and reach-avoid verification problem of interest in this paper. Before formulating them, let us introduce some basic notions used throughout this paper: for a function $v(\boldsymbol{x})$, $\bigtriangledown_{\boldsymbol{x}} v(\boldsymbol{x})$ denotes its gradient with respect to $\boldsymbol{x}$; $\mathbb{R}_{\geq 0}(\mathbb{R}_{>0})$ stands for the set of nonnegative (positive) real values in $\mathbb{R}$ with $\mathbb{R}$ being the set of real numbers; the closure of a set $\mathcal{X}$ is denoted by $\overline{\mathcal{X}}$, the complement by $\mathcal{X}^c$ and the boundary by $\partial\mathcal{X}$; $\wedge$ denotes conjunction, and $\forall$ and $\exists$ denote the universal and existential quantifiers, respectively; the ring of all multivariate polynomials in a variable $\boldsymbol{x}$ is denoted by $\mathbb{R}[\boldsymbol{x}]$; vectors are denoted by boldface letters, and the transpose of a vector $\boldsymbol{x}$ is denoted by $\boldsymbol{x}^\top$; $\sum[\boldsymbol{x}]$ is used to represent the set of sum-of-squares polynomials over variables $\boldsymbol{x}$, i.e.,

$$\sum[\boldsymbol{x}] = \{p \in \mathbb{R}[\boldsymbol{x}] \mid p = \sum_{i=1}^{k} q_i^2, q_i \in \mathbb{R}[\boldsymbol{x}], i = 1, \ldots, k\}.$$

A. Preliminaries

The continuous-time system of interest (or, **CS**) is a system whose dynamics are described by an ODE of the following form:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathbb{R}^n, \tag{1}$$

where $\dot{\boldsymbol{x}} = \frac{d\boldsymbol{x}(t)}{dt}$ and $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x}))^\top$ with $f_i(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$.

We denote the trajectory of system **CS** that originates from $\boldsymbol{x}_0 \in \mathbb{R}^n$ and is defined over the maximal time interval $[0, T_{\boldsymbol{x_0}})$ by $\boldsymbol{\phi}_{\boldsymbol{x}_0}(\cdot) : [0, T_{\boldsymbol{x_0}}) \to \mathbb{R}^n$. Consequently,

$$\boldsymbol{\phi}_{\boldsymbol{x}_0}(t) := \boldsymbol{x}(t), \forall t \in [0, T_{\boldsymbol{x_0}}), \text{ and } \boldsymbol{\phi}_{\boldsymbol{x}_0}(0) = \boldsymbol{x}_0,$$

where $T_{\boldsymbol{x}_0}$ is either a positive value (i.e., $T_{\boldsymbol{x_0}} \in \mathbb{R}_{>0}$) or ∞.

Given a bounded safe set $\mathcal{X}$, an initial set $\mathcal{X}_0$ and a target set $\mathcal{X}_r$, where

$$\mathcal{X} = \{\boldsymbol{x} \in \mathbb{R}^n \mid h(\boldsymbol{x}) < 0\} \text{ with } \partial\mathcal{X} = \{\boldsymbol{x} \in \mathbb{R}^n \mid h(\boldsymbol{x}) = 0\},$$
$$\mathcal{X}_0 = \{\boldsymbol{x} \in \mathbb{R}^n \mid l(\boldsymbol{x}) < 0\}, \text{ and } \mathcal{X}_r = \{\boldsymbol{x} \in \mathbb{R}^n \mid g(\boldsymbol{x}) < 0\}$$

with $l(\boldsymbol{x}), h(\boldsymbol{x}), g(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$, and $\mathcal{X}_0 \subseteq \mathcal{X}$ and $\mathcal{X}_r \subseteq \mathcal{X}$, the reach-avoid property of interest is defined as follows.

Definition 1 (Reach-Avoid Property): Given system **CS** with the safe set $\mathcal{X}$, initial set $\mathcal{X}_0$ and target set $\mathcal{X}_r$, we say that the reach-avoid property holds if for all initial conditions $\boldsymbol{x}_0 \in \mathcal{X}_0$, any trajectory $\boldsymbol{\phi}_{\boldsymbol{x}_0}(t)$ of system **CS** starting at $\boldsymbol{\phi}_{\boldsymbol{x}_0}(0) = \boldsymbol{x}_0$ can eventually enter the target set $\mathcal{X}_r$ eventually while remaining inside the safe set until the first target hit, i.e.,

$$\forall \boldsymbol{x}_0 \in \mathcal{X}_0.\exists T \in \mathbb{R}_{>0}.[\boldsymbol{\phi}_{\boldsymbol{x}_0}(T) \in \mathcal{X}_r \bigwedge \forall t \in [0, T].\boldsymbol{\phi}_{\boldsymbol{x}_0}(t) \in \mathcal{X}].$$

Since the reach-avoid property combines guarantees of safety by staying within the safe set $\mathcal{X}$ with the reachability property of reaching the target set $\mathcal{X}_r$ and thus can formalize many important engineering problems such as autonomous spacecraft rendezvous [9], its verification has turned out to be of fundamental importance in engineering. The problem of interest in this work is on reach-avoid verification, i.e., verifying that system **CS** satisfies the reach-avoid property in Definition 1. We attempt to solve this problem within the framework of optimization based methods. Generally, such methods are sound but incomplete.

In the following computations, all of constraints for reach-avoid verification are addressed via encoding them into semi-definite programs. The formulated semi-definite programs can be found in `https://arxiv.org/pdf/2208.08105`. In addtion, all of semi-definite programs are formulated using Matlab package YALMIP [18] and solved by employing the academic version of the semi-definite programming solver MOSEK [19].

B. Existing Methods

For the convenience of comparisons, in this subsection we recall the sets of quantified constraints in existing literature for verifying the reach-avoid property in Definition 1. The first one is from [22], while the other one is from [15].

Proposition 1: [22] Suppose that there exists a continuously differentiable function $v(\boldsymbol{x}) : \overline{\mathcal{X}} \to \mathbb{R}$ satisfying

$$v(\boldsymbol{x}) \leq 0, \forall \boldsymbol{x} \in \mathcal{X}_0 \tag{2}$$
$$v(\boldsymbol{x}) > 0, \forall \boldsymbol{x} \in \overline{\partial\mathcal{X} \setminus \partial\mathcal{X}_r}, \tag{3}$$
$$\bigtriangledown_{\boldsymbol{x}} v(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) < 0, \forall \boldsymbol{x} \in \overline{\mathcal{X} \setminus \mathcal{X}_r}, \tag{4}$$

Then the reach-avoid property in Definition 1 holds.

One of drawbacks of constraints (2)-(4) in reach-avoid verification is the strong requirement that the Lie derivative of $v(\boldsymbol{x})$ should be strictly decreasing along the trajectories of system **CS** over the set $\overline{\mathcal{X}\setminus\mathcal{X}_r}$. A straightforward consequence is that these constraints cannot deal with the case with an equilibrium being inside $\overline{\mathcal{X}\setminus\mathcal{X}_r}$, since $\boldsymbol{f}(\boldsymbol{x}_0)=0$ implies $\triangledown_{\boldsymbol{x}} v(\boldsymbol{x})\cdot\boldsymbol{f}(\boldsymbol{x})\mid_{\boldsymbol{x}=\boldsymbol{x}_0}=0$.

Besides, if the reach-avoid property in Definition 1 holds, the initial set $\mathcal{X}_0$ must be a subset of the reach-avoid set $\mathcal{RA}$, which is the set of all initial states guaranteeing the satisfaction of the reach-avoid property, i.e.

$$\mathcal{RA}=\left\{\boldsymbol{x}_0\in\mathbb{R}^n \;\middle|\; \begin{array}{l}\exists t\in\mathbb{R}_{\geq 0}.\boldsymbol{\phi}_{\boldsymbol{x}_0}(t)\in\mathcal{X}_r\\ \bigwedge\forall\tau\in[0,t].\boldsymbol{\phi}_{\boldsymbol{x}_0}(\tau)\in\mathcal{X}\end{array}\right\}.$$

Therefore, the method for computing under-approximations of the reach-avoid set $\mathcal{RA}$ can be used for reach-avoid verification. By adding the condition $v(\boldsymbol{x})<0,\forall\boldsymbol{x}\in\mathcal{X}_0$ into constraint [15, (18)], which is originally developed for under-approximating the reach-avoid set $\mathcal{RA}$, we can obtain a set of quantified constraints as shown in Proposition 2 for reach-avoid verification.

Proposition 2: Suppose that there exists a continuously differentiable function $v(\boldsymbol{x}):\overline{\mathcal{X}}\rightarrow\mathbb{R}$ and a continuous function $w(\boldsymbol{x}):\overline{\mathcal{X}}\rightarrow\mathbb{R}$ satisfying

$$v(\boldsymbol{x})<0,\forall\boldsymbol{x}\in\mathcal{X}_0 \tag{5}$$

$$\triangledown_{\boldsymbol{x}} v(\boldsymbol{x})\cdot\boldsymbol{f}(\boldsymbol{x})\leq\beta v(\boldsymbol{x}),\forall\boldsymbol{x}\in\overline{\mathcal{X}\setminus\mathcal{X}_r}, \tag{6}$$

$$w(\boldsymbol{x})\geq 0,\forall\boldsymbol{x}\in\overline{\mathcal{X}\setminus\mathcal{X}_r}, \tag{7}$$

$$w(\boldsymbol{x})\geq v(\boldsymbol{x})+1,\forall\boldsymbol{x}\in\overline{\mathcal{X}\setminus\mathcal{X}_r}, \tag{8}$$

$$v(\boldsymbol{x})\geq 0,\forall\boldsymbol{x}\in\partial\mathcal{X}, \tag{9}$$

where $\beta>0$ is a user-defined value, then the reach-avoid property in Definition 1 holds.

III. Reach-avoid Verification

This section presents our optimization based methods for reach-avoid verification. Based on a discount value function, which is defined based on trajectories of a switched system and introduced in Subsection III-A, two sets of quantified constraints are first respectively derived when the discount factor is respectively equal to zero and larger than zero. Once a solution (termed exponential or asymptotic guidance-barrier function) to any of these two sets of constraints is found, the reach-avoid property in Definition 1 is verified successfully. Furthermore, inspired by the set of constraints obtained when the discount factor is zero, a set of more expressive constraints is constructed for reach-avoid verification.

A. Induced Switched Systems

This subsection introduces a switched system, which is built upon system **CS**. This switched system is constructed by requiring the state of system **CS** to stay still when the complement of the safe set $\mathcal{X}$ is reached. For the sake of brevity, only trajectories of the induced switched system, also called **CSPS**, are introduced.

Definition 2: Given system **CSPS** with an initial state $\boldsymbol{x}_0\in\overline{\mathcal{X}}$, if there is a function $\boldsymbol{x}(\cdot):\mathbb{R}_{\geq 0}\rightarrow\mathbb{R}^n$ with $\boldsymbol{x}(0)=\boldsymbol{x}_0$ such that it satisfies the dynamics defined by $\dot{\boldsymbol{x}}=\widehat{\boldsymbol{f}}(\boldsymbol{x})$, where

$$\widehat{\boldsymbol{f}}(\boldsymbol{x}):=1_{\mathcal{X}}(\boldsymbol{x})\cdot\boldsymbol{f}(\boldsymbol{x}), \tag{10}$$

with $1_{\mathcal{X}}(\cdot):\overline{\mathcal{X}}\rightarrow\{0,1\}$ representing the indicator function of the set $\mathcal{X}$, i.e.,

$$1_{\mathcal{X}}(\boldsymbol{x}):=\begin{cases}1, & \text{if } \boldsymbol{x}\in\mathcal{X},\\ 0, & \text{if } \boldsymbol{x}\notin\mathcal{X},\end{cases}$$

then the trajectory $\widehat{\boldsymbol{\phi}}_{\boldsymbol{x}_0}(\cdot):\mathbb{R}_{\geq 0}\rightarrow\mathbb{R}^n$, induced by $\boldsymbol{x}_0$, of system **CSPS** is defined as follows:

$$\widehat{\boldsymbol{\phi}}_{\boldsymbol{x}_0}(t):=\boldsymbol{x}(t),\forall t\in\mathbb{R}_{\geq 0}.$$

It is observed that the set $\overline{\mathcal{X}}$ is an invariant set for system **CSPS**. Also, if $\boldsymbol{x}_0\in\mathcal{X}$ and there exists $T\geq\mathbb{R}_{\geq 0}$ such that $\widehat{\boldsymbol{\phi}}_{\boldsymbol{x}_0}(t)\in\mathcal{X}$ for $t\in[0,T]$, we have $\widehat{\boldsymbol{\phi}}_{\boldsymbol{x}_0}(t)=\boldsymbol{\phi}_{\boldsymbol{x}_0}(t),\forall t\in[0,T]$. Also, trajectories of system **CSPS** evolving in the viable set $\overline{\mathcal{X}}$ can be classified into three disjoint groups:

1) trajectories entering the set $\mathcal{X}_r$ in finite time. It is worth remarking here that these trajectories will not leave the safe set $\mathcal{X}$ before reaching the target set $\mathcal{X}_r$. Since $\widehat{\boldsymbol{\phi}}_{\boldsymbol{x}_0}(t)=\boldsymbol{\phi}_{\boldsymbol{x}_0}(t),\forall t\in[0,T]$, where $\boldsymbol{x}_0\in\mathcal{X}$ and $T\in\mathbb{R}_{\geq 0}$ is a time instant such that $\widehat{\boldsymbol{\phi}}_{\boldsymbol{x}_0}(t)\in\mathcal{X}$ for $t\in[0,T]$, we conclude that the set of initial states deriving these trajectories equals the reach-avoid set $\mathcal{RA}$;
2) trajectories entering the set $\partial\mathcal{X}$ in finite time, but never entering the target set $\mathcal{X}_r$;
3) trajectories staying in the set $\mathcal{X}\setminus\mathcal{X}_r$ for all time.

B. Discount Value Functions

The discount value function aforementioned is introduced in this subsection. With a non-negative discount factor β, the discount value function $V(\boldsymbol{x}):\overline{\mathcal{X}}\rightarrow\mathbb{R}$ with a non-negative discount factor β is defined in the following form:

$$V(\boldsymbol{x}):=\sup_{t\in\mathbb{R}_{\geq 0}} e^{-\beta t}1_{\mathcal{X}_r}(\widehat{\boldsymbol{\phi}}_{\boldsymbol{x}}(t)), \tag{11}$$

where $1_{\mathcal{X}_r}(\cdot):\mathbb{R}^n\rightarrow\{0,1\}$ is the indicator function of the target set $\mathcal{X}_r$. Obviously, $V(\boldsymbol{x})$ is bounded over the set $\overline{\mathcal{X}}$. Moreover, if $\beta=0$,

$$V(\boldsymbol{x})=\begin{cases}1, & \text{if } \boldsymbol{x}\in\mathcal{RA},\\ 0, & \text{otherwise.}\end{cases} \tag{12}$$

If $\beta>0$,

$$V(\boldsymbol{x})=\begin{cases}0, & \text{if } \boldsymbol{x}\in\overline{\mathcal{X}}\setminus\mathcal{RA},\\ e^{-\beta\tau_{\boldsymbol{x}}}, & \text{if } \boldsymbol{x}\in\mathcal{RA},\end{cases} \tag{13}$$

where $\tau_{\boldsymbol{x}}=\inf\{t\in\mathbb{R}_{\geq 0}\mid\widehat{\boldsymbol{\phi}}_{\boldsymbol{x}}(t)\in\mathcal{X}_r\}$ is the first hitting time of the target set $\mathcal{X}_r$.

From (12) and (13), we have that following proposition.

Proposition 3: When $\beta=0$, the one level set of $V(\cdot):\overline{\mathcal{X}}\rightarrow\mathbb{R}$ in (11) equals the reach-avoid set $\mathcal{RA}$, i.e.,$\{\boldsymbol{x}\in\overline{\mathcal{X}}\mid V(\boldsymbol{x})=1\}=\mathcal{RA}$. When $\beta>0$, the strict zero super level set of $V(\cdot):\overline{\mathcal{X}}\rightarrow\mathbb{R}$ in (11) equals $\mathcal{RA}$, i.e., $\{\boldsymbol{x}\in\overline{\mathcal{X}}\mid V(\boldsymbol{x})>0\}=\mathcal{RA}$.

In the following we respectively obtain two sets of quantified constraints for reach-avoid verification based on the discount value function $V(\boldsymbol{x})$ in (11) with $\beta>0$ and $\beta=0$. Through thorough analysis on these two sets of constraints, we further obtain a set of more expressive constraints for reach-avoid verification.

C. Exponential Guidance-barrier Functions

In this subsection we introduce the construction of quantified constraints based on the discount value function $V(\boldsymbol{x})$ in (11) with $\beta>0$, such that the reach-avoid verification problem is transformed into a problem of determining the existence of an exponential guidance-barrier function.

The set of constraints is derived from a system of equations admitting the value function $V(\boldsymbol{x})$ as solutions, which is formulated in Theorem 1.

Theorem 1: Given system **CSPS**, if there exists a continuously differential function $v(\boldsymbol{x}):\overline{\mathcal{X}}\rightarrow[0,1]$ such that

$$\triangledown_{\boldsymbol{x}} v(\boldsymbol{x})\cdot\boldsymbol{f}(\boldsymbol{x})=\beta v(\boldsymbol{x}),\forall\boldsymbol{x}\in\overline{\mathcal{X}\setminus\mathcal{X}_r}, \tag{14}$$

$$v(\boldsymbol{x})=0,\forall\boldsymbol{x}\in\partial\mathcal{X}, \tag{15}$$

$$v(\boldsymbol{x})=1,\forall\boldsymbol{x}\in\mathcal{X}_r, \tag{16}$$

then $v(\boldsymbol{x}) = V(\boldsymbol{x})$ for $\boldsymbol{x} \in \overline{\mathcal{X}}$ and thus $\{\boldsymbol{x} \in \overline{\mathcal{X}} \mid v(\boldsymbol{x}) > 0\} = \mathcal{RA}$, where $V(\cdot) : \overline{\mathcal{X}} \to \mathbb{R}$ is the value function with $\beta > 0$ in (11).

Proof: We first consider that $\boldsymbol{x} \in \mathcal{RA}$. If $\boldsymbol{x} \in \mathcal{X}_r$, $v(\boldsymbol{x}) = 1 = e^0$ according to constraint (16). Thus, we just consider $\boldsymbol{x} \in \mathcal{RA} \setminus \mathcal{X}_r$.

From (14), we have that $v(\boldsymbol{x}) = e^{-\beta\tau} v(\widehat{\boldsymbol{\phi}}_{\boldsymbol{x}}(\tau)), \forall \tau \in [0, \tau_{\boldsymbol{x}}]$, where $\tau_{\boldsymbol{x}}$ is the first hitting time of the target set $\mathcal{X}_r$. Due to constraint (16), we further have that $v(\boldsymbol{x}) = e^{-\beta\tau_{\boldsymbol{x}}}$.

Next, we consider that $\boldsymbol{x} \in \overline{\mathcal{X}} \setminus \mathcal{RA}$, but its resulting trajectory $\widehat{\boldsymbol{\phi}}_{\boldsymbol{x}}(\tau)$ will stay within the set $\mathcal{X} \setminus \mathcal{X}_r$ for all time. Due to constraint (14), we have that $v(\boldsymbol{x}) = e^{-\beta\tau} v(\widehat{\boldsymbol{\phi}}_{\boldsymbol{x}}(\tau))$ for $\tau \in \mathbb{R}_{\geq 0}$. Since $v(\cdot) : \overline{\mathcal{X}} \to \mathbb{R}$ is bounded, we have $v(\boldsymbol{x}) = 0$.

Finally, we consider that $\boldsymbol{x} \in \overline{\mathcal{X}} \setminus \mathcal{RA}$, but its resulting trajectory $\widehat{\boldsymbol{\phi}}_{\boldsymbol{x}}(\tau)$ will touch the set $\partial\mathcal{X}$ in finite time and never enter the target set $\mathcal{X}_r$. Let $\tau'_{\boldsymbol{x}} = \inf\{t \in \mathbb{R}_{\geq 0} \mid \widehat{\boldsymbol{\phi}}_{\boldsymbol{x}}(t) \in \partial\mathcal{X}\}$ be the first hitting time of the set $\partial\mathcal{X}$.

If $\boldsymbol{x} \in \partial\mathcal{X}$, $v(\boldsymbol{x}) = 0$ holds from constraint (15). Otherwise, $\tau'_{\boldsymbol{x}} > 0$. Further, from (14) we have $v(\boldsymbol{x}) = e^{-\beta\tau} v(\widehat{\boldsymbol{\phi}}_{\boldsymbol{x}}(\tau)), \forall \tau \in [0, \tau'_{\boldsymbol{x}})$. Regarding constraint (15), which implies $v(\widehat{\boldsymbol{\phi}}_{\boldsymbol{x}}(\tau'_{\boldsymbol{x}})) = 0$, we have $v(\boldsymbol{x}) = 0$.

Thus, $v(\boldsymbol{x}) = V(\boldsymbol{x})$ for $\boldsymbol{x} \in \overline{\mathcal{X}}$ according to (13) and $\mathcal{RA} = \{\boldsymbol{x} \in \overline{\mathcal{X}} \mid v(\boldsymbol{x}) > 0\}$. ■

Via relaxing the set of equations (14)-(16), we can obtain a set of inequalities for computing what we call exponential guidance-barrier function, whose existence ensures the satisfaction of the reach-avoid property in Definition 1. This set of inequalities is formulated in Proposition 4, in which inequalities (18) and (19) are obtained directly by relaxing equations (14) and (15), respectively.

Proposition 4: If there exists a continuously differentiable function $v(\boldsymbol{x}) : \overline{\mathcal{X}} \to \mathbb{R}$ such that

$$v(\boldsymbol{x}) > 0, \forall \boldsymbol{x} \in \mathcal{X}_0, \tag{17}$$

$$\triangledown_{\boldsymbol{x}} v(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) \geq \beta v(\boldsymbol{x}), \forall \boldsymbol{x} \in \overline{\mathcal{X} \setminus \mathcal{X}_r}, \tag{18}$$

$$v(\boldsymbol{x}) \leq 0, \forall \boldsymbol{x} \in \partial\mathcal{X}, \tag{19}$$

where $\beta > 0$ is a user-defined value, then the reach-avoid property in the sense of Definition 1 holds.

Comparing constraints (17)-(19) and (5)-(9), we find that the former is just a simplified version of the latter, i.e., if a function $v(\boldsymbol{x})$ satisfies constraints (5)-(9), $-v(\boldsymbol{x})$ satisfies (17)-(19). The former can be obtained by removing constraints (7) and (8) and reversing the inequality sign in the rest of constraints in the latter. Therefore, we did not give a proof of Proposition 4 here. Also, due to its more concise form, constraint (17)-(19) will be used for discussions and comparisons instead of constraint (5)-(9) in the sequel.

If an exponential guidance-barrier function $v(\boldsymbol{x})$ satisfying constraint (17)-(19) is found, the reach-avoid property in Definition 1 holds, which is justified via Proposition 4. Moreover, it is observed that the set $\mathcal{R} = \{\boldsymbol{x} \in \overline{\mathcal{X}} \mid v(\boldsymbol{x}) > 0\}$ is an under-approximation of the reach-avoid set, i.e., $\mathcal{R} \subseteq \mathcal{RA}$. Also, due to constraint (18), we conclude that the set $\mathcal{R}$ is an invariant for system **CS** until it enters the target set $\mathcal{X}_r$, i.e., the boundary $\partial\mathcal{R} = \{\boldsymbol{x} \in \overline{\mathcal{X}} \mid v(\boldsymbol{x}) = 0\}$ is a barrier, preventing system **CS** from leaving the set $\mathcal{R}$ and escorting system **CS** to the target set $\mathcal{X}_r$ safely; furthermore, we observe that if $v(\boldsymbol{x})$ satisfies constraint (17)-(19), it must satisfy

$$\triangledown_{\boldsymbol{x}} v(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) \geq \beta v(\boldsymbol{x}), \forall \boldsymbol{x} \in \overline{\mathcal{R} \setminus \mathcal{X}_r}, \tag{20}$$

which implies $v(\boldsymbol{\phi}_{\boldsymbol{x}_0}(t)) \geq e^{\beta t} v(\boldsymbol{x}_0), \forall t \in [0, \tau_{\boldsymbol{x}_0}]$, where $\boldsymbol{x}_0 \in \mathcal{R} \setminus \mathcal{X}_r$ and $\tau_{\boldsymbol{x}_0} = \inf\{t \in \mathbb{R}_{\geq 0} \mid \boldsymbol{\phi}_{\boldsymbol{x}_0}(t) \in \mathcal{X}_r\}$ is the first hitting time of the target set $\mathcal{X}_r$. Consequently, this constraint indicates that trajectories starting from $\overline{\mathcal{R} \setminus \mathcal{X}_r}$ will approach the target set $\mathcal{X}_r$ at an exponential rate of β. This is why we term a solution to the set of constraints (17)-(19) exponential guidance-barrier function. The above analysis also uncovers a necessary condition such that $v(\boldsymbol{x})$ is a solution to constraint (17)-(19), which is $\mathcal{R} \cap \mathcal{X}_r \neq \emptyset$.

The condition of entering the target set at an exponential rate is strict for many cases, limiting the application of constraint (17)-(19) to reach-avoid verification in practice. On the other hand, since the initial set $\mathcal{X}_0$ should be a subset of the set $\mathcal{R}$, the less conservative the set $\mathcal{R}$ is, the more likely the reach-avoid property is able to be verified. It is concluded from constraint (20) that the smaller β is, the less conservative the set $\mathcal{R}$ is inclined to be. This is illustrated in the following example.

Example 1: Consider an academic example,

$$\begin{cases} \dot{x} = -0.5x - 0.5y + 0.5xy \\ \dot{y} = -0.5y + 0.5 \end{cases} \tag{21}$$

with $\mathcal{X} = \{(x, y)^\top \mid x^2 + y^2 - 1 < 0\}$, $\mathcal{X}_r = \{(x, y)^\top \mid (x + 0.2)^2 + (y - 0.7)^2 - 0.02 < 0\}$ and $\mathcal{X}_0 = \{(x, y)^\top \mid 0.1 - x < 0, x - 0.5 < 0, -0.8 - y < 0, y + 0.5 < 0\}$.

In this example we use $\beta = 0.1$ and $\beta = 1$ to illustrate the effect of β on reach-avoid verification via solving constraint (17)-(19). The degree of all polynomials in the resulting semi-definite program is taken the same and is taken in order of $\{2, 4, 6, 8, 10, \ldots, 20\}$. When the reach-avoid property is verified successfully, the computations will terminate. The degree is respectively 14 for $\beta = 0.1$ and 20 for $\beta = 1$ when termination. Both of the computed sets $\mathcal{R}$ are showcased in Fig. 1, which almost collide with each other.

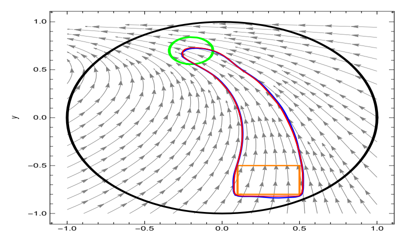

Fig. 1. Green, orange and black curve- $\partial\mathcal{X}_r$, $\partial\mathcal{X}_0$ and $\partial\mathcal{X}$; blue and red curve - $\partial\mathcal{R}$, which are computed via respectively solving constraints (17)-(19) with $\beta = 0.1$, and (17)-(19) with $\beta = 1$.

It is worth emphasizing here that although the discount factor β can arbitrarily approach zero from above, it cannot be zero in constraint (17)-(19), since a function $v(\boldsymbol{x})$ satisfying this constraint with $\beta = 0$ cannot rule out the existence of trajectories, which start from $\mathcal{X}_0$ and stay inside $\mathcal{X} \setminus \mathcal{X}_r$ for ever. Consequently, we do not recommend the use of too small β in practical numerical computations in order to avoid numerical issue (i.e., preventing the term $\beta V(\boldsymbol{x})$ in the right hand of constraint (18) from becoming zero numerically due to floating point errors).

Although a set of constraints for reach-avoid verification when $\beta = 0$ cannot be obtained directly from (17)-(19), we will obtain one from the discount function (11) with $\beta = 0$ in the sequel, expecting to remedy the shortcoming of the strict requirement of exponentially entering the set $\mathcal{X}_r$ when $\beta > 0$.

D. Asymptotic Guidance-barrier Functions

In this subsection we elucidate the construction of constraints fo reach-avoid verification based on the discount value function $V(\boldsymbol{x})$ in (11) with $\beta = 0$. In this case, the reach-avoid verification problem is transformed into a problem of determining the existence of an asymptotic guidance-barrier function.

The set of constraints is constructed via relaxing a system of equations admitting the value function $V(\boldsymbol{x})$ in (11) with $\beta = 0$ as solutions. These equations are presented in Theorem 2.

Theorem 2: Given system **CSPS**, if there exist continuously differential functions $v(\boldsymbol{x}) : \overline{\mathcal{X}} \to \mathbb{R}$ and $w(\boldsymbol{x}) : \overline{\mathcal{X}} \to \mathbb{R}$ satisfying

$$\triangledown_{\boldsymbol{x}} v(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) = 0, \forall \boldsymbol{x} \in \overline{\mathcal{X} \setminus \mathcal{X}_r}, \quad (22)$$
$$v(\boldsymbol{x}) = \triangledown_{\boldsymbol{x}} w(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}), \forall \boldsymbol{x} \in \overline{\mathcal{X} \setminus \mathcal{X}_r}, \quad (23)$$
$$v(\boldsymbol{x}) = 0, \forall \boldsymbol{x} \in \partial \mathcal{X}, \quad (24)$$
$$v(\boldsymbol{x}) = 1, \forall \boldsymbol{x} \in \mathcal{X}_r, \quad (25)$$

then $v(\boldsymbol{x}) = V(\boldsymbol{x})$ for $\boldsymbol{x} \in \overline{\mathcal{X}}$ and thus $\{\boldsymbol{x} \in \overline{\mathcal{X}} \mid v(\boldsymbol{x}) = 1\} = \mathcal{RA}$, where $V(\cdot) : \overline{\mathcal{X}} \to \mathbb{R}$ is the value function with $\beta = 0$ in (11).

Proof: From (22), we have that

$$v(\boldsymbol{x}) = v(\widehat{\boldsymbol{\phi}}_{\boldsymbol{x}}(\tau)), \forall \tau \in [0, \tau_{\boldsymbol{x}}], \quad (26)$$

where $\tau_{\boldsymbol{x}} \in \mathbb{R}_{\geq 0}$ is the time instant such that $\widehat{\boldsymbol{\phi}}_{\boldsymbol{x}}(\tau) \in \overline{\mathcal{X} \setminus \mathcal{X}_r}, \forall \tau \in [0, \tau_{\boldsymbol{x}}]$.

For $\boldsymbol{x} \in \mathcal{RA}$, we obtain $v(\boldsymbol{x}) = 1$ due to (25) and (26).

In the following we consider $\boldsymbol{x} \in \overline{\mathcal{X}} \setminus \mathcal{RA}$.

We first consider $\boldsymbol{x} \in \overline{\mathcal{X}} \setminus \mathcal{RA}$, but its trajectory $\widehat{\boldsymbol{\phi}}_{\boldsymbol{x}}(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ stays within the set $\mathcal{X} \setminus \mathcal{X}_r$. From (23), we have that

$$v(\widehat{\boldsymbol{\phi}}_{\boldsymbol{x}}(\tau)) = \triangledown_{\boldsymbol{y}} w(\boldsymbol{y}) \cdot \boldsymbol{f}(\boldsymbol{y}) \mid_{\boldsymbol{y}=\widehat{\boldsymbol{\phi}}_{\boldsymbol{x}}(\tau)}$$

for $\tau \in \mathbb{R}_{\geq 0}$. Thus, we have that

$$\int_0^t v(\widehat{\boldsymbol{\phi}}_{\boldsymbol{x}}(\tau)) d\tau = \int_0^t \triangledown_{\boldsymbol{y}} w(\boldsymbol{y}) \cdot \boldsymbol{f}(\boldsymbol{y}) \mid_{\boldsymbol{y}=\widehat{\boldsymbol{\phi}}_{\boldsymbol{x}}(\tau)} d\tau$$

for $t \in \mathbb{R}_{\geq 0}$ and further $v(\boldsymbol{x}) = \frac{w(\widehat{\boldsymbol{\phi}}_{\boldsymbol{x}}(t)) - w(\boldsymbol{x})}{t}$ for $t \in \mathbb{R}_{\geq 0}$. Since $w(\boldsymbol{x})$ is continuously differentiable function over $\overline{\mathcal{X}}$, it is bounded over $\boldsymbol{x} \in \overline{\mathcal{X}}$. Consequently, $v(\boldsymbol{x}) = 0$.

Next, we consider $\boldsymbol{x} \in \overline{\mathcal{X}} \setminus \mathcal{RA}$, but its trajectory $\widehat{\boldsymbol{\phi}}_{\boldsymbol{x}}(\tau)$ will touch $\partial \mathcal{X}$ in finite time and never enters the target set $\mathcal{X}_r$. For such $\boldsymbol{x}$, we can obtain that $v(\boldsymbol{x}) = 0$ due to constraints (26) and (24).

Thus, according to (12), $v(\boldsymbol{x}) = V(\boldsymbol{x})$ for $\boldsymbol{x} \in \overline{\mathcal{X}}$. Further, from Lemma 3, $\{\boldsymbol{x} \in \overline{\mathcal{X}} \mid v(\boldsymbol{x}) = 1\} = \mathcal{RA}$ holds. ∎

Based on the system of equations (22)-(25), we have a set of inequalities as shown in Proposition 5 for computing an asymptotic guidance-barrier function $v(\boldsymbol{x})$ to ensure the satisfaction of reach-avoid properties in the sense of Definition 1. In Proposition 5, inequalities (28), (29) and (30) are obtained directly by relaxing equations (22), (23) and (24), respectively.

Proposition 5: If there exist a continuously differentiable function $v(\boldsymbol{x}) : \overline{\mathcal{X}} \to \mathbb{R}$ and a continuously differentiable function $w(\boldsymbol{x}) : \overline{\mathcal{X}} \to \mathbb{R}$ satisfying

$$v(\boldsymbol{x}) > 0, \forall \boldsymbol{x} \in \mathcal{X}_0, \quad (27)$$
$$\triangledown_{\boldsymbol{x}} v(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) \geq 0, \forall \boldsymbol{x} \in \overline{\mathcal{X} \setminus \mathcal{X}_r}, \quad (28)$$
$$v(\boldsymbol{x}) - \triangledown_{\boldsymbol{x}} w(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) \leq 0, \forall \boldsymbol{x} \in \overline{\mathcal{X} \setminus \mathcal{X}_r}, \quad (29)$$
$$v(\boldsymbol{x}) \leq 0, \forall \boldsymbol{x} \in \partial \mathcal{X}, \quad (30)$$

then the reach-avoid property in the sense of Definition 1 holds.

Proof: Let $\mathcal{R} = \{\boldsymbol{x} \in \overline{\mathcal{X}} \mid v(\boldsymbol{x}) > 0\}$. We will show that $\mathcal{R} \subseteq \mathcal{RA}$. If it holds, we can obtain the conclusion since $\mathcal{X}_0 \subseteq \mathcal{RA}$, which is obtained from constraint (27).

Let $\boldsymbol{x}_0 \in \mathcal{R}$. Obviously, $\boldsymbol{x}_0 \in \mathcal{X}$ due to constraint (30). If $\boldsymbol{x}_0 \in \mathcal{X}_r$, $\boldsymbol{x}_0 \in \mathcal{RA}$ holds obviously. Therefore, in the following we assume $\boldsymbol{x}_0 \in \mathcal{R} \setminus \mathcal{X}_r$. We will prove that there exists $t \in \mathbb{R}_{\geq 0}$ satisfying $\boldsymbol{\phi}_{\boldsymbol{x}_0}(t) \in \mathcal{X}_r \bigwedge \forall \tau \in [0, t]. \boldsymbol{\phi}_{\boldsymbol{x}_0}(\tau) \in \mathcal{R}$.

Assume that there exists $t \in \mathbb{R}_{\geq 0}$ such that

$$\boldsymbol{\phi}_{\boldsymbol{x}_0}(t) \in \partial \mathcal{R} \bigwedge \forall \tau \in [0, t). \boldsymbol{\phi}_{\boldsymbol{x}_0}(\tau) \in \mathcal{R} \setminus \mathcal{X}_r.$$

From (28), we have that $v(\boldsymbol{\phi}_{\boldsymbol{x}_0}(\tau)) \geq v(\boldsymbol{x}_0) > 0$ for $\tau \in [0, t]$, contradicting that $v(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \partial \mathcal{R}$.

Therefore, either

$$\boldsymbol{\phi}_{\boldsymbol{x}_0}(\tau) \in \mathcal{R} \setminus \mathcal{X}_r, \forall \tau \in \mathbb{R}_{\geq 0} \quad (31)$$

or

$$\exists t \in \mathbb{R}_{\geq 0}. \boldsymbol{\phi}_{\boldsymbol{x}_0}(t) \in \mathcal{X}_r \bigwedge \forall \tau \in [0, t]. \boldsymbol{\phi}_{\boldsymbol{x}_0}(\tau) \in \mathcal{R} \quad (32)$$

holds.

Assume that (31) holds. From (28), we have that $v(\boldsymbol{\phi}_{\boldsymbol{x}_0}(\tau)) \geq v(\boldsymbol{x}_0) > 0, \forall \tau \in \mathbb{R}_{\geq 0}$.

From (29), we have that for $\tau \in \mathbb{R}_{\geq 0}$, $v(\boldsymbol{\phi}_{\boldsymbol{x}_0}(\tau)) \leq \triangledown_{\boldsymbol{x}} w(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) \mid_{\boldsymbol{x}=\boldsymbol{\phi}_{\boldsymbol{x}_0}(\tau)}$. Thus, $w(\boldsymbol{\phi}_{\boldsymbol{x}_0}(t)) - w(\boldsymbol{x}_0) \geq t v(\boldsymbol{x}_0), \forall t \in \mathbb{R}_{\geq 0}$, implying that $\lim_{t \to +\infty} w(\boldsymbol{\phi}_{\boldsymbol{x}_0}(t)) = +\infty$. Since $w(\cdot) : \overline{\mathcal{X}} \to \mathbb{R}$ is continuously differentiable and $\overline{\mathcal{X}}$ is compact, $w(\cdot) : \overline{\mathcal{X}} \to \mathbb{R}$ is bounded. This is a contradiction. Thus, (32) holds, implying that $\boldsymbol{x} \in \mathcal{RA}$ and thus $\mathcal{R} \subseteq \mathcal{RA}$. ∎

According to Proposition 5, the reach-avoid property in the sense of Definition 1 can be verified via searching for a feasible solution $v(\boldsymbol{x})$ to the set of constraints (27)-(30). Also, from the proof of Proposition 5, we find that the set $\mathcal{R} = \{\boldsymbol{x} \in \overline{\mathcal{X}} \mid v(\boldsymbol{x}) > 0\}$ will also be an invariant for system **CS** until it enters the target set $\mathcal{X}_r$, and $\mathcal{R} \cap \mathcal{X}_r \neq \emptyset$. Inspired by the notion of asymptotic stability in stability analysis, if system **CS** starting from the set $\mathcal{R} = \{\boldsymbol{x} \in \overline{\mathcal{X}} \mid v(\boldsymbol{x}) > 0\}$ with $v(\boldsymbol{x})$ satisfying constraint (17)-(19), enters the target set $\mathcal{X}_r$ at an exponential rate, then system **CS** starting from the set $\mathcal{R} = \{\boldsymbol{x} \in \overline{\mathcal{X}} \mid v(\boldsymbol{x}) > 0\}$ with $v(\boldsymbol{x})$ satisfying constraint (27)-(30), would enter the target set $\mathcal{X}_r$ in an asymptotic sense. This is the reason that we term a function $v(\boldsymbol{x})$ satisfying constraint (27)-(30) asymptotic guidance-barrier function.

Comparing constraints (17)-(19) and (27)-(30), it is easy to find that an auxillary function $w(\boldsymbol{x})$ is introduced when $\beta = 0$ in constraint (18). This constraint having $w(\boldsymbol{x})$, i.e., (29), excludes trajectories starting from $\mathcal{X}_0$ and staying inside $\mathcal{X} \setminus \mathcal{X}_r$ for ever. On the other hand, constraint (28) keeps trajectories, starting from $\mathcal{R}$, inside it before entering the target set $\mathcal{X}_r$. Also, we observe that constraint (17)-(19) can be derived from the set of constraints (27)-(30) via taking $w(\boldsymbol{x}) = \frac{1}{\beta} v(\boldsymbol{x})$. In such circumstances, constraint (28) is redundant and should be removed. Further, if there exists a function $v(\boldsymbol{x})$ satisfying constraint (20), there must exist a function $w(\boldsymbol{x})$, which can take $\frac{1}{\beta} v(\boldsymbol{x})$ for instance, such that

$$\begin{aligned} &\triangledown_{\boldsymbol{x}} v(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) \geq 0, \forall \boldsymbol{x} \in \overline{\mathcal{R} \setminus \mathcal{X}_r}, \\ &v(\boldsymbol{x}) - \triangledown_{\boldsymbol{x}} w(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) \leq 0, \forall \boldsymbol{x} \in \overline{\mathcal{R} \setminus \mathcal{X}_r} \end{aligned} \quad (33)$$

holds. Thus, constraint (33) is more expressive than (20) and consequently is more likely to produce less conservative set $\mathcal{R}$. We in the following continue to use the scenario in Example 1 to illustrate this.

Example 2: Consider the scenario in Example 1 again. We solve constraint (27)-(30) to verify the reach-avoid property. The reach-avoid property is verified when polynomials of degree 12 in the resulting semi-definite program are taken.

Remark 1: As done in verifying invariance of a set using barrier certificate methods [4], one simple application scenario, reflecting the advantage of constraint (33) over (20) further, is on verifying whether trajectories starting from a given open set $\widehat{\mathcal{R}}$, which may be designed a priori via the Monte-Carlo simulation method, will enter the target set $\mathcal{X}_r$ eventually while staying inside it before the first target hit, where $\widehat{\mathcal{R}} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \widehat{v}(\boldsymbol{x}) > 0\}$ with $\widehat{v}(\boldsymbol{x})$ being continuously differentiable and $\mathcal{X}_r \cap \widehat{\mathcal{R}} \neq \emptyset$. We are inclined to verifying whether there exists a continuously differentiable function $w(\boldsymbol{x})$ satisfying

$$\begin{aligned} &\triangledown_{\boldsymbol{x}} \widehat{v}(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) \geq 0, \forall \boldsymbol{x} \in \overline{\widehat{\mathcal{R}} \setminus \mathcal{X}_r}, \\ &\widehat{v}(\boldsymbol{x}) - \triangledown_{\boldsymbol{x}} w(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) \leq 0, \forall \boldsymbol{x} \in \overline{\widehat{\mathcal{R}} \setminus \mathcal{X}_r}, \end{aligned}$$

instead of verifying whether there exists $\beta > 0$ such that

$$\bigtriangledown_{\boldsymbol{x}} \widehat{v}(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) \geq \beta \widehat{v}(\boldsymbol{x}), \forall \boldsymbol{x} \in \overline{\widehat{\mathcal{R}} \setminus \mathcal{X}_r},$$

because the former is more expressive than the latter.

Although there are some benefits on the use of constraints (27)-(30) over (17)-(19) for reach-avoid verification, there is still a defect caused by constraint (28), possibly limiting the application of constraint (27)-(30) to some extent. Unlike constraint (18) in (17)-(19), constraint (28) not only requires the Lie derivative of function $v(\boldsymbol{x})$ along the flow of system **CS** to be non-negative over the set $\overline{\mathcal{R} \setminus \mathcal{X}_r}$, but also over $\overline{\mathcal{X} \setminus (\mathcal{X}_r \cup \mathcal{R})}$. One simple solution to remedy this defect is to combine constraints (27)-(30) and (17)-(19) together, and obtain a set of constraints which is more expressive. These constraints are presented in Proposition 6.

Proposition 6: If there exist continuously differentiable functions $v_1(\boldsymbol{x}), v_2(\boldsymbol{x}) : \overline{\mathcal{X}} \to \mathbb{R}$ and $w(\boldsymbol{x}) : \overline{\mathcal{X}} \to \mathbb{R}$ satisfying

$$v_1(\boldsymbol{x}) + v_2(\boldsymbol{x}) > 0, \forall \boldsymbol{x} \in \mathcal{X}_0, \tag{34}$$

$$\bigtriangledown_{\boldsymbol{x}} v_1(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) \geq 0, \forall \boldsymbol{x} \in \overline{\mathcal{X} \setminus \mathcal{X}_r}, \tag{35}$$

$$v_1(\boldsymbol{x}) - \bigtriangledown_{\boldsymbol{x}} w(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) \leq 0, \forall \boldsymbol{x} \in \overline{\mathcal{X} \setminus \mathcal{X}_r}, \tag{36}$$

$$v_1(\boldsymbol{x}) \leq 0, \forall \boldsymbol{x} \in \partial \mathcal{X}, \tag{37}$$

$$\bigtriangledown_{\boldsymbol{x}} v_2(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) \geq \beta v_2(\boldsymbol{x}), \forall \boldsymbol{x} \in \overline{\mathcal{X} \setminus \mathcal{X}_r}, \tag{38}$$

$$v_2(\boldsymbol{x}) \leq 0, \forall \boldsymbol{x} \in \partial \mathcal{X}, \tag{39}$$

where $\beta \in (0, +\infty)$, then the reach-avoid property in the sense of Definition 1 holds.

Proof: Let $\boldsymbol{x}_0 \in \mathcal{R} = \{\boldsymbol{x} \in \overline{\mathcal{X}} \mid v_1(\boldsymbol{x}) + v_2(\boldsymbol{x}) > 0\}$, we have that $\boldsymbol{x}_0 \in \{\boldsymbol{x} \in \overline{\mathcal{X}} \mid v_1(\boldsymbol{x}) > 0\}$ or $\boldsymbol{x}_0 \in \{\boldsymbol{x} \in \overline{\mathcal{X}} \mid v_2(\boldsymbol{x}) > 0\}$. Following Proposition 4 and 5, we have the conclusion. ■

Due to constraint (34), $v_1(\boldsymbol{x})$ may not be an asymptotic guidance-barrier function satisfying constraint (27)-(30). Similarly, $v_2(\boldsymbol{x})$ may not be an exponential guidance-barrier function satisfying constraint (17)-(19). Thus, constraint (34)-(39) is weaker than both of constraints (27)-(30) and (17)-(19). The set $\mathcal{R} = \{\boldsymbol{x} \in \overline{\mathcal{X}} \mid v_1(\boldsymbol{x}) + v_2(\boldsymbol{x}) > 0\}$ is a mix of states entering the target set $\mathcal{X}_r$ at an exponential rate and ones entering the target set $\mathcal{X}_r$ at an asymptotic rate, thus we term $v_1(\boldsymbol{x}) + v_2(\boldsymbol{x})$ asymptotic guidance-barrier function. However, we cannot guarantee that the set $\mathcal{R}$ still satisfies $\mathcal{R} \cap \mathcal{X}_r \neq \emptyset$ and it is an invariant for system **CS** until it enters the target set $\mathcal{X}_r$. If an initial state $\boldsymbol{x}_0 \in \mathcal{X}_0$ is a state such that $v_2(\boldsymbol{x}) > 0$, then the trajectory starting from it will stay inside the set $\mathcal{R}$ until it enters the target set $\mathcal{X}_r$, since

$$\begin{aligned} \frac{d(v_1 + v_2)}{dt} &= \bigtriangledown_{\boldsymbol{x}} v_1(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) + \bigtriangledown_{\boldsymbol{x}} v_2(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) \\ &\geq \beta v_2(\boldsymbol{x}), \forall \boldsymbol{x} \in \overline{\mathcal{X} \setminus \mathcal{X}_r} \end{aligned}$$

holds; otherwise, we cannot have such a conclusion. Instead, we have that $\mathcal{R}_i = \{\boldsymbol{x} \in \overline{\mathcal{X}} \mid v_i(\boldsymbol{x}) > 0\}$ satisfies $\mathcal{R}_i \cap \mathcal{X}_r \neq \emptyset$ and is an invariant for system **CS** until it enters the target set $\mathcal{X}_r$, if $\mathcal{R}_i \neq \emptyset$, where $i \in \{1, 2\}$. Let's illustrate this via an example.

Example 3: Consider the scenario in Example 1 again, but solve constraint (34)-(39) with $\beta = 2$ for reach-avoid verification. The reach-avoid property is verified when polynomials of degree 12 are taken. The computed $\mathcal{R}$ is shown in Fig. 2. For this case, the set $\mathcal{R}_2 = \{\boldsymbol{x} \in \overline{\mathcal{X}} \mid v_2(\boldsymbol{x}) > 0\}$ is empty. It is observed from Fig. 2 that the set $\mathcal{R}$ does not intersect $\mathcal{X}_r$, and system **CS** leaves it before entering the target set $\mathcal{X}_r$. However, the set $\mathcal{R}_1$ is an invariant for system **CS** until it enters the target set $\mathcal{X}_r$ and $\mathcal{R}_1 \cap \mathcal{X}_r \neq \emptyset$.

The other more sophisticated solution of enhancing constraint (27)-(30) is to replace constraint (28) with

$$\bigtriangledown_{\boldsymbol{x}} v(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) \geq \alpha(\boldsymbol{x}), \forall \boldsymbol{x} \in \overline{\mathcal{X} \setminus \mathcal{X}_r},$$

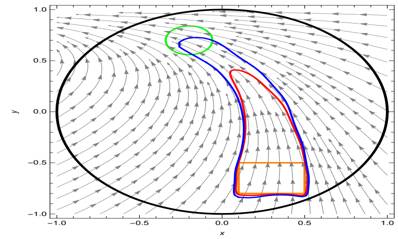

Fig. 2. Green, orange and black curve- $\partial \mathcal{X}_r$, $\partial \mathcal{X}_0$ and $\partial \mathcal{X}$; blue and red curve - $\partial \mathcal{R}_1$ and $\partial \mathcal{R}$ computed via solving constraint (34)-(39).

where $\alpha(\cdot) : \overline{\mathcal{X}} \to \mathbb{R}$ is a continuous function satisfying $\alpha(\boldsymbol{x}) \geq 0$ over $\{\boldsymbol{x} \in \overline{\mathcal{X} \setminus \mathcal{X}_r} \mid v(\boldsymbol{x}) \geq 0\}$. One instance for $\alpha(\boldsymbol{x})$ is $\beta(\boldsymbol{x}) v(\boldsymbol{x})$, where $\beta(\cdot) : \overline{\mathcal{X}} \to [0, +\infty)$. The new constraints are formulated in Proposition 7.

Proposition 7: If there exists a continuously differentiable function $v(\boldsymbol{x}) : \overline{\mathcal{X}} \to \mathbb{R}$, a continuous function $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ satisfying $\alpha(\boldsymbol{x}) \geq 0$ over $\{\boldsymbol{x} \in \overline{\mathcal{X} \setminus \mathcal{X}_r} \mid v(\boldsymbol{x}) \geq 0\}$, and a continuously differentiable function $w(\boldsymbol{x}) : \overline{\mathcal{X}} \to \mathbb{R}$ satisfying

$$v(\boldsymbol{x}) > 0, \forall \boldsymbol{x} \in \mathcal{X}_0, \tag{40}$$

$$\bigtriangledown_{\boldsymbol{x}} v(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) \geq \alpha(\boldsymbol{x}), \forall \boldsymbol{x} \in \overline{\mathcal{X} \setminus \mathcal{X}_r}, \tag{41}$$

$$v(\boldsymbol{x}) - \bigtriangledown_{\boldsymbol{x}} w(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) \leq 0, \forall \boldsymbol{x} \in \overline{\mathcal{X} \setminus \mathcal{X}_r}, \tag{42}$$

$$v(\boldsymbol{x}) \leq 0, \forall \boldsymbol{x} \in \partial \mathcal{X}, \tag{43}$$

then the reach-avoid property in the sense of Definition 1 holds.

Proof: Let $\mathcal{R} = \{\boldsymbol{x} \in \overline{\mathcal{X}} \mid v(\boldsymbol{x}) > 0\}$. From constraint (41), we have that if $\boldsymbol{\phi}_{\boldsymbol{x}_0}(t) \in \overline{\mathcal{R} \setminus \mathcal{X}_r}$, $\bigtriangledown_{\boldsymbol{x}} v(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) \mid_{\boldsymbol{x} = \boldsymbol{\phi}_{\boldsymbol{x}_0}(t)} \geq 0$. Then, following the arguments in the proof of Proposition 5, we have the conclusion. ■

Constraint (40)-(43) is less strict than constraint (27)-(30) in that the former only requires the Lie derivative of $v(\boldsymbol{x})$ along the flow of system **CS** to be non-negative over the set $\overline{\mathcal{R} \setminus \mathcal{X}_r}$ rather than $\overline{\mathcal{X} \setminus \mathcal{X}_r}$, due to constraint (41). Constraint (40)-(43) does not impose any restrictions on the Lie derivative of $v(\boldsymbol{x})$ along the flow of system **CS** over the set $\overline{\mathcal{X} \setminus \mathcal{R}}$. Moreover, it is more expressive since it degenerates to constraint (27)-(30) when $\alpha(\cdot) \equiv 0$, and it is more expressive than constraint (17)-(19), since the former degenerates to the latter when $\alpha(\boldsymbol{x}) = \beta v(\boldsymbol{x})$ and $w(\boldsymbol{x}) = \frac{1}{\beta} v(\boldsymbol{x})$. Besides, the set $\mathcal{R}$ obtained via solving constraint (40)-(43) will be an invariant for system **CS** until it enters the target set $\mathcal{X}_r$, and $\mathcal{R} \cap \mathcal{X}_r \neq \emptyset$.

Example 4: Consider the system in Example 1 with $\mathcal{X} = \{(x, y)^\top \mid x^2 + y^2 - 1 < 0\}$, $\mathcal{X}_r = \{(x, y)^\top \mid (x + 0.2)^2 + (y - 0.7)^2 - 0.02 < 0\}$ and $\mathcal{X}_0 = \{(x, y)^\top \mid 0.1 - x < 0, x - 0.5 < 0, -0.8 - y < 0, y + 0.4 < 0\}$.

In this example we use $\alpha(v(\boldsymbol{x})) = x^2 v(\boldsymbol{x})$ to illustrate the benefits of constraints (40)-(43) on reach-avoid verification. The degree of all polynomials in the resulting semi-definite program is taken the same and in order of $\{2, 4, 6, 8, 10, \ldots, 20\}$. When the reach-avoid property is verified successfully, the computations terminate. The degree is 12 for termination. In contrast, the degree is 14 when using constraints (17)-(19) with $\beta = 0.1$ and (27)-(30) to verify the reach-avoid property. Consequently, these experiments further support our analysis that constraint (40)-(43) is more expressive and can provide more chances for verifying the reach-avoid property successfully.

IV. EXAMPLES

We further demonstrate the theoretical development and performance of the proposed conditions on several examples, i.e., Examples 5-9. In the computations, the degree of unknown polynomials in the resulting semi-definite programs is taken the same and in order of $\{2,4,6,8,10,\dots,20\}$. When the reach-avoid property is verified successfully, the computations terminate. A return of 'Successfully solved (MOSEK)' from YALMIP will denote that a feasible solution is found, and the reach-avoid property is successfully verified.

Example 5: Consider the scenario in Example 4. As analyzed in Subsection III-C, the smaller the discount factor β is in constraint (17)-(19), the more likely the reach-avoid property is able to be verified. Thus, in this example, we supplement some experiments involving constraint (17)-(19) with $\beta < 0.1$ for more comprehensive and fair comparisons with the proposed methods in the present work. In these experiments, $\beta = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}$ are used. For all of these experiments, the computations terminate when the degree takes 14. All the results, including the ones in Example 4, further validate the benefits of constraint (40)-(43) over constraints (27)-(30) and (17)-(19) in terms of stronger expressiveness.

We also experimented using constraint (2)-(4), and the reach-avoid property is verified when the degree is 14.

Example 6 (Van der Pol Oscillator): Consider the reversed-time Van der Pol oscillator given by

$$\begin{cases} \dot{x} = -2y \\ \dot{y} = 0.8x + 10(x^2 - 0.21)y \end{cases}$$

with $\mathcal{X} = \{(x,y)^\top \mid x^2 + y^2 - 1 < 0\}$, $\mathcal{X}_0 = \{(x,y)^\top \mid -x < -0.6, x < 0.8, -y < 0, y < 0.2\}$ and $\mathcal{X}_r = \{(x,y)^\top \mid x^2 + y^2 - 0.01 < 0\}$.

The reach-avoid property is verified when the degree is 8, 8 and 12 for constraints (2)-(4), (27)-(30) and (17)-(19) with $\beta = 0.1$, respectively. We did not obtain any positive verification result from constraint (17)-(19) with $\beta = 1$. However, it can be improved by solving constraint (34)-(39) with $\beta = 1$ and the reach-avoid property is verified when the degree is 8. Further, if constraint (40)-(43) is used with $\alpha(\boldsymbol{x}) = x^2 v(\boldsymbol{x})$, the degree is 6. Some of the computed sets $\mathcal{R}$ are illustrated in Fig. 3.

Besides, we also experimented using constraints (17)-(19) and (34)-(39) with $\beta = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}$ for more comprehensive and fair comparisons. The computations terminate when the degree takes 8 for all of these experiments.

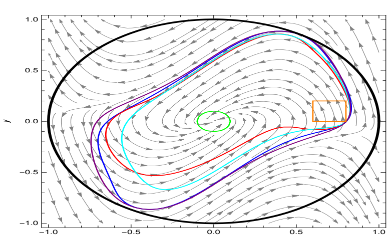

Fig. 3. Green, black and orange curve- $\partial\mathcal{X}_r$, $\partial\mathcal{X}$ and $\partial\mathcal{X}_0$; blue, red, cyan and purple curve - $\partial\mathcal{R}$, which is computed via solving constraints (27)-(30) when the degree is 8, (17)-(19) when the degree is 12 and $\beta = 0.1$, (34)-(39) when the degree is 8 and $\beta = 1$, and (40)-(43) when the degree is 6 and $\alpha(\boldsymbol{x}) = x^2 v(\boldsymbol{x})$.

Example 7: Consider the following system from [24],

$$\begin{cases} \dot{x} = -0.42x - 1.05y - 2.3x^2 - 0.56xy - x^3 \\ \dot{y} = 1.98x + xy \end{cases}$$

with $\mathcal{X} = \{\boldsymbol{x} \in \mathbb{R}^2 \mid (x)^2 + (y)^2 - 4 < 0\}$, $\mathcal{X}_0 = \{\boldsymbol{x} \in \mathbb{R}^2 \mid (x-1.2)^2 + (y-0.8)^2 - 0.1 < 0\}$ and $\mathcal{X}_r = \{\boldsymbol{x} \in \mathbb{R}^2 \mid (x+1.2)^2 + (y+0.5)^2 - 0.3 < 0\}$.

The reach-avoid property in the sense of Definition 1 is not verified using constraint (2)-(4). Actually, it cannot be verified via solving constraint (2)-(4), since there exists an equilibrium in the set $\mathcal{X} \setminus \mathcal{X}_r$.

The reach-avoid property is verified when the degree is 10 for constraints (27)-(30), and (17)-(19) with $\beta = 1$ and $\beta = 0.1$. If constraint(40)-(43) is used with $\alpha(\boldsymbol{x}) = (2-y)v(\boldsymbol{x})$, the degree is 8. Some of the computed sets $\mathcal{R}$ are illustrated in Fig. 4.

Like Example 6, we also experimented using constraints (17)-(19) and (34)-(39) with $\beta = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}$ for more comprehensive and fair comparisons. The computations terminate when the degree takes 10 for all of these experiments.

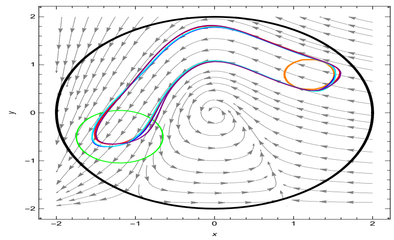

Fig. 4. Green, black and orange curve- $\partial\mathcal{X}_r$, $\partial\mathcal{X}$ and $\partial\mathcal{X}_0$; blue, red, cyan and purple curve - $\partial\mathcal{R}$, which is computed via solving constraints (27)-(30) when the degree is $\mathbf{10}$, (17)-(19) when the degree is $\mathbf{10}$ and $\beta = \mathbf{1}$, (17)-(19) when the degree is $\mathbf{10}$ and $\beta = \mathbf{0.1}$, and(40)-(43) when the degree is $\mathbf{8}$ and $\alpha(\boldsymbol{x}) = (\mathbf{2} - y)v(\boldsymbol{x})$.

Example 8: Consider the following system from [24],

$$\begin{cases} \dot{x} = y \\ \dot{y} = -(1-x^2)x - y \end{cases}$$

with $\mathcal{X} = \{\boldsymbol{x} \in \mathbb{R}^2 \mid \frac{x^2}{4} + \frac{y^2}{9} - 1 < 0\}$, $\mathcal{X}_0 = \{\boldsymbol{x} \in \mathbb{R}^2 \mid (x+1)^2 + (y-1.5)^2 - 0.25 < 0\}$ and $\mathcal{X}_r = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x^2 + y^2 - 0.01 < 0\}$.

The reach-avoid property is not verified using constraint (2)-(4), and is verified when the degree is 10 for constraint (27)-(30). We did not obtain any positive verification result from solving constraint (17)-(19) with $\beta = 1$ and $\beta = 0.1$. However, this negative situation can be improved by solving constraint (34)-(39) with $\beta = 1$ and $\beta = 0.1$, and the reach-avoid property is verified when the degree is 10. If constraint(40)-(43) is used with $\alpha(\boldsymbol{x}) = (x+y)^2 v(\boldsymbol{x})$, the degree is 6. Furthermore, if constraint (40)-(43) is used with $\alpha(\boldsymbol{x}) = x^4 v(\boldsymbol{x})$, the degree is 4.

Analogously, we also experimented using constraint (17)-(19) and (34)-(39) with $\beta = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}$ for more comprehensive and fair comparisons. The computations terminate when the degree is 10 for all of these experiments.

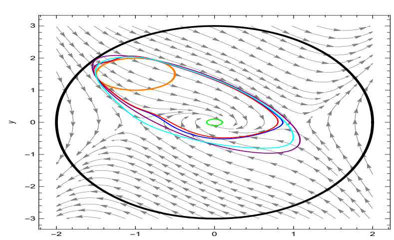

Fig. 5. Green, black and orange curve- $\partial\mathcal{X}_r$, $\partial\mathcal{X}$ and $\partial\mathcal{X}_0$; blue, red, cyan and purple curve - $\partial\mathcal{R}$, which is computed via solving constraints (27)-(30) when the degree is $\mathbf{10}$, (34)-(39) when the degree is $\mathbf{10}$ and $\beta = \mathbf{1}$, (40)-(43) when the degree is $\mathbf{6}$ and $\alpha(\boldsymbol{x}) = (x+y)^2 v(\boldsymbol{x})$, and(40)-(43) when the degree is $\mathbf{4}$ and $\alpha(\boldsymbol{x}) = x^4 v(\boldsymbol{x})$.

Example 9 (Dubin's Car): Consider the Dubin's car: $\dot{a} = v\cos(\theta)$, $\dot{b} = v\sin(\theta)$, $\dot{\theta} = \omega$, where $v = 1$ and $w = 2$. By the change of variables, $x = \theta$, $y = a\cos(\theta) + b\sin(\theta)$, $z = -2(a\sin(\theta) - b\cos(\theta)) + \theta y$ with $u_1 = \omega$ and $u_2 = v -$

$\omega(a\sin(\theta) - b\cos(\theta))$, it is transformed into polynomial dynamics:

$$\begin{cases} \dot{x} = u_1, \\ \dot{y} = u_2, \\ \dot{z} = yu_1 - xu_2. \end{cases} \tag{44}$$

with $u_1 = 2, u_2 = 1+z-xy$, $\mathcal{X} = \{\boldsymbol{x} \in \mathbb{R}^3 \mid x^2+y^2+z^2-4 < 0\}$, $\mathcal{X}_0 = \{\boldsymbol{x} \in \mathbb{R}^3 \mid (x+0.6)^2 + y^2 + (z+0.6)^2 - 0.02 < 0\}$ and $\mathcal{X}_r = \{\boldsymbol{x} \in \mathbb{R}^3 \mid x^2+y^2+z^2-4 < 0, (x-1.0)^2 - (y+0.5)^2 + (z+0.1)^2 - 0.1 < 0\}$.

The reach-avoid property is verified when degree is 8 for all of constraints (2)-(4), (27)-(30), (17)-(19) with $\beta \in \{1, 0.1, \ldots, 10^{-6}\}$, (34)-(39) with $\beta \in \{1, 0.1, \ldots, 10^{-6}\}$. However, if constraint(40)-(43) is used with $\alpha(\boldsymbol{x}) = (1-x)^2 v(\boldsymbol{x})$, the degree is 6.

Examples above, i.e., Example 5-9, indicate that when the discount factor is small, constraint (17)-(19) has the same performance with constraint (27)-(30), although it performs worse when the discount factor is large. On the other hand, constraint (34)-(39) indeed is able to improve the performance of constraint (17)-(19) when the discount factor is large, but it does not improve constraint (27)-(30) and its performance will be the same with constraint (17)-(19) when the discount factor is small. However, constraint (40)-(43) outperforms the former three, i.e., constraints (17)-(19), (27)-(30) and (34)-(39), and constraint (2)-(4). It is indeed more expressive and has more feasible solutions, providing more chances for verifying the reach-avoid property in the sense of Definition 1 successfully. Besides, from Example 8 we observe that the performance of constraint(40)-(43) is affected by the choice of the function $\alpha(\boldsymbol{x})$, and an appropriate choice will be more conducive to the reach-avoid verification. However, how to determine such a function in an optimal sense is still an open problem, which will be investigated in the future work. In practice, engineering experiences and insights may facilitate the choice.

In the present work we only demonstrate the performance of all of quantified constraints by relaxing them into semi-definite constraints and addressing them within the semi-definite programming framework, which could be solved efficiently via interior point methods in polynomial time. It is worth remarking here that besides semi-definite programs for implementing these constraints, other methods such as counterexample-guided inductive synthesis methods combining machine learning and SMT solving techniques (e.g., [1]) can also be used to solve these constraints. We did not show their performance in this present work and leave these investigations for ones of interest.

V. Conclusion and Future Work

In this paper we studied the reach-avoid verification problem of continuous-time systems within the framework of optimization based methods. At the beginning of our method, two sets of quantified inequalities were derived respectively based on a discount value function with the discount factor being larger than zero and equal to zero, such that the reach-avoid verification problem is transformed into a problem of searching for exponential/asymptotic guidance-barrier functions. The set of constraints associated with asymptotic guidance-barrier functions is completely novel and has certain benefits over the other one, which is a simplified version of the one in existing literature. Furthermore, we enhanced the new set of constraints such that it is more expressive than the aforementioned two sets of constraints, providing more chances to verify the satisfaction of reach-avoid properties successfully. When the datum involved are polynomials, i.e., the initial set, safe set and target set are semi-algebraic, and the system has polynomial dynamics, the problem of solving these sets of constraints can be efficiently addressed using convex optimization. Finally, several examples demonstrated the theoretical developments and benefits of the proposed constraints.

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