Invariant-Based Verification and Synthesis for Hybrid Systems

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Outline

1. Background

2. Talk1: Preliminaries
   - Polynomials and Polynomial Ideals
   - First-order Theory of Reals
   - Continuous Dynamical Systems
   - Hybrid Automata

3. Talk2: Computing Invariants for Hybrid Systems
   - Generating Continuous Invariants in Simple Case
   - Generating Continuous Invariants in General Case
   - Generating Semi-algebraic Global Invariants
   - Abstraction of Elementary Hybrid Systems by Variable Transformation
   - An Industrial Case Study: Soft Landing

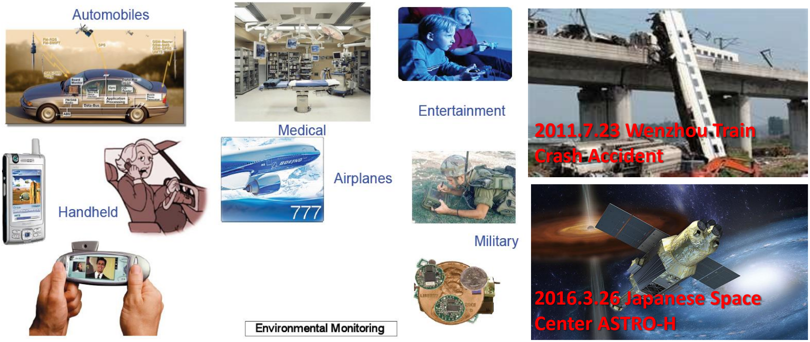
4. Talk3: Controller Synthesis
   - Controller Synthesis with Safety
   - Controller Synthesis with Safety and Optimality
   - An Industrial Case Study: The Oil Pump Control Problem

5. Conclusions

Hybrid systems

Hybrid systems exhibit combinations of discrete jumps and continuous evolution.

Examples



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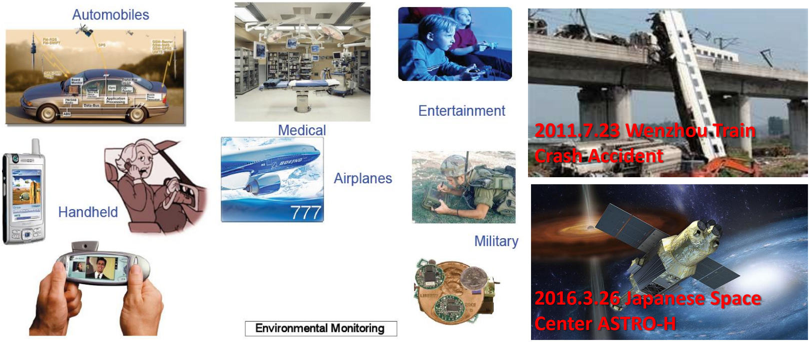

Issues of hybrid systems

- **Modelling:**
  - To establish a model for the system to be developed with precise mathematical semantics
  - Have to consider: **concurrency, deterministic vs nondeterministic, continuous vs discrete, communication, static vs dynamic (mobility, adaptability), qualitative vs quantitative (predicability), real-time**, ...
- **Simulation:**
  - To obtain a possible execution of the model upto a finite time horizon using numerical methods
  - Well accepted in industrial practice
- **Verification:**
  - Using mathematical approach to prove if a model satisfies the desired properties (specification)
  - Main methods include: **model-checking, theorem proving, abstract interpretation**

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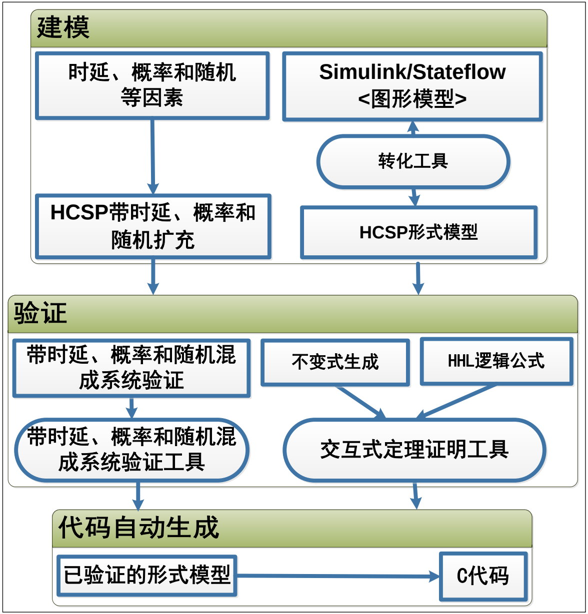

Schedule and References

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- Talk 1: Preliminaries
- Talk 2: Differential invariant generation
- Talk 3: Controller synthesis

References

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- L. Zou, M. Fraenzle, N. Zhan and P. Mosaad (2015): Automatic stability and safety verification for delay differential equations. Proc. of CAV 2015, LNCS.

Outline

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Polynomial ideal

Polynomial ideal

- A subset $I \subseteq \mathbb{K}[\mathbf{x}]$ is called an **ideal** if the following conditions are satisfied:
  1. $0 \in I$;
  2. If p, $g \in I$, then $p + g \in I$;
  3. If $p \in I$ and $h \in \mathbb{K}[\mathbf{x}]$, then $hp \in I$.
- Let $g_1, g_2, \ldots, g_s \in \mathbb{K}[\mathbf{x}]$, then $\langle g_1, g_2, \ldots, g_s \rangle \hat{=} \{\sum_{i=1}^{s} h_i g_i : h_1, h_2, \ldots, h_s \in \mathbb{K}[\mathbf{x}]\}$ is an ideal *generated* by $g_1, g_2, \ldots, g_s$.
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Hilbert Basis Theorem

Every ideal $I \subseteq \mathbb{K}[\mathbf{x}]$ has a finite basis, that is, $I = \langle g_1, g_2, \ldots, g_s \rangle$ for some $g_1, g_2, \ldots, g_s \in \mathbb{K}[\mathbf{x}]$.

Ascending Chain Theorem

For any ascending chain of ideals $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k \subseteq \cdots$ in $\mathbb{K}[\mathbf{x}]$, there exists an $N \in \mathbb{N}$ such that $I_k = I_N$ for any $k \geq N$.

Outline

First-order Theory $T(\mathbb{R})$ of Reals

Syntax

- The language of $T(\mathbb{R})$ consists of
  - variables: $x, y, z, \ldots, x_1, x_2, \ldots$, which are interpreted over $\mathbb{R}$;
  - relational symbols: $>, <, \geq, \leq, =, \neq$;
  - Boolean connectives: $\wedge, \vee, \neg, \rightarrow, \leftrightarrow, \ldots$; and
  - quantifiers: $\forall, \exists$.
- A *term* of $T(\mathbb{R})$ over a finite set of variables $\{x_1, x_2, \ldots, x_n\}$ is a polynomial $p \in \mathbb{R}[x_1, x_2, \ldots, x_n]$.
- An *atomic formula* of $T(\mathbb{R})$ is of the form $p \rhd 0$, where $\rhd$ is any relational symbol.
- A *quantifier-free formula* (QFF) of $T(\mathbb{R})$ is a Boolean combination of atomic formulas.
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- A theory $\mathcal{T}$ is said to have **quantifier elimination property**, if for any formula φ in $\mathcal{T}$, there exists a quantifier-free formula φ_{QF} which only contains *free* variables of φ such that $\varphi \Leftrightarrow \varphi_{QF}$.
- $T(\mathbb{R})$ admits **quantifier elimination**.
- The ***decidability*** of $T(\mathbb{R})$

Example

$$\begin{aligned} \exists x. ax^2 + bx + c = 0 \quad \Leftrightarrow \quad & a = b = c = 0 \vee \\ & (a = 0 \wedge b \neq 0) \vee \\ & (a \neq 0 \wedge b^2 - 4ac \geq 0) \end{aligned}$$

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Quantifier Elimination

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- A theory $\mathcal{T}$ is said to have **quantifier elimination property**, if for any formula φ in $\mathcal{T}$, there exists a quantifier-free formula φ_{QF} which only contains *free* variables of φ such that $\varphi \Leftrightarrow \varphi_{QF}$.
- $T(\mathbb{R})$ admits **quantifier elimination**.
- The ***decidability*** of $T(\mathbb{R})$

Example

$$\begin{aligned} \exists x. ax^2 + bx + c = 0 \quad \Leftrightarrow \quad & a = b = c = 0 \vee \\ & (a = 0 \wedge b \neq 0) \vee \\ & (a \neq 0 \wedge b^2 - 4ac \geq 0) \end{aligned}$$

Quantifier Elimination (Cont'd)

Semi-algrbraic Set

- A subset $A \subseteq \mathbb{R}^n$ is called a ***semi-algebraic set*** (SAS), if there exists a QFF $\phi \in T(\mathbb{R})$, such that $A = \{\mathbf{x} \in \mathbb{R}^n \mid \phi(\mathbf{x}) \text{ is true}\}$.
- SASs are closed under common set operations:
  - $\mathcal{A}(\phi_1) \cap \mathcal{A}(\phi_2) = \mathcal{A}(\varphi_1 \wedge \varphi_2)$;
  - $\mathcal{A}(\phi_1) \cup \mathcal{A}(\phi_2) = \mathcal{A}(\varphi_1 \vee \phi_2)$;
  - $\mathcal{A}(\phi_1)^c = \mathcal{A}(\neg\phi_1)$;
  - $\mathcal{A}(\phi_1) \setminus \mathcal{A}(\phi_2) = \mathcal{A}(\phi_1) \cap \mathcal{A}(\phi_2)^c = \mathcal{A}(\phi_1 \wedge \neg\phi_2)$.
- Any SAS can be represented by a QFF in the form of $\phi(\mathbf{x}) \widehat{=} \bigvee_{k=1}^{K} \bigwedge_{j=1}^{J_k} p_{kj}(\mathbf{x}) \triangleright 0$, where $p_{kj}(\mathbf{x}) \in \mathbb{Q}[\mathbf{x}]$ and $\triangleright \in \{\geq, >\}$.

Semi-algebraic Template

A ***semi-algebraic template*** with degree d is of the form

$$\phi(\mathbf{u}, \mathbf{x}) \widehat{=} \vee_{k=1}^{K} \wedge_{j=1}^{J_k} p_{kj}(\mathbf{u}_{kj}, \mathbf{x}) \triangleright 0.$$

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Survey of QE Algorithms

- **Tarski's algorithm** [Tarski 51]: the first one, but its complexity is nonelementary, impratical, simplified by Seidenberg [Seidenberg 54].
- **Collins' algorithm** [Collins 76]: based on **cylindrical algebraic decomposition (CAD)**, **double exponential** in the number of variables, improved by Hoon Hong [Hoon Hong 92] by combining with SAT engine **partial cylindrical algebraic decomposition (PCAD)**, implemented in many computer algebra tools, e.g., **QEBCAD, REDLOG, ...**
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Outline

Continuous Dynamical Systems

- A **continuous dynamical systems (CDS)** is of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a ***vector field***.

- If $\mathbf{f}$ in (1) satisfies **local Lipschitz condition**, then given $\mathbf{x}_0 \in \mathbb{R}^n$, there exists a unique solution $\mathbf{x}(\mathbf{x}_0; t) : (a, b) \to \mathbb{R}^n$ such that $\mathbf{x}(\mathbf{x}_0; 0) = \mathbf{x}_0$ and $\forall t \in (a, b). \frac{\mathrm{d}\mathbf{x}(\mathbf{x}_0;t)}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}(\mathbf{x}_0; t))$.
- If $\mathbf{f}$ in (1) satisfies **global Lipschitz condition**, then the **existence, uniqueness** and **completeness** of solutions to (1) can be guaranteed.
- The k-th **Lie derivatives** $L^k_{\mathbf{f}}\sigma : \mathbb{R}^n \to \mathbb{R}$ of σ along $\mathbf{f}$ is defined by:
  - $L^0_{\mathbf{f}}\sigma(\mathbf{x}) = \sigma(\mathbf{x})$,
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  where $\nabla\varrho(\mathbf{x}) \widehat{=} \left(\frac{\partial\varrho(\mathbf{x})}{\partial x_1}, \frac{\partial\varrho(\mathbf{x})}{\partial x_2}, \ldots, \frac{\partial\varrho(\mathbf{x})}{\partial x_n}\right)$ and $(\cdot, \cdot)$ is the *inner product* of two vectors.

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Outline

Hybrid Automaton

A **hybrid automaton** (HA) is a system $\mathcal{H} \mathrel{\widehat{=}} (Q, X, f, D, E, G, R, \Xi)$, where

- $Q = \{q_1, \ldots, q_m\}$ is a finite set of modes;
- $X = \{x_1, \ldots, x_n\}$ is a finite set of continuous state variables, with $\mathbf{x} = (x_1, \ldots, x_n)$ ranging over $\mathbb{R}^n$; $Q \times \mathbb{R}^n$ is the state space of $\mathcal{H}$;
- $f : Q \to (\mathbb{R}^n \to \mathbb{R}^n)$ assigns to each mode $q \in Q$ a vector field $\mathbf{f}_q$;
- $D : Q \to 2^{\mathbb{R}^n}$ assigns to each mode $q \in Q$ a domain $D_q \subseteq \mathbb{R}^n$;
- $E \subseteq Q \times Q$ is a set of discrete transitions;
- $G : E \to 2^{\mathbb{R}^n}$ assigns to each transition $e \in E$ a switching guard $G_e \subseteq \mathbb{R}^n$.
- R assigns to each transition $e \in E$ a reset function R_e: $\mathbb{R}^n \to \mathbb{R}^n$;
- Ξ assigns to each $q \in Q$ a set of initial states $\Xi_q \subseteq \mathbb{R}^n$.

Hybrid Trajectories Accepted by HA [Tomlin et al. 00]

Definition (Hybrid Time Set)

A hybrid time set is a sequence of time intervals $\tau = \{I_i\}_{i=0}^{N}$ (N can be ∞) s.t.

- $I_i = [\tau_i, \tau_i']$ with $\tau_i \leq \tau_i' = \tau_{i+1}$ for all $i < N$;
- if $N < \infty$, then $I_N = [\tau_N, \tau_N'\rangle$ is a right-closed or right-open nonempty interval (τ_N' may be ∞);
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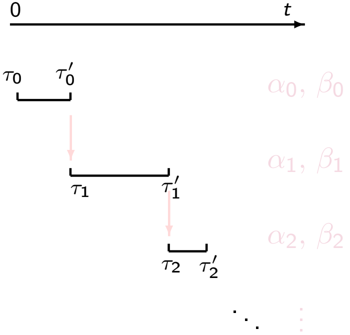

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A hybrid trajectory is a triple $\omega = (\tau, \alpha, \beta)$, where $\tau = \{I_i\}_{i=0}^{N}$ is a hybrid time set and $\alpha = \{\alpha_i : I_i \to Q\}$ and $\beta = \{\beta_i : I_i \to \mathbb{R}^n\}$ are two sequences of functions satisfying

1. Initial condition: $\alpha_0[0] = q_0$ and $\beta_0[0] = \mathbf{x}_0$;
2. Discrete transition: for all $i < \langle\tau\rangle$, $e = (\alpha_i(\tau_i'), \alpha_{i+1}(\tau_{i+1})) \in E$, $\beta_i(\tau_i') \in G_e$ and $\beta_{i+1}(\tau_{i+1}) = R_e(\beta_i(\tau_i'))$;
3. Continuous evolution: for all $i \leq \langle\tau\rangle$ with $\tau_i < \tau_i'$, if $q = \alpha_i(\tau_i)$, then
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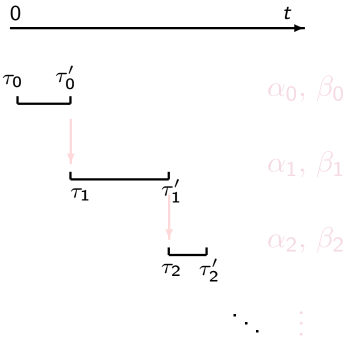

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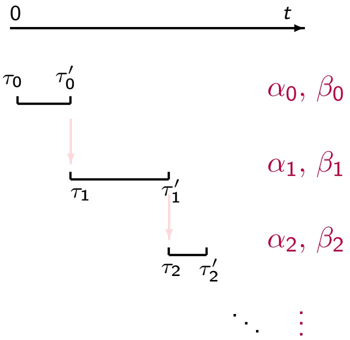

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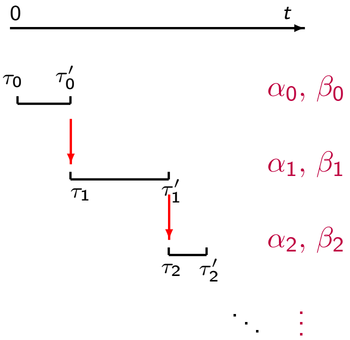
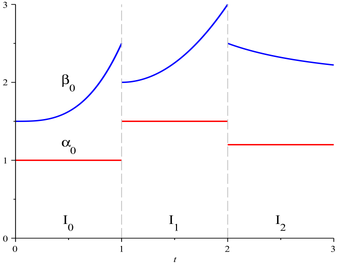


Hybrid Trajectories Accepted by HA (Cont'd) [Tomlin et al. 00]

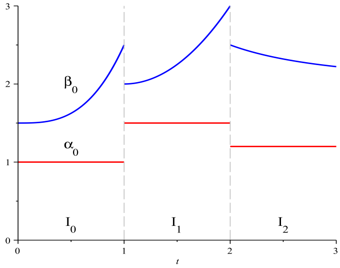


A hybrid trajectory (τ, α, β) is called *infinite* if

- $\langle \tau \rangle = N$ is ∞, or
- $\|\tau\| = \sum_{i=0}^{N} (\tau_i' - \tau_i)$ is ∞.

A hybrid automaton is called *non-blocking* if there is an infinite trajectory starting from any initial state (q_0, x_0), and *blocking* otherwise.

Hybrid Trajectories Accepted by HA (Cont'd) [Tomlin et al. 00]



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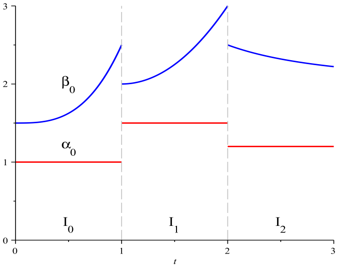
Hybrid Trajectories Accepted by HA (Cont'd) [Tomlin et al. 00]



A hybrid trajectory (τ, α, β) is called *infinite* if

- $\langle\tau\rangle = N$ is ∞, or
- $\|\tau\| = \sum_{i=0}^{N}(\tau_i' - \tau_i)$ is ∞.

A hybrid automaton is called *non-blocking* if there is an infinite trajectory starting from any initial state (q_0, x_0), and *blocking* otherwise.

Reachable Set of HA

Definition (Reachable Set)

Given an HA $\mathcal{H}$, the **reachable set** $\mathcal{R}_{\mathcal{H}}$ of $\mathcal{H}$ consists of those $(q, \mathbf{x})$ for which there exists a finite sequence

$$(q_0, \mathbf{x}_0), (q_1, \mathbf{x}_1), \ldots, (q_l, \mathbf{x}_l)$$

such that $(q_0, \mathbf{x}_0) \in \Xi_{\mathcal{H}}$, $(q_l, \mathbf{x}_l) = (q, \mathbf{x})$, and for any $0 \leq i \leq l - 1$, one of the following two conditions holds:

- **(Discrete Jump):** $e = (q_i, q_{i+1}) \in E$, $\mathbf{x}_i \in G_e$ and $\mathbf{x}_{i+1} = R_e(\mathbf{x}_i)$; or
- **(Continuous Evolution):** $q_i = q_{i+1}$, and there exists a $\delta \geq 0$ s.t. the solution $\mathbf{x}(\mathbf{x}_i; t)$ to $\dot{\mathbf{x}} = \mathbf{f}_{q_i}$ satisfies
  - $\mathbf{x}(\mathbf{x}_i; t) \in D_{q_i}$ for all $t \in [0, \delta]$; and
  - $\mathbf{x}(\mathbf{x}_i; \delta) = \mathbf{x}_{i+1}$.

Outline

Continuous vs Global Invariants

Note that

- **Hybrid systems** consists of a set of CDSs, a set of transitions between these CDSs, and a transition may be equipped with a guard and reset
- **Invariant** plays a key role in analysis, verification, synthesis of hybrid systems
- **Global invariant** keeps invariant during continuous and discrete evolutions
- **Continuous invariant** keeps invariant in a mode
- Interplay between global and continuous invariant
- Both can be reduced to constraint solving
- Continuous invariant (differential invariant) generation is more complicated

Global Invariant

Definition (Global Invariant)

An invariant of an HA $\mathcal{H}$ maps to each $q \in Q$ a subset $I_q \subseteq \mathbb{R}^n$, such that for all $(q, \mathbf{x}) \in \mathcal{R}_{\mathcal{H}}$ (the reachable set), we have $\mathbf{x} \in I_q$.

Definition (Inductive Invariant)

Given an HA $\mathcal{H}$, an **inductive invariant** maps to each $q \in Q$ a subset $I_q \subseteq \mathbb{R}^n$, such that the following conditions are satisfied:

1. $\Xi_q \subseteq I_q$ for all $q \in Q$;
2. for any $e = (q, q') \in E$, if $\mathbf{x} \in I_q \cap G_e$, then $\mathbf{x}' = R_e(\mathbf{x}) \in I_{q'}$;
3. for any $q \in Q$ and any $\mathbf{x}_0 \in I_q$, if there exists a $\delta \geq 0$ s.t. the solution $\mathbf{x}(\mathbf{x}_0; t)$ to $\dot{\mathbf{x}} = \mathbf{f}_q$ satisfies: (i) $\mathbf{x}(\mathbf{x}_0; \delta) = \mathbf{x}'$; and (ii) $\mathbf{x}(\mathbf{x}_0; t) \in D_q$ for all $t \in [0, \delta]$, then $\mathbf{x}' \in I_q$.

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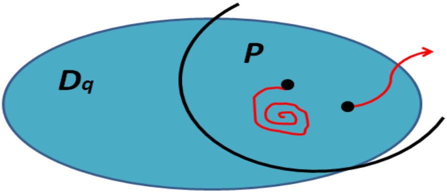
Continuous Invariant

Definition (Continuous Invariant see also [Platzer & Clarke 08] **)**

Given $(D_q, \mathbf{f}_q)$, we call $P \subseteq \mathbb{R}^n$ a **continuous invariant** of $(D_q, \mathbf{f}_q)$ if for all $\mathbf{x}_0 \in P$ and all $T \geq 0$,

$$(\forall t \in [0, T].\, \mathbf{x}(t) \in D_q) \implies (\forall t \in [0, T].\, \mathbf{x}(t) \in P) \ .$$



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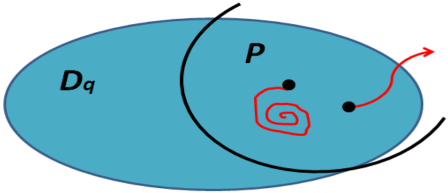


- A continuous invariant of a PDS is called a ***semi-algebraic invariant*** (SAI) if it is a semi-algebraic set.

Related Work

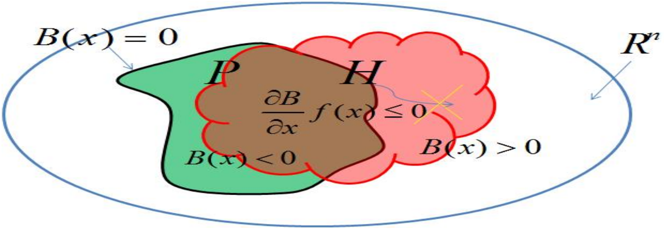
- **Barrier-certificate** [Prajna&Jadbadbaie 2004, Plazer&Clarke 2008]
  - Basic idea: Let $\mathcal{D} = \{\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})\}$ and $H = \{h(\mathbf{x}) \geq 0\}$. A function $B : \mathbb{R}^n \to \mathbb{R}$ is a barrier certificate if it is differentiable and satisfying

$$\forall \mathbf{x} \in H. \frac{\partial B}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) \leq 0.$$

or

$$\forall \mathbf{x} \in H(B(x) = 0 \Rightarrow \frac{\partial B}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) < 0).$$

Let $P := \{\mathbf{x} \mid B(\mathbf{x}) \leq 0\}$. Then P is an invariant of $(\mathcal{D}, H)$.



Related Work (Cont'd)

- **Boundary method** [Taly, Gulwani&Tiwari, VMCAI 2009]
  Let $\mathcal{D} = \{\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})\}$ and $H = \{h(\mathbf{x}) \geq 0\}$. If $P := \{\mathbf{x} \mid p(\mathbf{x}) \geq 0\}$ has the following property: For each $\mathbf{x}$ s.t. $p(\mathbf{x}) = 0$, there is a $\delta > 0$ s.t.

  $$\forall \mathbf{y} : (p(\mathbf{y}) = 0 \wedge \|\mathbf{y} - \mathbf{x}\| < \delta \Rightarrow L_{\mathbf{f}} p(\mathbf{y}) \geq 0 \wedge \frac{\partial p}{\partial \mathbf{y}} \neq \mathbf{0}),$$

  then P is an invariant of $(\mathcal{D}, H)$.
  - It imposes a strong assumption on the boundary.
- **Ideal fixed point method** [Sankaranarayanan, HSCC 2010]
  - Basic idea: If an ideal $\mathcal{I} \subseteq \mathcal{R}[\mathbf{x}]$ has the property:
    1. $(\forall p \in \mathcal{I}, \mathbf{x} \in H) p(\mathbf{x}) = 0$,
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Related Work (Cont'd)

Open Problem

- **Open problem** [Sankaranarayanan, HSCC 2010, Taly&Tiwari, FSTTCS 2009]: **Can we find a complete method to generate all semi-algebraic invariants of a polynomial dynamical system?**
- We addressed this problem and gave an affirmative answer in [Liu, Zhan&Zhao 2011].

Outline

Basic Idea

- Let $(D, \mathbf{f})$ be a PDS, $\mathbf{x}(t)$ is a trajectory of $(D, \mathbf{f})$ from $\mathbf{x}_0$, and $P \widehat{=} p(\mathbf{x}) \geq 0$. Then P be a differential invariant of $(D, \mathbf{f})$ iff

$$\forall \mathbf{x}_0 \in \partial P \cap D, \exists \epsilon > 0, \forall t \in [0, \epsilon]. p(\mathbf{x}(t)) \geq 0 \tag{2}$$

- $p(\mathbf{x}(t))$'s Taylor's expansion at $t = 0$

$$p(\mathbf{x}(t)) = L_{\mathbf{f}}^1 p(\mathbf{x}_0).t + L_{\mathbf{f}}^2 p(\mathbf{x}_0).\frac{t^2}{2!} + \cdots L_{\mathbf{f}}^i p(\mathbf{x}_0).\frac{t^i}{i!} + \cdots$$

- (2) holds iff
  1. either for all $i \geq 0, L_{\mathbf{f}}^i p(\mathbf{x}_0) = 0$
  2. or there is some $k > i \geq 0$, such that $L_{\mathbf{f}}^i p(\mathbf{x}_0) = 0$ and $L_{\mathbf{f}}^k p(\mathbf{x}_0) > 0$.
- The *pointwise rank* of p with respect to $\mathbf{f}$ as the function $\gamma_{p,\mathbf{f}} : \mathbb{R}^n \to \mathbb{N} \cup \{\infty\}$ defined by

$$\gamma_{p,\mathbf{f}}(\mathbf{x}) = \min\{k \in \mathbb{N} \mid L_{\mathbf{f}}^k p(\mathbf{x}) \neq 0\}$$

if such k exists, and $\gamma_{p,\mathbf{f}}(\mathbf{x}) = \infty$ otherwise.

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Example

Let $\mathbf{f} = (-x, y)$ and
$p(x, y) = x + y^2$. Then

$$
\begin{aligned}
L^0_{\mathbf{f}} p(x,y) &= x + y^2 \\
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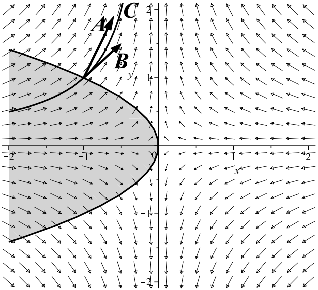

I: 1-order Lie Derivative and Gradient

Consider point $(-1, 1)$ (see the picture),

- The points on the parabola $p(x, y) = 0$ with zero energy, and the points in the white area have positive energy, i.e. $p(x, y) > 0$.
- B denotes the evolution direction of $\mathbf{f}$ at the point.
- A is the gradient $\nabla p|_{(-1,1)}$ of $p(x, y)$.
- $L^1_{\mathbf{f}} p|_{(-1,1)} = 3$ predicts that the trajectory starting at $(-1, 1)$ will enter the white area.

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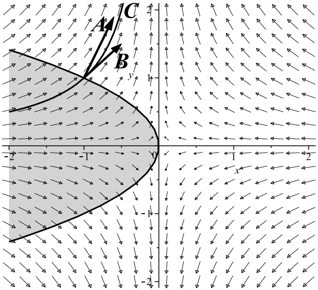

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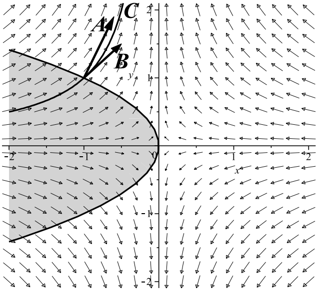

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Example

Let $\mathbf{f}(x, y) = (-2y, x^2)$ and $h(x, y) = x + y^2$. Then

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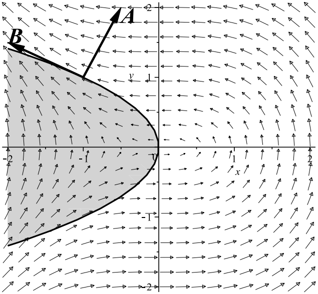

II: Demand for Higher Order Lie Derivative

Also consider point $(-1, 1)$ on $h(x, y) = 0$ (see the picture),

- the gradient of h is $(1, 2)$ (vector A);
- the evolution direction is $(-2, 1)$ (vector B);
- their inner product is zero, i.e., $L_{\mathbf{f}}^1 h(-1, 1) = 0$, thus it is impossible to predict the tendency of the trajectory starting from $(-1, 1)$ via the 1-order Lie derivative;
- By a simple computation, $L_{\mathbf{f}}^2 h(-1, 1) = 8$. Hence $\gamma_{h,\mathbf{f}}(-1, 1) = 2$.

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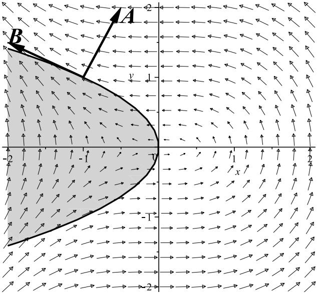

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Theoretical Results

Theorem (Rank Theorem)

Given a polynomial p and a PVF $\mathbf{f}$, there is a natural number $N_{p,\mathbf{f}}$ such that for any $\mathbf{x} \in \mathbb{R}^n$, if $\gamma_{p,\mathbf{f}}(\mathbf{x}) < \infty$, then $\gamma_{p,\mathbf{f}}(\mathbf{x}) \leq N_{p,\mathbf{f}}$.

Theorem (Parametric Rank Theorem)

Given a parametric polynomial $p(\mathbf{u}, \mathbf{x})$ and a PVF $\mathbf{f}$, there is an integer $N_{p,\mathbf{f}} \in \mathbb{N}$ such that $\gamma_{p_{\mathbf{u}_0},\mathbf{f}}(\mathbf{x}) < \infty$ implies $\gamma_{p_{\mathbf{u}_0},\mathbf{f}}(\mathbf{x}) \leq N_{p,\mathbf{f}}$ for all $\mathbf{x} \in \mathbb{R}^n$ and all $\mathbf{u}_0 \in \mathbb{R}^w$.

Theorem (Criterion Theorem)

Given a polynomial p, $p(\mathbf{x}) \geq 0$ is an SCI of the PCCDS $(h(\mathbf{x}) \geq 0, \mathbf{f})$ iff

$$\theta(h, p, \mathbf{f}, \mathbf{x}) \widehat{=} (p(\mathbf{x}) = 0 \wedge \pi(p, \mathbf{f}, \mathbf{x})) \rightarrow \pi(h, \mathbf{f}, \mathbf{x}), \tag{3}$$

holds for all $\mathbf{x} \in \mathbb{R}^n$, where

$$\pi^{(i)}(p, \mathbf{f}, \mathbf{x}) \;\widehat{=}\; \left(\bigwedge_{0 \leq j < i} L^j_{\mathbf{f}} p(\mathbf{x}) = 0 \right) \wedge L^i_{\mathbf{f}} p(\mathbf{x}) < 0,$$

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$$\begin{aligned} \pi^{(i)}(p, \mathbf{f}, \mathbf{x}) \quad &\widehat{=} \quad \left(\bigwedge_{0 \leq j < i} L^j_{\mathbf{f}} p(\mathbf{x}) = 0 \right) \wedge L^i_{\mathbf{f}} p(\mathbf{x}) < 0\,, \\ \pi(p, \mathbf{f}, \mathbf{x}) \quad &\widehat{=} \quad \bigvee_{0 \leq i \leq N_{p,\mathbf{f}}} \pi^{(i)}(p, \mathbf{f}, \mathbf{x})\,. \end{aligned}$$

Algorithm

I. **First, set a simple semi-algebraic template $P \,\widehat{=}\, p(\mathbf{u}, \mathbf{x}) \geq 0$ using a parametric polynomial $p(\mathbf{u}, \mathbf{x})$.**

II. Then apply QE to the formula $\forall \mathbf{x}.\theta(h, p, \mathbf{f}, \mathbf{x})$. In practice, QE may be applied to a formula $\forall \mathbf{x}.(\theta \wedge \phi)$, where ϕ is a formula imposing some additional constraint on the SCI P. If the output of QE is *false*, then there is no SCI in the form of the predefined P; otherwise, a constraint on $\mathbf{u}$, denoted by $R(\mathbf{u})$, will be returned.

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Running Example

Consider a PDS $(D = -x - y^2 \geq 0, \mathbf{f}(x, y) = (-2y, x^2))$.
Apply procedure (I-III), we have:

I Set a template $P \mathrel{\widehat{=}} p(\mathbf{u}, \mathbf{x}) \geq 0$ with $p(\mathbf{u}, \mathbf{x}) \mathrel{\widehat{=}} ay(x - y)$, where $\mathbf{u} \mathrel{\widehat{=}} (a)$. By a simple computation we get $N_{p,\mathbf{f}} = 2$.

II Compute the corresponding formula

$$\begin{aligned}\theta(h, p, \mathbf{f}, \mathbf{x}) \quad &\widehat{=} \quad p = 0 \wedge (\pi^{(0)}_{p,\mathbf{f},\mathbf{x}} \vee \pi^{(1)}_{p,\mathbf{f},\mathbf{x}} \vee \pi^{(2)}_{p,\mathbf{f},\mathbf{x}}) \longrightarrow \\ &\quad (\pi^{(0)}_{h,\mathbf{f},\mathbf{x}} \vee \pi^{(1)}_{h,\mathbf{f},\mathbf{x}} \vee \pi^{(2)}_{h,\mathbf{f},\mathbf{x}})\end{aligned}$$

where

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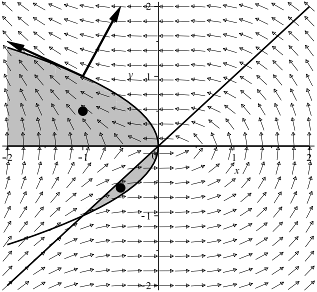
Running Example (Cont'd)

III In addition, we require the two points $\{(-1, 0.5), (-0.5, -0.6)\}$ to be contained in P. Then apply **QE** to the formula

$$\forall x \forall y. (\theta(h, p, \mathbf{f}, \mathbf{x}) \wedge 0.5a(-1 - 0.5) \geq 0 \wedge -0.6a(-0.5 + 0.6) \geq 0).$$

The result is $a \leq 0$.

IV Just pick $a = -1$, and then $-xy + y^2 \geq 0$ is an SCI of $(D, \mathbf{f})$. The grey part of Picture III is the intersection of the invariant P and domain D.



III: SCI in Simple Case

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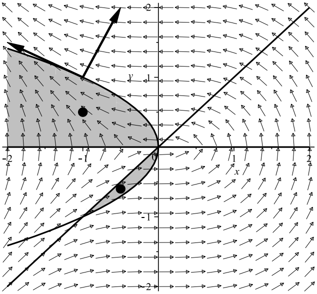

III: SCI in Simple Case

Outline

General Case

- **Problem:** Consider a PDS $(D, \mathbf{f})$ with

$$D = \bigvee_{i=1}^{I} \bigwedge_{j=1}^{J_i} p_{ij}(\mathbf{x}) \triangleright 0,$$

and $\mathbf{f} \in \mathbb{Q}^n[\mathbf{x}]$, where $\triangleright \in \{\geq, >\}$, to generate SAIs automatically with a general template

$$P = \bigvee_{k=1}^{K} \bigwedge_{l=1}^{L_k} p_{kl}(\mathbf{u}_{kl}, \mathbf{x}) \triangleright 0\,, \ \triangleright \in \{\geq, >\}$$

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- **Basic idea** The procedure is essentially same as in the simple case, but have to sophisticatedly handle the complex combinations due to the complicated boundaries.

Theorem (Main Result)

A semi-algebraic template $P(\mathbf{u}, \mathbf{x})$ *defined by*

$$\bigvee_{k=1}^{K} \left(\bigwedge_{j=1}^{j_k} p_{kj}(\mathbf{u}_{kj}, \mathbf{x}) \geq 0 \quad \wedge \bigwedge_{j=j_k+1}^{J_k} p_{kj}(\mathbf{u}_{kj}, \mathbf{x}) > 0 \right)$$

is a CI of the PCCDS $(D, \mathbf{f})$ *with*

$$D \mathrel{\widehat{=}} \bigvee_{m=1}^{M} \left(\bigwedge_{l=1}^{l_m} p_{ml}(\mathbf{x}) \geq 0 \quad \wedge \bigwedge_{l=l_m+1}^{L_m} p_{ml}(\mathbf{x}) > 0 \right),$$

iff $\mathbf{u}$ *satisfies*

$$\forall \mathbf{x}. \Big(\big(P \wedge D \wedge \Phi_D \rightarrow \Phi_P\big) \wedge \big(\neg P \wedge D \wedge \Phi_D^{Iv} \rightarrow \neg \Phi_P^{Iv}\big) \Big),$$

where

Theorem (Main Result (Cont'd))

$$\Phi_D \widehat{=} \bigvee_{m=1}^{M} \left(\bigwedge_{l=1}^{l_m} \psi_0^+(p_{ml}, \mathbf{f}) \wedge \bigwedge_{l=l_m+1}^{L_m} \psi^+(p_{ml}, \mathbf{f}) \right),$$

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- Let $\mathbf{f}(x, y) = (-2y, x^2)$ and $D \mathrel{\widehat{=}} \mathbb{R}^2$.
- Take a template: $P(\mathbf{u}, \mathbf{x}) \mathrel{\widehat{=}} x - a \geq 0 \vee y - b > 0$ with $\mathbf{u} = (a, b)$.
- So, P is an SCI of $(D, \mathbf{f})$ iff a, b satisfy

$$\forall x \forall y. (P \to \zeta) \wedge (\neg P \to \neg \xi),$$

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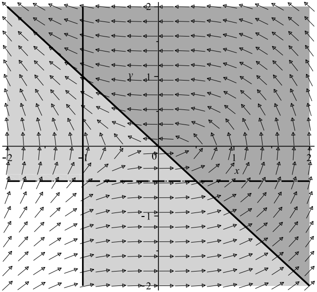
Running Example (Cont'd)

- In addition, we require the set $x + y \geq 0$ to be contained in P.
- By applying **QE**, we get $a + b \leq 0 \wedge b \leq 0$.
- Let $a = -1$ and $b = -0.5$, and we obtain an SCI $P \hat{=} x + 1 \geq 0 \vee y + 0.5 > 0$.



IV: SCI in General Case

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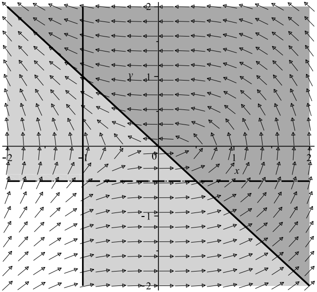

IV: SCI in General Case

Outline

Algorithm

I. Predefine a familiy of semi-algebraic templates $I_q(\mathbf{u}, \mathbf{x})$ with degree bound d for each $q \in Q$, as the SCI to be generated at mode q.

II. Translate conditions for the family of $I_q(\mathbf{u}, \mathbf{x})$ to be a GI of $\mathcal{H}$, i.e.
- $\Xi_q \subseteq I_q$ for all $q \in Q$;
- for any $e = (q, q') \in E$, if $\mathbf{x} \in I_q \cap G_e$, then $\mathbf{x}' = R_e(\mathbf{x}) \in I_{q'}$;
- for any $q \in Q$, I_q is a CI of $(D_q, \mathbf{f}_q)$

into a set of first-order real arithmetic formulas, i.e.

(1) $\forall \mathbf{x}. (\Xi_q \rightarrow I_q(\mathbf{u}, \mathbf{x}))$ for all $q \in Q$;

(2) $\forall \mathbf{x}, \mathbf{x}'. (I_q(\mathbf{u}, \mathbf{x}) \wedge G_e \wedge \mathbf{x}' = R_e(\mathbf{x}) \rightarrow I_{q'}(\mathbf{u}, \mathbf{x}'))$ for all $q \in Q$ and all $e = (q, q') \in E$;

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For safety property $\mathcal{S}$, there may be a fourth set of formulas:

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   - for any $q \in Q$, I_q is a CI of $(D_q, \mathbf{f}_q)$

   into a set of first-order real arithmetic formulas, i.e.

   (1) $\forall \mathbf{x}. \big(\Xi_q \rightarrow I_q(\mathbf{u}, \mathbf{x})\big)$ for all $q \in Q$;

   (2) $\forall \mathbf{x}, \mathbf{x}'. \big(I_q(\mathbf{u}, \mathbf{x}) \wedge G_e \wedge \mathbf{x}' = R_e(\mathbf{x}) \rightarrow I_{q'}(\mathbf{u}, \mathbf{x}')\big)$ for all $q \in Q$ and all $e = (q, q') \in E$;

   (3) $\forall \mathbf{x}. \big((I_q(\mathbf{u}, \mathbf{x}) \wedge D_q \wedge \Phi_{D_q} \rightarrow \Phi_{I_q}) \wedge (\neg I_q(\mathbf{u}, \mathbf{x}) \wedge D_q \wedge \Phi^{\mathrm{Iv}}_{D_q} \rightarrow \neg \Phi^{\mathrm{Iv}}_{I_q})\big)$, for each $q \in Q$.

   For safety property $\mathcal{S}$, there may be a fourth set of formulas:

   (4) $\forall \mathbf{x}. (I_q(\mathbf{u}, \mathbf{x}) \longrightarrow S_q)$ for all $q \in Q$.

III. Take the conjunction of all the formulas in Step 2 and apply QE to get a QFF $\phi(\mathbf{u})$. Then choose a specific $\mathbf{u}_0$ from $\phi(\mathbf{u})$ with a tool like **Z3**, and the set of instantiations $I_{q,\mathbf{u}_0}(\mathbf{x})$ form a GI of $\mathcal{H}$.

Algorithm

I. Predefine a familiy of semi-algebraic templates $I_q(\mathbf{u}, \mathbf{x})$ with degree bound d for each $q \in Q$, as the SCI to be generated at mode q.

II. Translate conditions for the family of $I_q(\mathbf{u}, \mathbf{x})$ to be a GI of $\mathcal{H}$, i.e.
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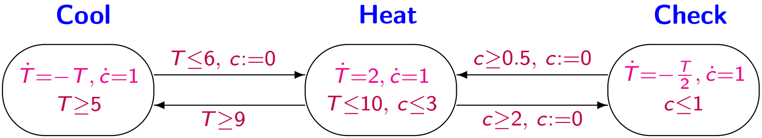
Running Example

- The Thermostat can be described by the HA in following figure.



- To verify that under the initial condition $\Xi_{\mathcal{H}} \mathrel{\widehat{=}} \{q_{\rm ht}\} \times X_0$ with $X_0 \mathrel{\widehat{=}} c = 0 \wedge 5 \leq T \leq 10$, $S \mathrel{\widehat{=}} T \geq 4.5$ is satisfied at all modes.

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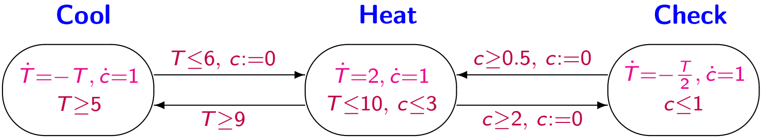

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Running Example (Cont'd)

- Firstly, predefine the following set of templates:
  - $I_{q_{\rm ht}} \widehat{=} T + a_1 c + a_0 \geq 0 \wedge c \geq 0$;
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$$10a_3 - 9 \leq 0 \wedge 2a_3 - 1 \geq 0 \wedge a_1 + 2 = 0 \wedge a_0 + 2a_1 + 9 = 0 \wedge a_2 - a_0 = 0\,.$$

- By choosing $a_0 = -5, a_1 = -2, a_2 = -5, a_3 = \frac{1}{2}$, obtain the following SGI
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Outline

Elementary Functions

$$\begin{aligned} f, g \quad ::= \quad & c \mid x \mid f+g \mid f-g \mid f \times g \mid \\ & \frac{f}{g} \mid f^a \mid e^f \mid \ln(f) \mid \sin(f) \mid \cos(f) \ , \end{aligned}$$

- $c \in \mathbb{R}$, $a \in \mathbb{Q}$, $x \in \{x, y, z, \ldots, x_1, x_2, \ldots, x_n\}$
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Univariate Basic Elementary Functions: $\dot{x} = f(x)$

- $f(x) = \frac{1}{x}$: let $v = \frac{1}{x}$, and thus $\dot{v} = -\frac{\dot{x}}{x^2}$, so (1) is transformed to

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Compositional and Multivariate Functions

- Compositional: if $f(x) = \ln(2+\sin x)$, then let

$$\begin{cases} v & = \sin x \\ u & = \cos x \\ w & = \ln(2+v) = \ln(2+\sin x) \\ z & = \frac{1}{2+v} = \frac{1}{2+\sin x} \end{cases},$$

so (1) is transformed to

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- Multivariate: analogous.

Abstracting EDSs

Abstracting EDS $\mathcal{C}_{\mathbf{x}} \widehat{=} (\Xi_{\mathbf{x}}, \mathbf{f}_{\mathbf{x}}, D_{\mathbf{x}})$ to PDS $C_{\mathbf{y}} \widehat{=} (\Xi_{\mathbf{y}}, \mathbf{f}_{\mathbf{y}}, D_{\mathbf{y}})$

(S1) Introduce new variables to replace all non-polynomial terms in $\mathbf{f}_{\mathbf{x}}$, $\Xi_{\mathbf{x}}$ and $D_{\mathbf{x}}$, and obtain a collection of replacement equations $\mathbf{v} = \Gamma(\mathbf{x})$.

(S2) Differentiate both sides of $\mathbf{v} = \Gamma(\mathbf{x})$ w.r.t. time, and then replace all newly appearing non-polynomial terms by introducing fresh variables.

(S3) Repeat (S2) until no more variables need to be introduced. For simplicity, still denote the final set of replacement equations by $\mathbf{v} = \Gamma(\mathbf{x})$.

(S4) Define the simulation map as $\Theta(\mathbf{x}) = (\mathbf{x}, \Gamma(\mathbf{x}))$. Then use $\mathbf{v} = \Gamma(\mathbf{x})$ to construct $\Xi_{\mathbf{y}}$ and $D_{\mathbf{y}}$ as illustrated by the following example.

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Abstracting EDSs: An Example

Consider the EDS $\mathcal{C}_{\mathbf{x}} \mathrel{\widehat{=}} (\Xi_{\mathbf{x}}, \mathbf{f}_{\mathbf{x}}, D_{\mathbf{x}})$, where

- $\Xi_{\mathbf{x}} \mathrel{\widehat{=}} (x+0.5)^2 + (y-0.5)^2 - 0.16 \leq 0$;
- $D_{\mathbf{x}} \mathrel{\widehat{=}} -2 \leq x \leq 2 \wedge -2 \leq y \leq 2$; and
- $\mathbf{f}_{\mathbf{x}}$ defines the ODE

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} e^{-x} + y - 1 \\ -\sin^2(x) \end{pmatrix} .$$

Abstracting EDSs: An Example

- (S1-S3): by the replacement relations $\mathbf{v} = \Gamma(\mathbf{x})$

$$(v_1, v_2, v_3) = (\sin x, e^{-x}, \cos x)$$

we get the transformed polynomial ODE (i.e. $\mathbf{f_y}$)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \end{pmatrix} = \begin{pmatrix} v_2 + y - 1 \\ -v_1^2 \\ v_3(v_2 + y - 1) \\ -v_2(v_2 + y - 1) \\ -v_1(v_2 + y - 1) \end{pmatrix},$$

Abstracting EDSs: An Example

- (S4): the simulation map is $\Theta(x, y) = (x, y, \sin x, e^{-x}, \cos x)$
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Polynomial Approximation via Taylor Model

- $D_{\mathbf{x}} \widehat{=} -2 \leq x \leq 2 \wedge -2 \leq y \leq 2$
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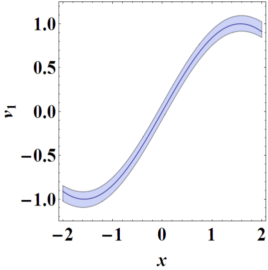

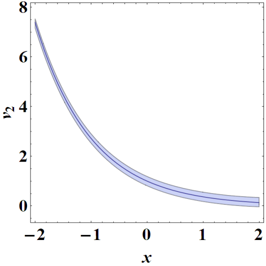

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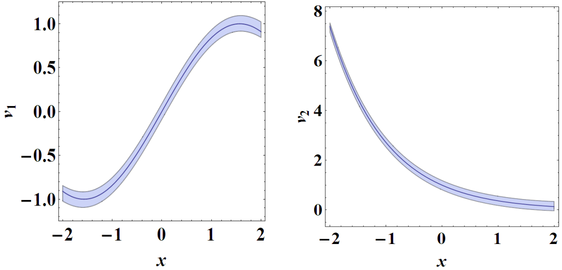

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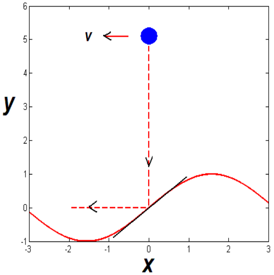
Abstracting EHSs

- abstracting EHS $\mathcal{H}_{\mathbf{x}} \,\widehat{=}\, (Q, X, f_{\mathbf{x}}, D_{\mathbf{x}}, E, G_{\mathbf{x}}, R_{\mathbf{x}}, \Xi_{\mathbf{x}})$ by PHS $\mathcal{H}_{\mathbf{y}} \,\widehat{=}\, (Q, Y, f_{\mathbf{y}}, D_{\mathbf{y}}, E, G_{\mathbf{y}}, R_{\mathbf{y}}, \Xi_{\mathbf{y}})$
- just extend the abstraction approach for EDSs to take into account guard constraints and reset functions
- treat each mode of a HA separately by constructing an individual abstraction map for each of them

Abstracting EHSs: An Example

- Bouncing ball on a sine-waved surface
- $Q = \{q\}$; $X = \{x, y, v_x, v_y\}$;
- $E = \{e\}$ with $e = (q, q)$;
- $D_{\mathbf{x},q} \mathrel{\widehat{=}} y \geq \sin x$; $G_{\mathbf{x},e} \mathrel{\widehat{=}} y = \sin x$;
- $\Xi_{\mathbf{x},q} \mathrel{\widehat{=}} y \geq 4.9 \wedge y \leq 5.1 \wedge x = 0 \wedge v_x = -1 \wedge v_y = 0$;
- $\mathbf{f}_{\mathbf{x},q} = \begin{cases} \dot{x} & = & v_x \\ \dot{y} & = & v_y \\ \dot{v}_x & = & 0 \\ \dot{v}_y & = & -9.8 \end{cases}$
- $R_{\mathbf{x},e}(x, y, v_x, v_y) \mathrel{\widehat{=}} \{(x, y, v_x', v_y')\}$ with

$$\begin{cases} v_x' & = & \frac{(\sin x)^2 \cdot v_x + 2(\cos x)\cdot v_y}{1+(\cos x)^2} \\ v_y' & = & \frac{2(\cos x)\cdot v_x - (\sin x)^2 \cdot v_y}{1+(\cos x)^2} \end{cases}.$$



Abstracting EHSs: An Example

- Bouncing ball on a sine-waved surface
- $Q = \{q\}$; $X = \{x, y, v_x, v_y\}$;
- $E = \{e\}$ with $e = (q, q)$;
- $D_{\mathbf{x},q} \mathrel{\widehat{=}} y \geq \sin x$; $G_{\mathbf{x},e} \mathrel{\widehat{=}} y = \sin x$;
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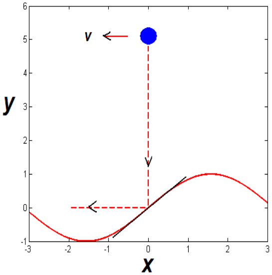
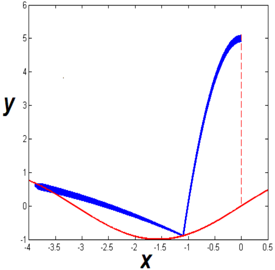


Abstracting EHSs: An Example

- replacement equations: $(u_1, u_2, u_3) = (\sin x, \cos x, \frac{1}{1+(\cos x)^2})$,
- flowpipe computation for the abstract system using Flow* (not applicable on the original system)



The Verification Problem

Consider the EDS $\mathcal{C}_{\mathbf{x}} \mathrel{\widehat{=}} (\Xi_{\mathbf{x}}, \mathbf{f}_{\mathbf{x}}, D_{\mathbf{x}})$, where

- $\Xi_{\mathbf{x}} \mathrel{\widehat{=}} (x+0.5)^2 + (y-0.5)^2 - 0.16 \leq 0$;
- $D_{\mathbf{x}} \mathrel{\widehat{=}} -2 \leq x \leq 2 \wedge -2 \leq y \leq 2$; and
- $\mathbf{f}_{\mathbf{x}}$ defines the ODE

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} e^{-x} + y - 1 \\ -\sin^2(x) \end{pmatrix} .$$

- verify the safety of $\mathcal{C}_{\mathbf{x}}$ w.r.t. an unsafe region $\bar{S}_{\mathbf{x}} \mathrel{\widehat{=}} (x-0.7)^2 + (y+0.7)^2 - 0.09 \leq 0$

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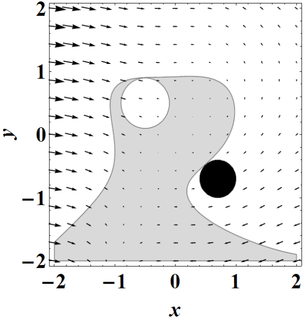
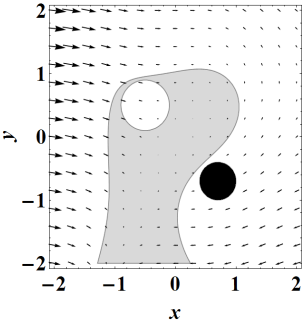
Generating Polynomial Invariants

- $(v_1, v_2, v_3) = (\sin x, e^{-x}, \cos x)$
- Assume a polynomial invariant template of degree 5 without fresh variables

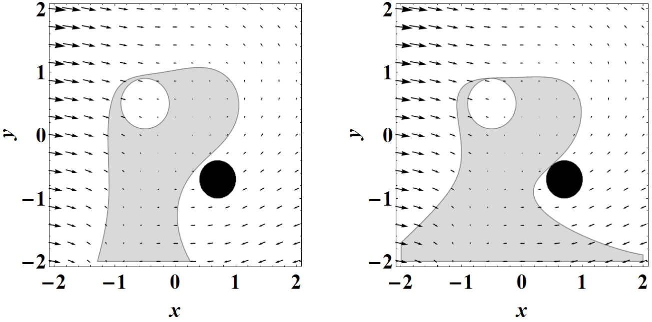


Generating Elementary Invariants

- $(v_1, v_2, v_3) = (\sin x, e^{-x}, \cos x)$
- Assume a polynomial invariant template of degree 4 with fresh variables



Comparison



Outline

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Problem Decription

- A safety requirement $\mathcal{S}$ assigns to each mode $q \in Q$ a safe region $S_q \subseteq \mathbb{R}^n$, i.e. $\mathcal{S} = \bigcup_{q \in Q}(\{q\} \times S_q)$.

Switching controller synthesis for safety [Asarin et al. 00]

Given a hybrid automaton $\mathcal{H}$ and a safety property S, find a hybrid automaton $\mathcal{H}' = (Q, X, f, D', E, G')$ such that

(r1) Refinement: for any $q \in Q$, $D'_q \subseteq D_q$, and for any $e \in E$, $G'_e \subseteq G_e$;

(r2) Safety: for any trajectory ω that $\mathcal{H}'$ accepts, if $(q, \mathbf{x})$ is on ω, then $\mathbf{x} \in S_q$;

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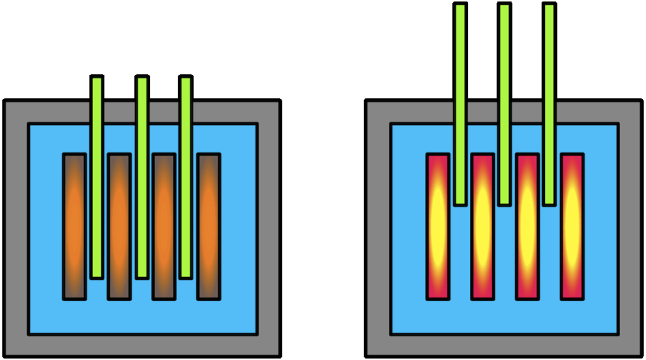
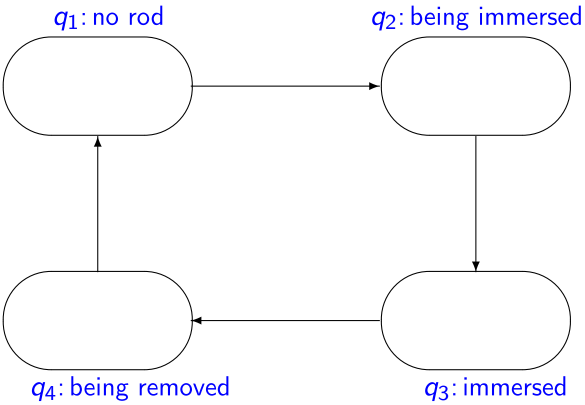
A Nuclear Reactor Example

The nuclear reactor system consists of a reactor core and a cooling rod which is immersed into and removed out of the core periodically to keep the temperature of the core in a certain range.



A Nuclear Reactor Example (Cont'd)

- x: temperature;
- p: proportion immersed



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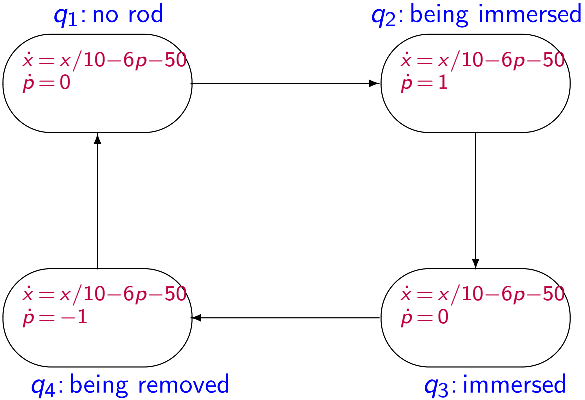

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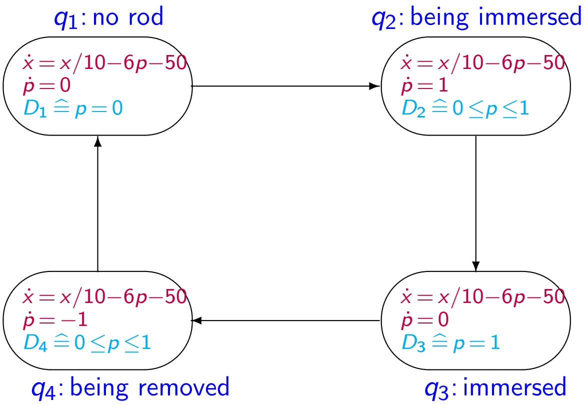

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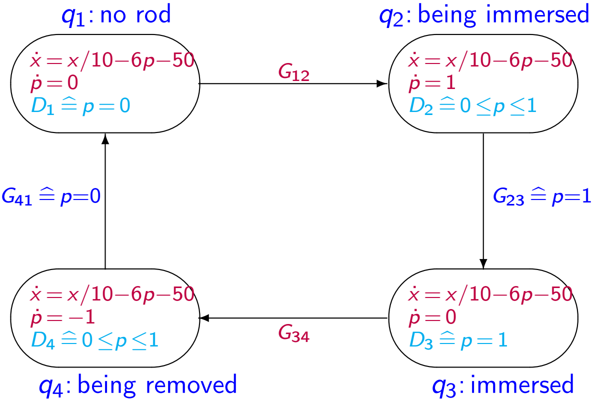

Switching Controller Synthesis for the Reactor

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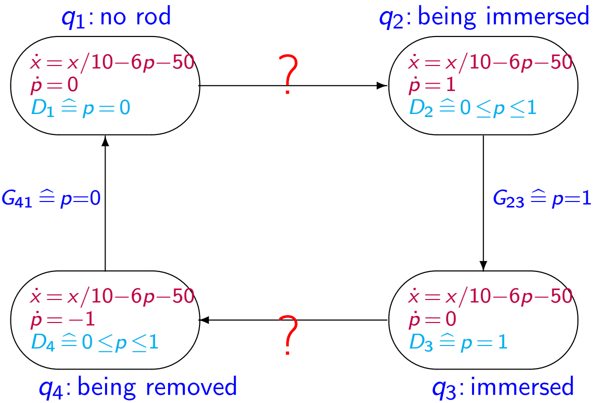
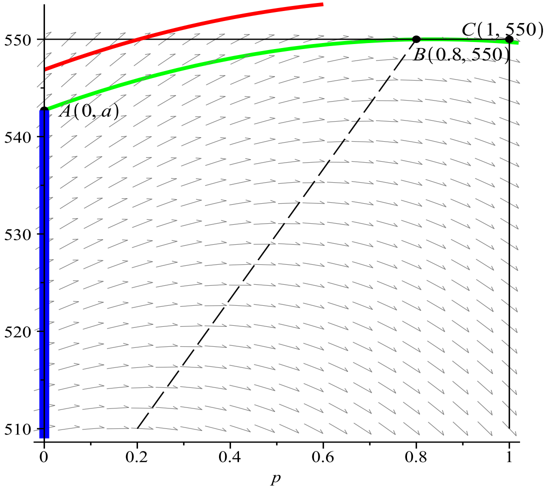


Bad Switching Violates Safety Property

Transition from mode q_1 to q_2



Solution to the Controller Synthesis Problem

Abstract Solution

Let $\mathcal{H}$ be a hybrid system and $\mathcal{S}$ be a safety property. If we can find a family of $D'_q \subseteq \mathbb{R}^n$ such that

(c1) for all $q \in Q$, $D'_q \subseteq D_q \cap S_q$;

(c2) for all $q \in Q$, D'_q is a continuous invariant of (H_q, f_q) with

$$H_q \widehat{=} \Big(\bigcup_{e=(q,q')\in E} G'_e \Big)^c ,$$

where $G'_e \widehat{=} G_e \cap D'_{q'}$ for $e = (q, q')$, then the family of G'_e form a safe switching controller.

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Template-Based Synthesis Framework

(s1) Template assignment: assign to each $q \in Q$ a template D'_q as the continuous invariant to be generated at mode q;

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Heuristics for Predefining Templates by Qualitative Analysis

Using qualitative analysis to identify critical points for predefining templates

- Infer the evolution behavior (increasing or decreasing) of continuous variables in each mode from the ODEs
- Identify modes (called critical) at which the evolution behavior of a continuous variable changes, and thus the maximal (or minimal) value of this continuous variable can be achieved
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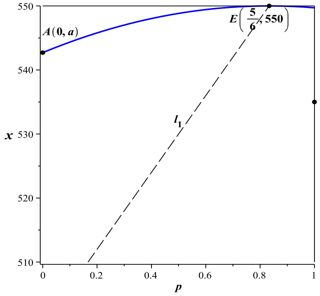
Revisiting the Running Example



For the running example,

- At D_{q_2}, temperature x achieves maximal value when crossing $l_1 \widehat{=} x/10 - 6p - 50 = 0$.
- $E(5/6, 550)$ at q_2 is obtained by taking the intersection of l_1 and safety upper bound $x = 550$
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- Compute a parabola $x - 550 - \frac{36}{25}(a - 550)(p - \frac{5}{6})^2 = 0$ through A and E as part of the template D'_{q_2}

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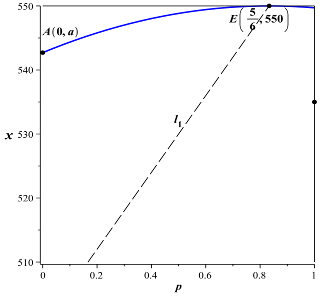

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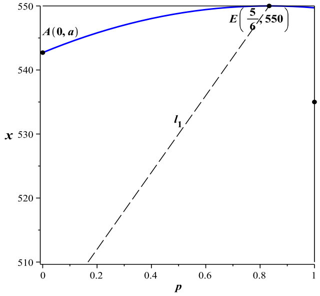

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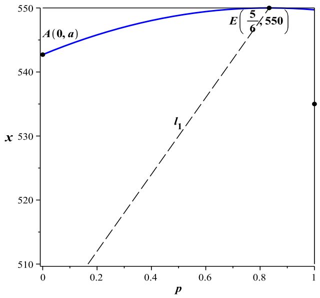

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Revisiting the Running Example (Cont'd)

The set of parameters: a, b, c, d

- $D'_1 \mathrel{\widehat{=}} p = 0 \wedge 510 \leq x \leq a$
- $D'_2 \mathrel{\widehat{=}} 0 \leq p \leq 1 \wedge x - b \geq p(d - b) \wedge$
  $x - 550 - \frac{36}{25}(a - 550)(p - \frac{5}{6})^2 \leq 0$
- $D'_3 \mathrel{\widehat{=}} p = 1 \wedge d \leq x \leq 550$
- $D'_4 \mathrel{\widehat{=}} 0 \leq p \leq 1 \wedge x - a \leq p(c - a) \wedge$
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- $G'_{12} \mathrel{\widehat{=}} p = 0 \wedge b \leq x \leq a$
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Revisiting the Running Example (Cont'd)

- $a = \frac{6575}{12} \wedge b = \frac{4135}{8} \wedge c = \frac{4345}{8} \wedge d = \frac{6145}{12}$.
- From this result we get that the cooling rod should be immersed before temperature rises to $\frac{6575}{12} = 547.92$, and removed before temperature drops to $\frac{6145}{12} = 512.08$.
- By solving differential equations explicitly, the corresponding exact bounds are 547.97 and 512.03

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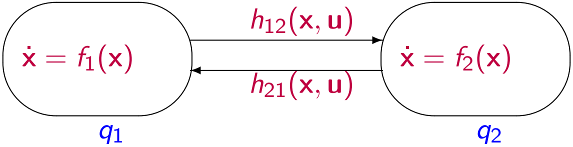
Outline

Problem Description



- Given a hybrid system $\mathcal{H}$ in which transition conditions h_{ij} are not determined but parameterized by $\mathbf{u}$, a vector of control parameters
- Our task is to determine $\mathbf{u}$ such that $\mathcal{H}$ can make discrete jumps at desired points, thus guaranteeing that
  - a safety property $\mathcal{S}$ is satisfied, i.e. $\mathbf{x} \in \mathcal{S}$ at any time
  - an optimization goal, e.g. $\min_{\mathbf{u}} g(\mathbf{u})$, is achieved

Our Approach – Step 1

Derive constraint $D(\mathbf{u})$ on $\mathbf{u}$ from the safety requirements $\mathcal{S}$

- Compute
  - the exact reachable set $\mathrm{Reach}_{\mathcal{H}}(\mathbf{x}, \mathbf{u})$ of $\mathcal{H}$, or
  - an inductive invariant $\mathrm{Inv}_{\mathcal{H}}(\mathbf{x}, \mathbf{u})$

  as polynomial formulas
- Suppose $\mathcal{S}$ is also modeled by polynomial formulas, then $D(\mathbf{u})$ can be obtained by applying QE to

$$\forall \mathbf{x}.\ \Big(\mathrm{Reach}_{\mathcal{H}}(\mathbf{x}, \mathbf{u}) \longrightarrow \mathcal{S}\Big)$$

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Encode the optimization problem (suppose the objective function g is a polynomial) over constraint $D(\mathbf{u})$ into a quantified first-order polynomial formula $\mathbf{Qu}.\varphi(\mathbf{u}, z)$ by introducing a fresh variable z

- Minimize u^2 on $[-1, 1]$
- Introduce a fresh variable z:
  $u \geq -1 \wedge u \leq 1 \wedge \; u^2 \leq z$
- Projection to the z-axis:
  $\exists u.(u \geq -1 \wedge u \leq 1 \wedge u^2 \leq z)$
- After QE: $z \geq 0$, which means

$$\min_{u \in [-1,1]} u^2 = 0$$

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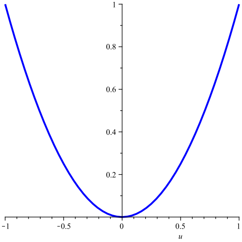

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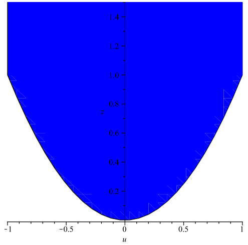

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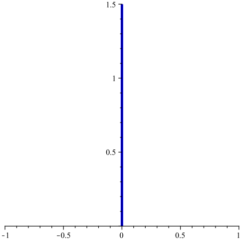

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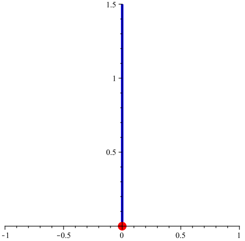

Encoding Optimization Criteria

Lemma

Suppose $g_1(\mathbf{u}_1)$, $g_2(\mathbf{u}_1, \mathbf{u}_2)$, $g_3(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ *are polynomials, and* $D_1(\mathbf{u}_1)$, $D_2(\mathbf{u}_1, \mathbf{u}_2)$, $D_3(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ *are nonempty compact semi-algebraic sets. Then there exist* c_1, c_2, $c_3 \in \mathbb{R}$ *s.t.*

$$\exists \mathbf{u}_1.(D_1 \wedge g_1 \leq z) \Leftrightarrow z \geq c_1 \tag{4}$$

$$\forall \mathbf{u}_2.\big(\exists \mathbf{u}_1.D_2 \Rightarrow \exists \mathbf{u}_1.(D_2 \wedge g_2 \leq z)\big) \Leftrightarrow z \geq c_2 \tag{5}$$

$$\exists \mathbf{u}_3.\Big((\exists \mathbf{u}_1 \mathbf{u}_2.D_3) \wedge \forall \mathbf{u}_2.\big(\exists \mathbf{u}_1.D_3 \Rightarrow \exists \mathbf{u}_1.(D_3 \wedge g_3 \leq z)\big)\Big) \Leftrightarrow z \rhd c_3 \tag{6}$$

where $\rhd \in \{>, \geq\}$, *and* c_1, c_2, c_3 *satisfy*

$$c_1 = \min_{\mathbf{u}_1} g_1(\mathbf{u}_1) \quad \text{over } D_1(\mathbf{u}_1)\,, \tag{7}$$

$$c_2 = \sup_{\mathbf{u}_2} \min_{\mathbf{u}_1} g_2(\mathbf{u}_1, \mathbf{u}_2) \quad \text{over } D_2(\mathbf{u}_1, \mathbf{u}_2)\,, \tag{8}$$

$$c_3 = \inf_{\mathbf{u}_3} \sup_{\mathbf{u}_2} \min_{\mathbf{u}_1} g_3(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) \quad \text{over } D_3(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)\,. \tag{9}$$

Our Approach – Step 3

Eliminate quantifiers in $\mathbf{Qu}.\varphi(\mathbf{u}, z)$ and from the result we can retrieve the optimal value and the corresponding optimal controller $\mathbf{u}$

- Combine exact QE with numeric computation: (discretization of existentially quantified variables)

$$\exists \mathbf{x} \in A.\, \varphi(\mathbf{x}) \approx \bigvee_{\mathbf{y} \in F_A} \varphi(\mathbf{y}) \quad ,$$

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Outline

A Reported Case Study

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- Provided by the HYDAC ELECTRONIC GMBH company within the European project *Quasimodo*
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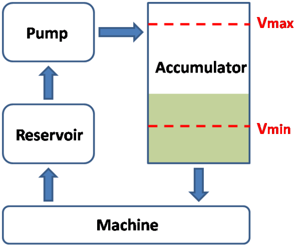
The System

- The system is composed of a machine, an accumulator, a reservoir and a pump
- The machine consumes oil out of the accumulator; the pump adds oil from the reservoir into the accumulator



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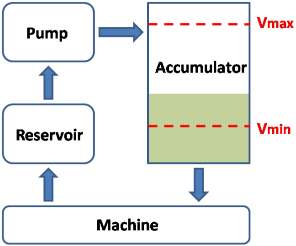
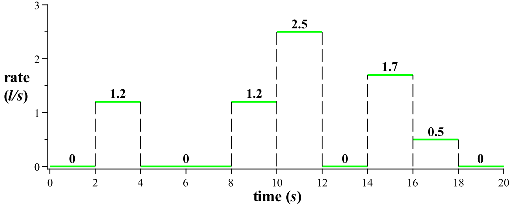


The Consumption Rate

- The oil consumption is periodic. The length of one consumption cycle is 20s (second)
- The profile of consumption rate in one cycle is depicted by



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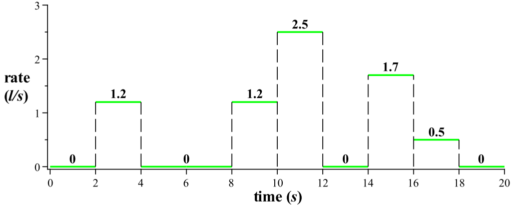

The Pump

- The power of the pump is 2.2 l/s (liter/second)
- 2-second latency: if the pump is switched on (t_{2k+1}) or off (t_{2k+2}) at time points

$$0 \leq t_1 \leq t_2 \leq \cdots \leq t_i \leq t_{i+1} \leq \cdots ,$$

then

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Control Objectives

Determine the t_i's in order to

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$$v(t) \in [V_{\min}, V_{\max}], \quad \forall t \in [0, \infty)$$

  - $v(t)$ denotes the oil volume in the accumulator at time t
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and considering the energy cost and wear of the system,

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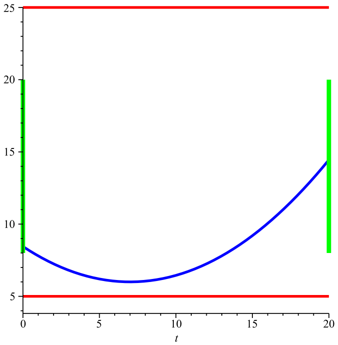
Localize the Controller

- $0 \leq t_1 \leq t_2 \leq \cdots \leq t_i \leq t_{i+1} \leq \cdots$
- Employing the periodicity
- Stable interval $[L, U] \subseteq [V_{\min}, V_{\max}]$



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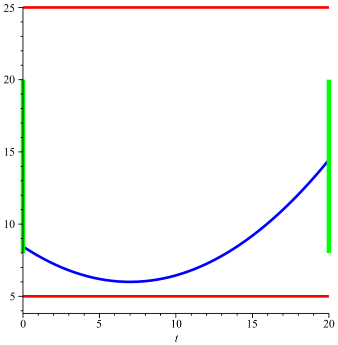

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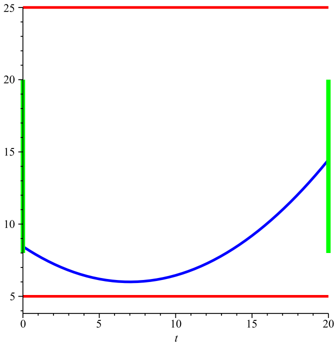
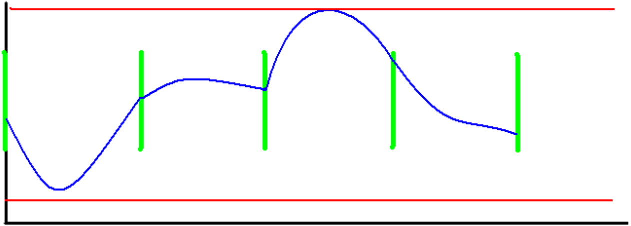


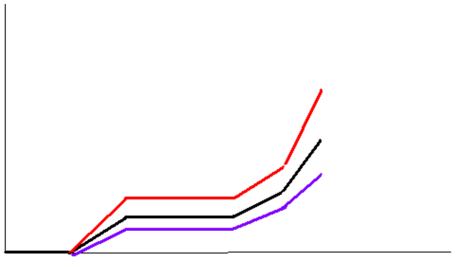
Repeated Cycles



Step 1: Modeling Oil Consumption

time	[2,4]	[8,10]	[10,12]	[14,16]	[16,18]
rate	1.2	1.2	2.5	1.7	0.5

- fluctuation of consumption rate: $f = 0.1$



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$$
C_1 \mathrel{\widehat{=}}
\begin{array}{lcl}
(0 \le t \le 2 & \longrightarrow & V_{out} = 0) \\
\wedge\, (2 \le t \le 4 & \longrightarrow & 1.1(t-2) \le V_{out} \le 1.3(t-2)) \\
\wedge\, (4 \le t \le 8 & \longrightarrow & 2.2 \le V_{out} \le 2.6) \\
\wedge\, (8 \le t \le 10 & \longrightarrow & 2.2 + 1.1(t-8) \le V_{out} \le 2.6 + 1.3(t-8)) \\
\wedge\, (10 \le t \le 12 & \longrightarrow & 4.4 + 2.4(t-10) \le V_{out} \le 5.2 + 2.6(t-10)) \\
\wedge\, (12 \le t \le 14 & \longrightarrow & 9.2 \le V_{out} \le 10.4) \\
\wedge\, (14 \le t \le 16 & \longrightarrow & 9.2 + 1.6(t-14) \le V_{out} \le 10.4 + 1.8(t-14)) \\
\wedge\, (16 \le t \le 18 & \longrightarrow & 12.4 + 0.4(t-16) \le V_{out} \le 14 + 0.6(t-16)) \\
\wedge\, (18 \le t \le 20 & \longrightarrow & 13.2 \le V_{out} \le 15.2)
\end{array}
$$

Step 1: Modeling the Pump

- We will first assume that the pump is activated at most twice in one cycle: t_1, t_2, t_3, t_4
- $t_{i+1} - t_i \geq 2$:

$$C_2 \mathrel{\widehat{=}} \begin{array}{l} \quad (t_1 \geq 2 \wedge t_2 - t_1 \geq 2 \wedge t_3 - t_2 \geq 2 \wedge t_4 - t_3 \geq 2 \wedge t_4 \leq 20) \\ \vee (t_1 \geq 2 \wedge t_2 - t_1 \geq 2 \wedge t_2 \leq 20 \wedge t_3 = 20 \wedge t_4 = 20) \\ \vee (t_1 = 20 \wedge t_2 = 20 \wedge t_3 = 20 \wedge t_4 = 20) \end{array}.$$

- $2.2 l/s$

$$C_3 \mathrel{\widehat{=}} \begin{array}{lcl} \quad (0 \leq t \leq t_1 & \longrightarrow & V_{in} = 0) \\ \wedge (t_1 \leq t \leq t_2 & \longrightarrow & V_{in} = 2.2(t - t_1)) \\ \wedge (t_2 \leq t \leq t_3 & \longrightarrow & V_{in} = 2.2(t_2 - t_1)) \\ \wedge (t_3 \leq t \leq t_4 & \longrightarrow & V_{in} = 2.2(t_2 - t_1) + 2.2(t - t_3)) \\ \wedge (t_4 \leq t \leq 20 & \longrightarrow & V_{in} = 2.2(t_2 + t_4 - t_1 - t_3)) \end{array}.$$

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Step 1: Modeling the Pump

- We will first assume that the pump is activated at most twice in one cycle: t_1, t_2, t_3, t_4
- $t_{i+1} - t_i \geq 2$:

$$C_2 \mathrel{\widehat{=}} \begin{array}{l} (t_1 \geq 2 \wedge t_2 - t_1 \geq 2 \wedge t_3 - t_2 \geq 2 \wedge t_4 - t_3 \geq 2 \wedge t_4 \leq 20) \\ \vee (t_1 \geq 2 \wedge t_2 - t_1 \geq 2 \wedge t_2 \leq 20 \wedge t_3 = 20 \wedge t_4 = 20) \\ \vee (t_1 = 20 \wedge t_2 = 20 \wedge t_3 = 20 \wedge t_4 = 20) \end{array}.$$

- $2.2l/s$

$$C_3 \mathrel{\widehat{=}} \begin{array}{lll} (0 \leq t \leq t_1 & \longrightarrow & V_{in} = 0) \\ \wedge (t_1 \leq t \leq t_2 & \longrightarrow & V_{in} = 2.2(t - t_1)) \\ \wedge (t_2 \leq t \leq t_3 & \longrightarrow & V_{in} = 2.2(t_2 - t_1)) \\ \wedge (t_3 \leq t \leq t_4 & \longrightarrow & V_{in} = 2.2(t_2 - t_1) + 2.2(t - t_3)) \\ \wedge (t_4 \leq t \leq 20 & \longrightarrow & V_{in} = 2.2(t_2 + t_4 - t_1 - t_3)) \end{array}.$$

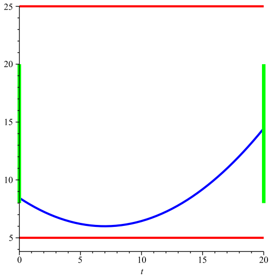
Step 1: Encoding Safety Requirements

- Oil volume in the accumulator:

$$C_4 \mathrel{\widehat{=}} v = v_0 + V_{in} - V_{out} \quad .$$

- Inductiveness and safety (considering robustness):

$$\begin{aligned} C_5 &\mathrel{\widehat{=}} \quad t = 20 \longrightarrow L{+}0.2 \leq v \leq U{-}0.2 \\ C_6 &\mathrel{\widehat{=}} \quad 0 \leq t \leq 20 \longrightarrow V_{\min}{+}0.2 \leq v \leq V_{\max}{-}0.2 \quad . \end{aligned}$$



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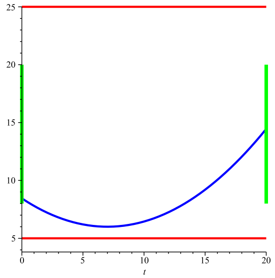

Step 1: Encoding Safety Requirements (Cont'd)

$$\mathcal{S} \mathrel{\widehat{=}} \forall t, v, V_{in}, V_{out}.(C_1 \wedge C_3 \wedge C_4 \longrightarrow C_5 \wedge C_6)\,.$$

- C_1: oil consumed
- C_3: oil pumped
- C_4: oil in the accumulator
- C_5: inductiveness
- C_6: (local) safety

$$C_8 \mathrel{\widehat{=}} \forall v_0.\Big(C_7 \longrightarrow \exists t_1 t_2 t_3 t_4.\big(C_2 \wedge \mathcal{S}\big)\Big)\,.$$

- $C_7 \mathrel{\widehat{=}} L \leq v_0 \leq U$
- C_2: 2-second latency

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Deriving Constraints

Applying QE to

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Deriving Constraints (Cont'd)

$$C_{10} \mathrel{\widehat{=}} C_2 \wedge C_7 \wedge C_9 \wedge \mathcal{S}\,.$$

- C_2: 2-second latency
- $C_7 : L \leq v_0 \leq U$
- C_9: constraint on L, U
- $\mathcal{S}$: safety and inductiveness

After QE:

$$\mathcal{D}(L, U, v_0, t_1, t_2, t_3, t_4) \mathrel{\widehat{=}} \bigvee_{i=1}^{92} D_i$$

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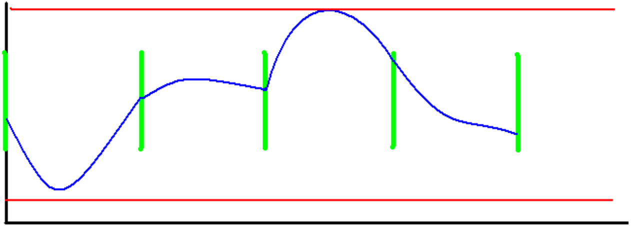
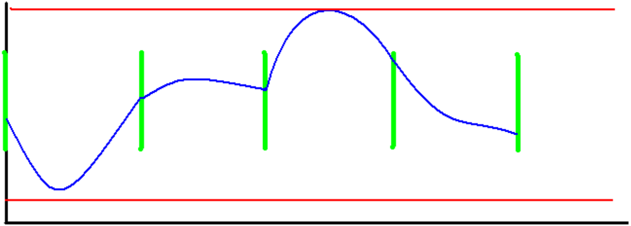
Step 2: Optimization Criterion



$\mathrm{R_o}$ (*optimality*): minimize the average accumulated oil volume in the limit, i.e. minimize

$$\lim_{T\to\infty} \frac{1}{T} \int_{t=0}^{T} v(t)\mathrm{d}t$$

Optimization Criterion (Contd.)



- R'_{o} : $\min\limits_{[L,U]} \max\limits_{v_0 \in [L,U]} \min\limits_{\mathbf{t}} \frac{1}{20} \int_{t=0}^{20} v(t)\mathrm{d}t$.

Step 2: Encoding the Optimization Criterion

Cost function:

$$
\begin{aligned}
g(v_0, t_1, t_2, t_3, t_4) &\mathrel{\widehat{=}} \frac{1}{20}\int_{t=0}^{20} v(t)\mathrm{d}t \\
&= \frac{20v_0+1.1(t_1^2-t_2^2+t_3^2-t_4^2-40t_1+40t_2-40t_3+40t_4)-132.2}{20}
\end{aligned}
$$

R'_{o} can be encoded into

$$\exists L, U.\Big(C_9 \wedge \forall v_0.\big(C_7 \longrightarrow \exists t_1 t_2 t_3 t_4.(\mathcal{D} \wedge g \leq z)\big)\Big),$$

which is equivalent to $z \geq z^*$ or $z > z^*$

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Step 3: Performing QE

$$\exists L, U. \Big(C_9 \wedge \forall v_0. \big(C_7 \longrightarrow \exists t_1 t_2 t_3 t_4. (\mathcal{D} \wedge g \leq z) \big) \Big)$$

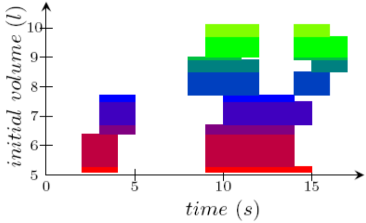
- the inner $\exists$: *qudratic programming*
- the outer $\exists$: discretization

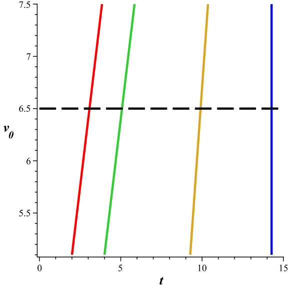
$$L \geq 5.1 \wedge U \leq 24.9 \wedge U - L \geq 2.4$$

- the middle $\forall$: divide and conquer

Optimal Controllers with 2 Activations

- In [Cassez et al hscc09], the optimal value 7.95 is obtained at interval [5.1,8.3]
- Using our approach, the optimal value is 7.53 (a 5% improvement) and the corresponding interval is $[5.1, 7.5]$
- Comparison of local optimal controllers: (the left one comes from [Cassez et al hscc09])





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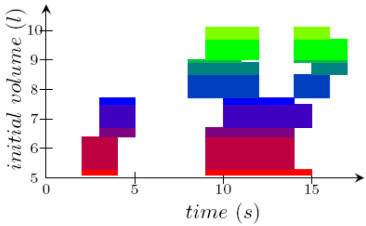

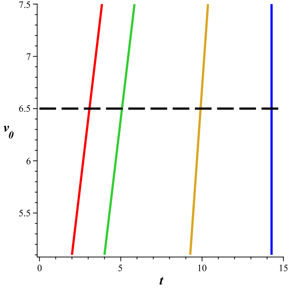

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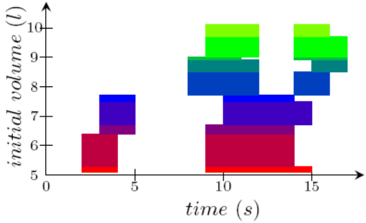
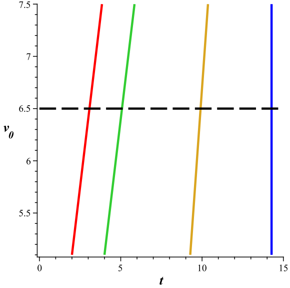
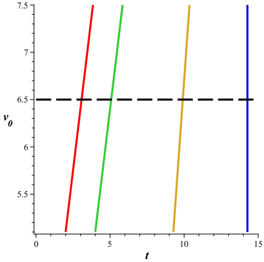




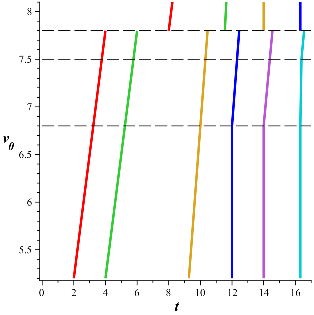
Local Optimal Controllers — 2 Activations



$$t_1 = \frac{10v_0 - 25}{13} \wedge t_2 = \frac{10v_0 + 1}{13} \wedge t_3 = \frac{10v_0 + 153}{22} \wedge t_4 = \frac{157}{11}$$

Improvement by Increasing Activations

- The pump is allowed to be switched on at most 3 times in one cycle
- The optimal average accumulated oil volume 7.35 (a 7.5% improvement) is obtained at interval $[5.2, 8.1]$
- The local optimal controllers corresponding to $v_0 \in [5.2, 8.1]$:



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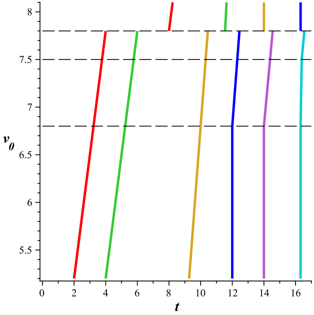

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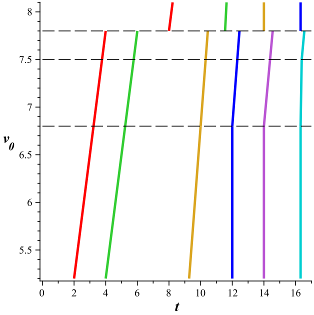
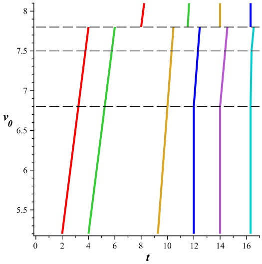


Local Optimal Controllers — 3 Activations



$$\begin{cases} t_1=\frac{10v_0-26}{13} \wedge t_2=\frac{10v_0}{13} \wedge t_3=\frac{5v_0+76}{11} \wedge t_4=12 \wedge t_5=14 \wedge t_6=\frac{359}{22} & v_0\in[5.2,6.8) \\ t_1=\frac{10v_0-26}{13} \wedge t_2=\frac{10v_0}{13} \wedge t_3=\frac{5v_0+76}{11} \wedge t_4=\frac{5v_0+98}{11} \wedge t_5=\frac{5v_0+92}{9} \wedge t_6=\frac{20v_0+3095}{198} & v_0\in[6.8,7.5) \\ t_1=\frac{10v_0-26}{13} \wedge t_2=\frac{10v_0}{13} \wedge t_3=\frac{5v_0+76}{11} \wedge t_4=\frac{5v_0+98}{11} \wedge t_5=\frac{5v_0+92}{9} \wedge t_6=\frac{5v_0+110}{9} & v_0\in[7.5,7.8) \\ t_1=\frac{10v_0+26}{13} \wedge t_2=\frac{45v_0+1300}{143} \wedge t_3=14 \wedge t_4=\frac{359}{22} \wedge t_5=20 \wedge t_6=20 & v_0\in[7.8,8.1] \end{cases}$$

Three Activations are Enough

Proposition

For each admissible $[L, U]$, each $v_0 \in [L, U]$, and any local control strategy s_4 with at least 4 activations subject to R_{lu}, R_i and R_{ls}, there exists a local control strategy s_3 subject to R_{lu}, R_i and R_{ls} with 3 activations such that

$$\frac{1}{20}\int_{t=0}^{20} v_{s_3}(t)\mathrm{d}t < \frac{1}{20}\int_{t=0}^{20} v_{s_4}(t)\mathrm{d}t$$

where $v_{s_3}(t)$ (resp. $v_{s_4}(t)$) is the oil volume in the accumulator at t with s_3 (resp. s_4).

Outline

Conclusions

- Hybrid systems attracts more and more interests with the development of **safety critical** embedded systems
- Invariant plays an important role in the study **(formal verification, controller synthesis)** of hybrid systems
- Semi-algebraic inductive invariant checking for polynomial continuous/hybrid systems is **decidable**
- Use parametric polynomials and symbolic computation to automatically discover invariants, and to perform optimization
  - rigorous
  - high complexity (may be combined with numeric computation)
  - Non-polynomial systems transformed to polynomials ones
- Case studies show good prospect of proposed methods

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Thank you!

Questions?