Kolmogorov complexity

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August 28, 2021

Background of KC

The notion of algorithmic complexity (algorithmic entropy) appeared in the 1960s in between the theory of computation, probability theory, and information theory.

Kolmogorov complexity:

For a string x consider all its descriptions, the length of the shortest string y among them is called the Kolmogorov complexity of x with respect to D: $C_D(x) = \min\{l(y) \mid D(y) = x\}$.

Martin-Löf random

X is Martin-Löf random iff there is a constant c such that $K(X{\upharpoonright}_n) \geq n - c$ for all n, where $X{\upharpoonright}_n$ denotes the initial segment of X of length n and K denotes prefix-free Kolmogorov complexity.

Two examples

Formal language theory:

Show $L = \{a^k b^k | k > 0\}$ not regular.

Combinatorics:

There is a tournament (complete directed graph) T of n players that contains no large transitive subtournaments ($> 1 + 2\log n$).

Pumping lemma

Let L be a regular language. Then there exists an integer $p \geq 1$ depending only on L such that every string w in L of length at least p (p is called the "pumping length") can be written as $w = xyz$ (i.e., w can be divided into three substrings), satisfying the following conditions:

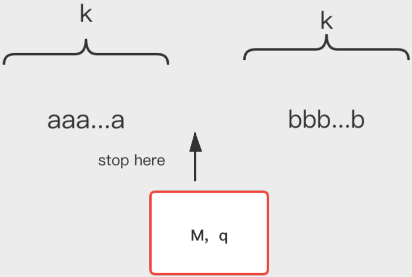
- $|y| \geq 1$
- $|xy| \leq p$
- $\forall n \geq 0,\ xy^n z \in L$

Note:

The pumping lemma is a necessary condition of regular language, which is often used to prove that a particular language is non-regular.

Proof using KC

By contradiction, assume that DFA M accepts L. Choose k so that $C(k) \gg 2|M|$. Simulate M.



$C(k) < |M| + q + O(1) < 2|M|$. Contradiction.

Tournament proof by KC

There is a tournament (complete directed graph) T of n players that contains no large transitive subtournaments ($> 1 + 2\log n$).

- Choose a random T.
- One bit codes an edge. $C(T) \geq n(n-1)/2$.
- If there is a large transitive subtournament, then a large number of edges are given for free!
- $C(T) < n(n-1)/2 - (\text{subgraph\_edges}) + \text{overhead}$

Let k represent the number of nodes in this subgraph. We know that subgraph_edges $= k(k-1)/2$ and overhead $= k * \log n$, so to avoid contradiction, $k < 2\log n + 1$.

Current research

The goal is to give an answer to the problem appears in Miller and Nies (2006).

For each X there exists a Martin-Löf random Y such that $X \leq_K Y$?

Miller and Nies (2006) and Miller and Yu (2011) also asked:

Is there a $\leq_K$-maximal and/or a $\ll_K$-maximal real?

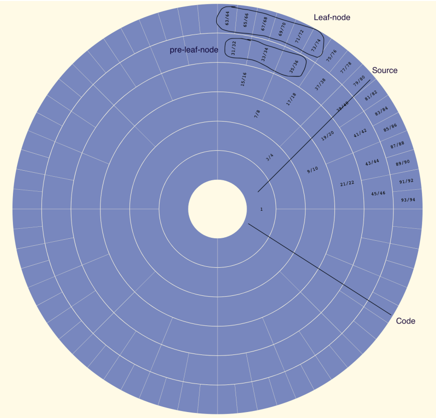
Definition:

$A \leq_{rK} B$ if $\exists d \forall n.\ K(A{\restriction}_n \mid B{\restriction}_n) < d$.

$A \leq_K B$ if $\exists d \forall n.\ K(A{\restriction}_n) \leq K(B{\restriction}_n) + d$.

$A \ll_K B$ if $\lim_n K(A{\restriction}_n) - K(B{\restriction}_n) = \infty$.

Current result



There exists a constructing strategy for spoiler to defeat the greedy coder with no more than $n(n-1)+1+\frac{(n-4)(n-3)(n-2)}{3}$ hits in the tree of height n.

- 踏实勤奋
- 集思广益
- 劳逸结合

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