Taming Delays in Dynamical Systems

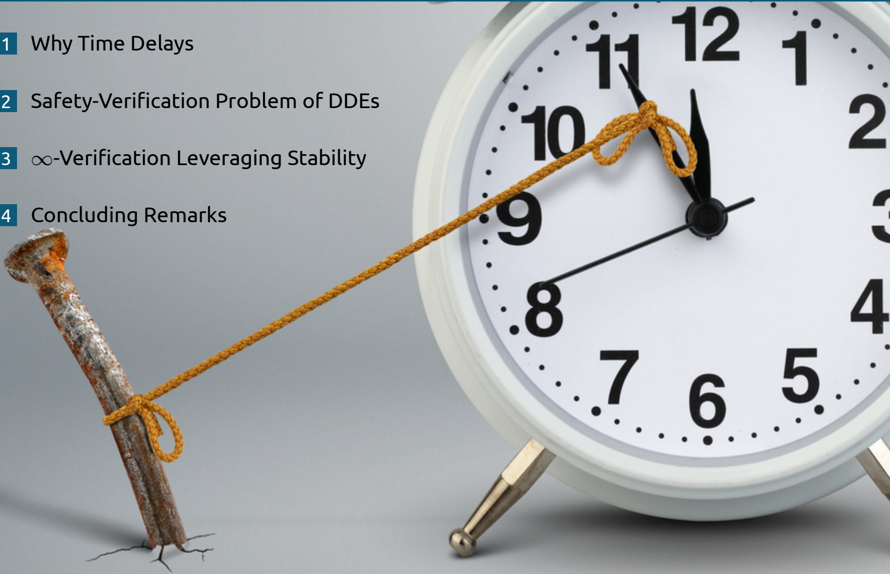
Unbounded Verification of Delay Differential Equations

Shenghua Feng[1], **Mingshuai Chen**[1], Naijun Zhan[1], Martin Fränzle[2], Bai Xue[1]

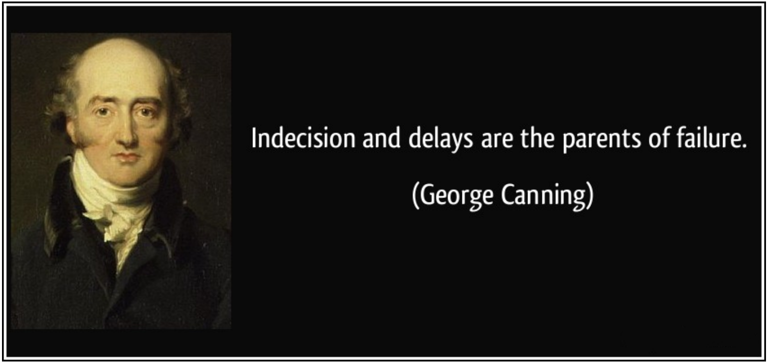
[1] Institute of Software, Chinese Academy of Sciences
[2] Dpt. of Computing Science, C. v. Ossietzky Universität Oldenburg

New York City · July 2019

Outline



Advice by a Wise Man



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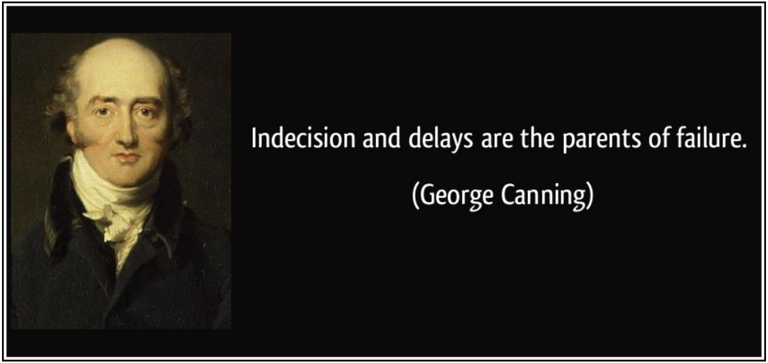

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- Only relevant to ordinary people's life ?
- Or to scientists, in particular comp. sci. and control folks, too ?

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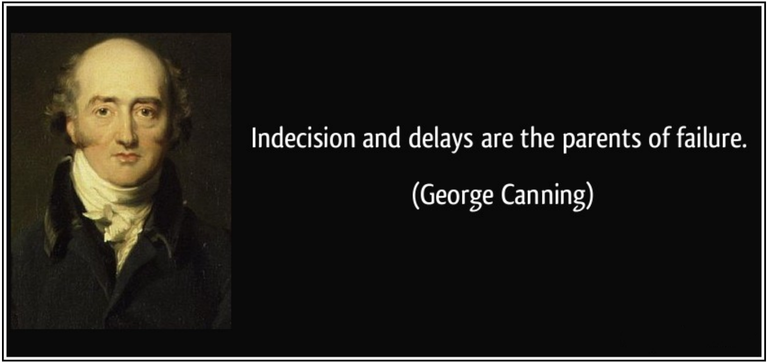

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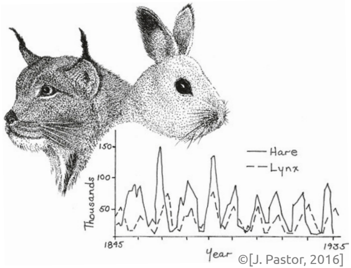
Remember that Canning briefly controlled Great Britain !

Delayed Coupling in Differential Dynamics

Vito Volterra



Predator-prey dynamics

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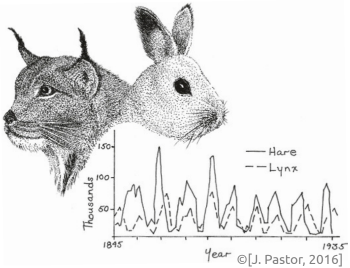

Predator-prey dynamics

"Despite [...] very satisfactory state of affairs as far as [ordinary] differential equations are concerned, we are nevertheless forced to turn to the study of more complex equations. Detailed studies of the real world impel us, albeit reluctantly, to take account of the fact that the rate of change of physical systems depends not only on their present state, but also on their past history."

[Richard Bellman and Kenneth L. Cooke, 1963]

Shall I Care about Delays ?

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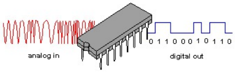
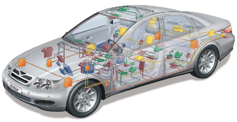
We are no better :
As soon as computer scientists enter the scene, serious delays are ahead...

Is Instantaneous Coupling Realistic?



Digital control needs A/D and D/A conversion, which induces latency in signal forwarding.

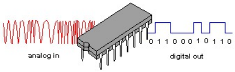
Digital signal processing, especially in complex sensors like CV, needs processing time, adding signal delays.



Networked control introduces communication latency into the feedback control loop.

Harvesting, fusing, and forwarding data through sensor networks enlarge the latter by orders of magnitude.

Is Instantaneous Coupling Realistic? – No.



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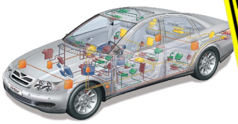


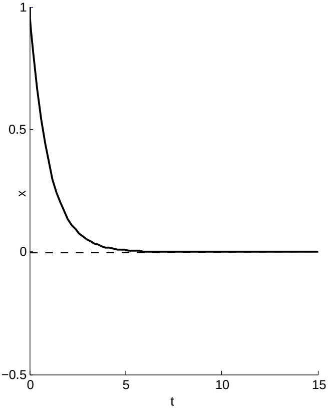
...s communication latency in ... loop.

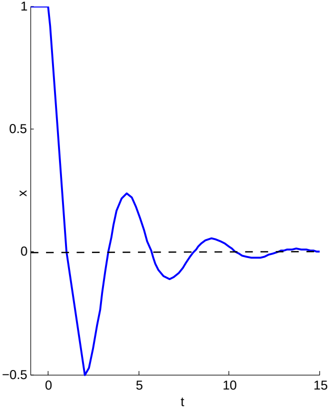
Harvesting, fusing, and forwarding data through sensor networks enlarge the latter by orders of magnitude.

Do Delays Have Observable Effect?

$$\begin{cases} \dot{x}(t) = -x(t) \\ x(0) = 1 \end{cases}$$

$$\begin{cases} \dot{x}(t) = -x(t-1) \\ x([-1,0]) \equiv 1 \end{cases}$$





Do Delays Have Observable Effect? – Yes, they have.

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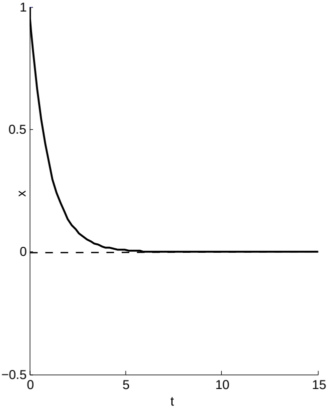
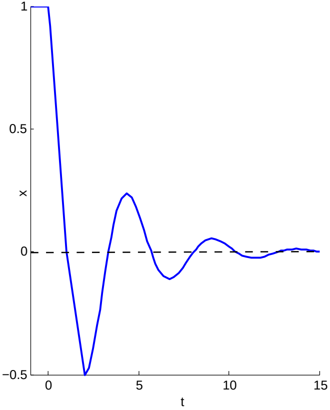
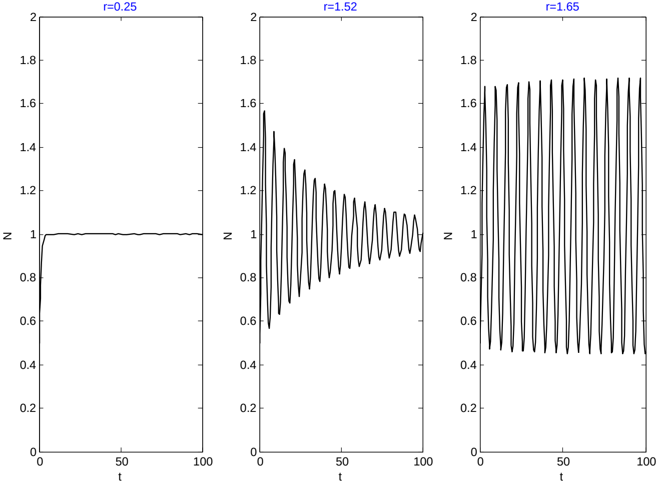




May the Effects be Harmful ?

- Delayed logistic equation [G. Hutchinson, 1948] :

$$\dot{N}(t) = N(t)[1 - N(t - r)]$$



May the Effects be Harmful ? – Yes, delays may well annihilate control performance.

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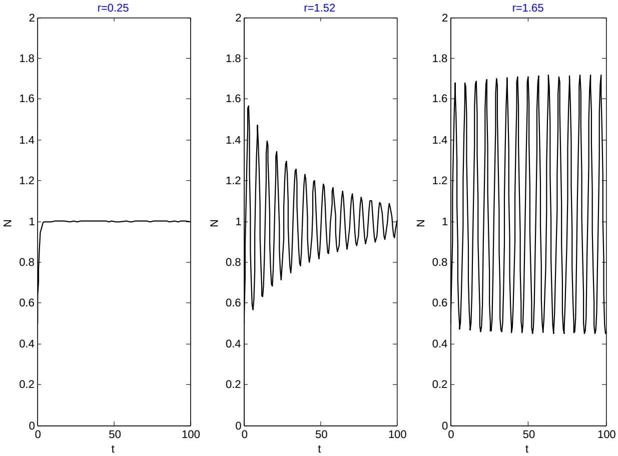

Consequences

- Delays in feedback control loops are ubiquitous.
- They may well invalidate the safety/stability/...certificates obtained by verifying delay-free abstractions of the feedback control systems.

Automatic verification methods addressing feedback delays in dynamical systems should therefore abound!

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- Delays in feedback control loops are ubiquitous.
- They may well invalidate the safety/stability/…certificates obtained by verifying delay-free abstractions of the feedback control systems.

Automatic verification methods addressing feedback delays in dynamical systems should therefore abound !

Surprisingly, they don't :

1. S. Prajna, A. Jadbabaie : *Meth. f. safety verification of time-delay syst.* (CDC'05)
2. L. Zou, M. Fränzle, N. Zhan, P.N. Mosaad : *Autom. verific. of stabil. and safety* (CAV'15)
3. H. Trinh, P.T. Nam, P.N. Pathirana, H.P. Le : *On bwd.s and fwd.s reachable sets bounding for perturbed time-delay systems* (Appl. Math. & Comput. 269, '15)
4. Z. Huang, C. Fan, S. Mitra : *Bounded invariant verification for time-delayed nonlinear networked dynamical systems* (NAHS'16)
5. P.N. Mosaad, M. Fränzle, B. Xue : *Temporal logic verification for DDEs* (ICTAC'16)
6. M. Chen, M. Fränzle, Y. Li, P.N. Mosaad, N. Zhan : *Validat. simul.-based verific.* (FM'16)
7. B. Xue, P.N. Mosaad, M. Fränzle, M. Chen, Y. Li, N. Zhan : *Safe approx. of reachable sets for DDEs* (FORMATS'17)
8. E. Goubault, S. Putot, L. Sahlman : *Approximating flowpipes for DDEs* (CAV'18)

(plus a handful of related versions)

Delay Differential Equations (DDEs)

$$\begin{cases} \dot{\mathbf{x}}(t) &= \boldsymbol{f}(\mathbf{x}(t), \mathbf{x}(t-r_1), \ldots, \mathbf{x}(t-r_k)), \quad t \in [0, \infty) \\ \mathbf{x}(t) &= \phi(t), \quad t \in [-r_{\max}, 0] \end{cases}$$

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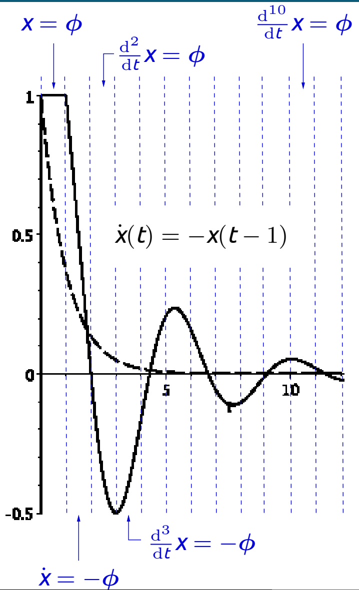
The unique *solution* (*trajectory*) : $\xi_\phi(t) : [-r_{\max}, \infty) \mapsto \mathbb{R}^n$.

Why DDEs are Hard(er)



DDEs constitute a model of system dynamics beyond "state snapshots" :

- They feature "functional state" instead of state in the $\mathbb{R}^n$.
- Thus providing rather infallible, infinite-dimensional memory of the past.

N.B. : More complex transformations may be applied to the initial segment f_0 according to the DDE's right-hand side. f_0 will nevertheless hardly ever vanish from the state space.

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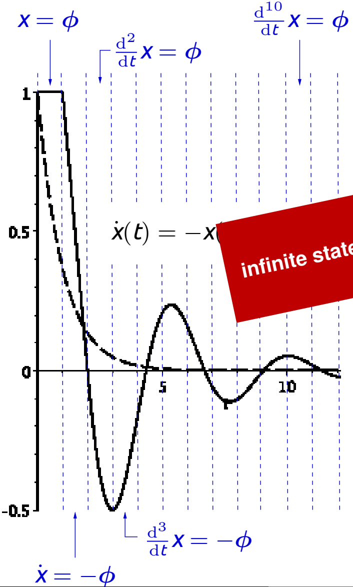

Try only if infinite state no longer is scary enough to you!

DDEs con... model of system ... "state snapshots" :

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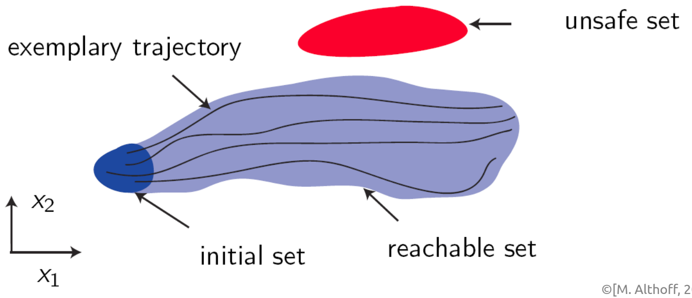
Safety Verification Problem

Given $T \in \mathbb{R}, \mathcal{X} \subseteq \mathbb{R}^n, \mathcal{U} \subseteq \mathbb{R}^n$, weather

$$\forall \phi \in \{\phi \mid \phi(t) \in \mathcal{X}, \forall t \in [-r_{\max}, 0]\} : \quad \left(\bigcup_{t \leq T} \xi_{\phi}(t)\right) \cap \mathcal{U} = \emptyset \quad ?$$



©[M. Althoff, 2010]

- System is T-safe, if no trajectory enters $\mathcal{U}$ within $[-r_{\max}, T]$; Unbounded : $T = \infty$.

Stability of Linear Dynamics by Spectral Analysis

For linear DDEs :

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{x}(t - r)$$

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The characteristic equation :

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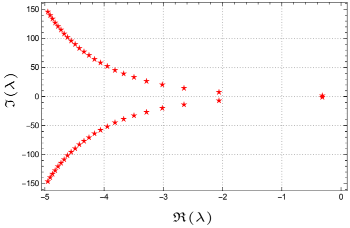

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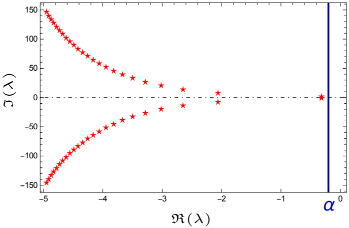

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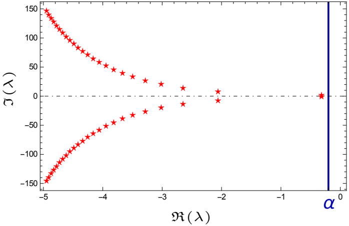
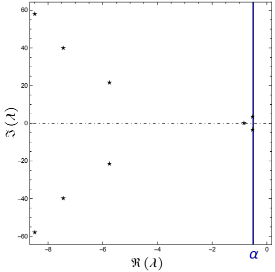


Globally exponentially stable if $\forall \lambda \colon \Re(\lambda) < 0$, i.e.,

$$\exists K > 0.\, \exists \alpha < 0 \colon \left\| \xi_\phi(t) \right\| \leq K \|\phi\| \,\mathrm{e}^{\alpha t}, \quad \forall t \geq 0,\ \forall \phi \in \mathcal{C}_r$$

Reducing Unbounded Verification to Bounded One

1. Identify the rightmost eigenvalue (and hence α) and construct K.
2. Compute T^* based on the exponential estimation spanned by α and K.
3. Reduce to bounded verifi., i.e., $\forall T > T^*$, ∞-safe $\iff$ T-safe.



$$K = \hat{K}\left(1 + \|B\| \int_0^r e^{-\alpha\tau}\, d\tau\right) \|\mathcal{X}\|$$

$$\hat{K} = \frac{1}{2\pi}\left(\int_{-M}^{M} \left\|\mathcal{O}\left(\frac{1}{(\alpha+i\nu)^2}\right)\right\| d\nu + \frac{8n}{M}\left(\|A\| + \|B\|\, e^{-r\alpha}\right)\right) + 1_0(\alpha)$$

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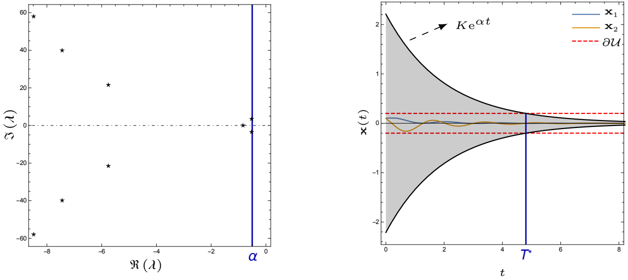

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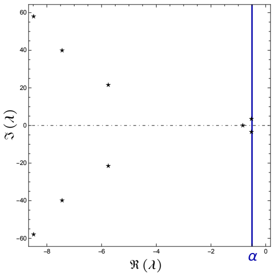

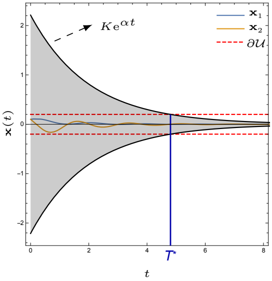

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Stability of Nonlinear Dynamics by Linearization

For nonlinear DDEs :

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \boldsymbol{f}(\mathbf{x}(t), \mathbf{x}(t-r)) \\ &= A\mathbf{x} + B\mathbf{y} + \boldsymbol{g}(\mathbf{x}, \mathbf{y}), \text{ with } A = \boldsymbol{f}_{\mathbf{x}}(\mathbf{0}, \mathbf{0}), B = \boldsymbol{f}_{\mathbf{y}}(\mathbf{0}, \mathbf{0}) \end{aligned}$$

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The linearization yields

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The linearization yields

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{x}(t-r)$$

Locally exponentially stable if $\forall \lambda\colon \Re(\lambda) < 0$, i.e.,

$$\exists \delta > 0.\, \exists K > 0.\, \exists \alpha < 0\colon\ \|\phi\| \leq \delta \implies \left\|\boldsymbol{\xi}_\phi(t)\right\| \leq K\|\phi\|\,\mathrm{e}^{\alpha t/2}, \quad \forall t \geq 0$$

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1. Identify the rightmost eigenvalue (and hence α), then construct K and δ.
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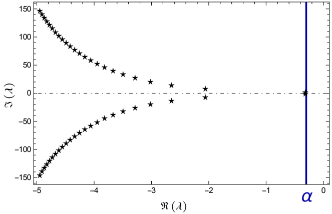

$$\delta = \min\left\{\delta_\epsilon,\ \delta_\epsilon / \left(\hat{K}e^{-r\alpha}\ (1 + \|B\| \int_0^r e^{-\alpha\tau}\, d\tau)\right)\right\}$$

$$\delta_\epsilon = \hat{K}e^{-r\alpha}\ (1 + \|B\| \int_0^r e^{-\alpha\tau}\, d\tau)\ \|\phi\|\ e^{\epsilon\hat{K}e^{-r\alpha}t + \alpha t}$$

$$\epsilon \leq -\alpha/(2\hat{K}e^{-r\alpha})$$

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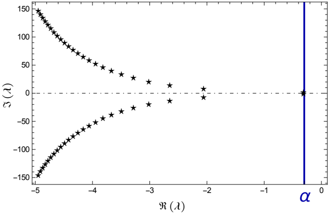

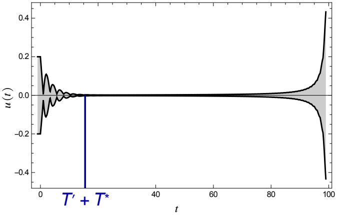

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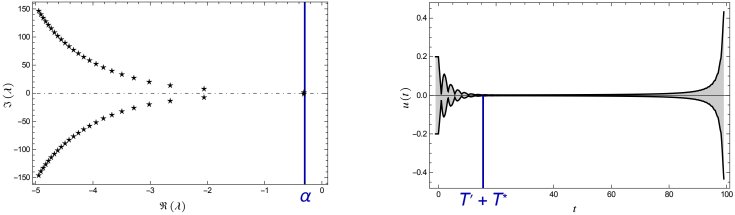

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Concluding Remarks

Problem : We face

- increasingly wide-spread use of networked distributed sensing and control,
- substantial feedback delays thus affecting dynamical control schemes,
- delays impact control performance in continuous evolution.

Status : We present

- a constructive method for computing a delay-dependent enclosure of the infinite-horizon reachable set of a DDE featuring exponential stability,
- a reduction of the verification problem over an unbounded temporal horizon to that over a bounded one,
- an extension of the scope of existing bounded verification techniques to unbounded verification tasks.

Future Work : We plan to

- exploit more permissive forms of stabilities, e.g. asymptotical stability,
- investigate more general forms of DDEs, e.g., with time-varying, or distributed (i.e., a weighted average of) delays,
- refine the enclosure by, e.g., topologically partitioning the initial functional space.