Verification of Delayed Differential Dynamics

Based on Validated Simulation

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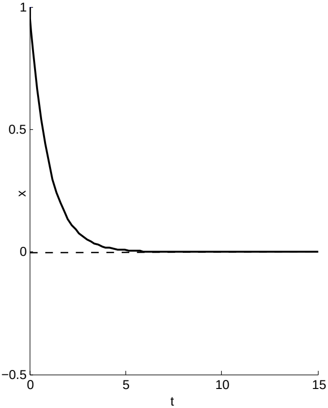
Motivation : Why Delays ?

$$\begin{cases} \dot{x}(t) = -x(t) \\ x(0) = 1 \end{cases}$$



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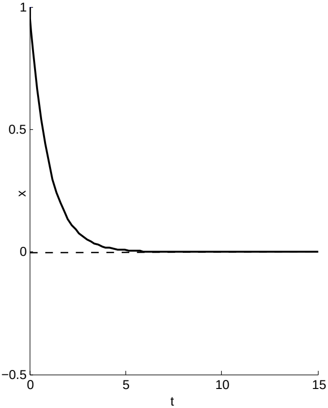
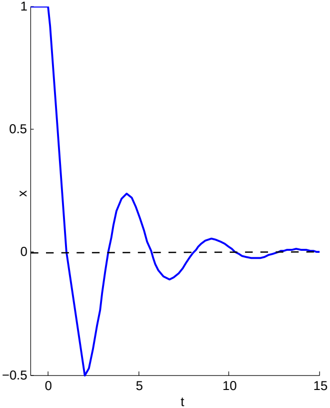
$$\begin{cases} \dot{x}(t) = -x(t) \\ x(0) = 1 \end{cases}$$

$$\begin{cases} \dot{x}(t) = -x(t-1) \\ x([-1,0]) \equiv 1 \end{cases}$$





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- Delayed logistic equation [G. Hutchinson, 1948] :

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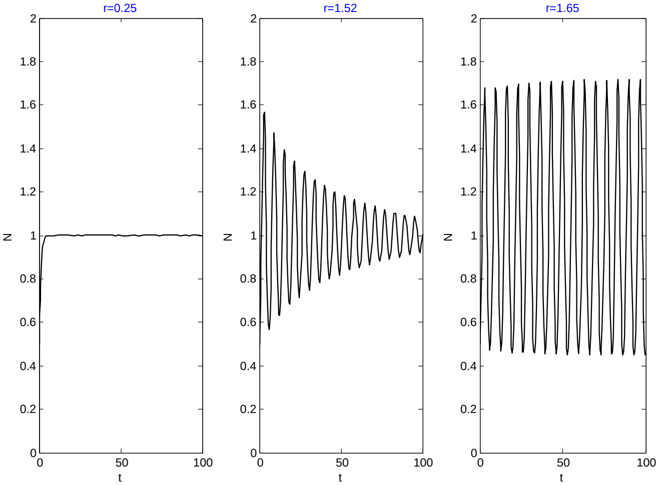

Outline

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Delayed Dynamical Systems

Delayed Dynamical Systems

$$\begin{cases} \dot{\mathbf{x}}(t) &= \boldsymbol{f}(\mathbf{x}(t), \mathbf{x}(t-r_1), \ldots, \mathbf{x}(t-r_k)), \quad t \in [0, \infty) \\ \mathbf{x}(t) &\equiv \mathbf{x}_0 \in \Theta, \quad t \in [-r_{\max}, 0] \end{cases}$$

The unique *solution* (*trajectory*) : $\xi_{\mathbf{x}_0}(t) : [-r_{\max}, \infty) \mapsto \mathbb{R}^n$.

Safety Verification Problem [1]

Given $T \in \mathbb{R}$, $\mathcal{X}_0 \subseteq \Theta, \mathcal{U} \subseteq \mathbb{R}^n$, whether

$$\forall \mathbf{x}_0 \in \mathcal{X}_0 : \quad \left(\bigcup_{t \leq T} \xi_{\mathbf{x}_0}(t)\right) \cap \mathcal{U} = \emptyset \quad ?$$

1. The figure is taken from [M. Althoff, 2010].

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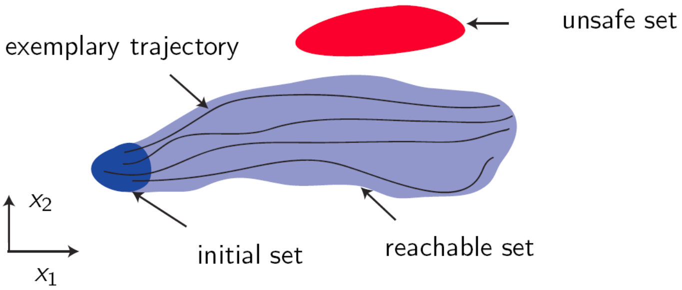

- System is safe, if no trajectory enters the unsafe set.

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Outline

Basic Idea [2]

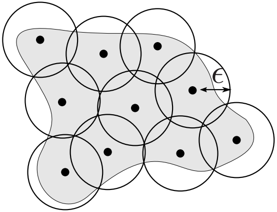

Figure : A finite ϵ-cover of the initial set of states.

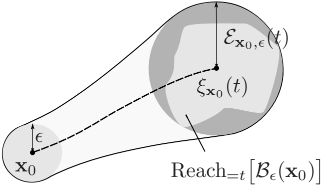

Figure : An Over-approximation of the reachable set by bloating the simulation.

2. Figures are taken from [A. DonzDonzé & O. Maler, 2007].

Verification Algorithm

Algorithm 1: Simulation-based Verification for Delayed Dynamical Systems

input : The dynamics $\boldsymbol{f}(\mathbf{x}, \mathbf{u})$, delay term r, initial set $\mathcal{X}_0$, unsafe set $\mathcal{U}$, time bound T, precision ϵ.

/* initialization */

$\mathcal{R} \leftarrow \emptyset$; $\delta \leftarrow$ *dia*$(\mathcal{X}_0)/2$; $\tau \leftarrow \tau_0$;

$\mathcal{X} \leftarrow \delta$-Partition$(\mathcal{X}_0)$;

while $\mathcal{X} \neq \emptyset$ **do**

    if $\delta < \epsilon$ **then**

        return (UNKNOWN, $\mathcal{R}$);

    for $\mathcal{B}_\delta(\mathbf{x}_0) \in \mathcal{X}$ **do**

        $\langle \mathbf{t}, \mathbf{y}, \mathbf{d} \rangle \leftarrow$ Simulation$(\mathcal{B}_\delta(\mathbf{x}_0), \boldsymbol{f}(\mathbf{x}, \mathbf{u}), r, \tau, T)$;

        $\mathcal{T} \leftarrow \bigcup_{n=0}^{N-1}$ *conv*$(\mathcal{B}_{\mathbf{d}_n}(\mathbf{y}_n) \cup \mathcal{B}_{\mathbf{d}_{n+1}}(\mathbf{y}_{n+1}))$;

        if $\mathcal{T} \cap \mathcal{U} = \emptyset$ **then**

            $\mathcal{X} \leftarrow \mathcal{X} \backslash \mathcal{B}_\delta(\mathbf{x}_0)$; $\mathcal{R} \leftarrow \mathcal{R} \cup \mathcal{T}$;

        else if $\exists i.\ \mathcal{B}_{\mathbf{d}_i}(\mathbf{y}_i) \subseteq \mathcal{U}$ **then**

            return (UNSAFE, $\mathcal{T}$);

        else

            $\mathcal{X} \leftarrow \mathcal{X} \backslash \mathcal{B}_\delta(\mathbf{x}_0)$; $\mathcal{X} \leftarrow \mathcal{X} \cup \frac{\delta}{2}$-Partition$(\mathcal{B}_\delta(\mathbf{x}_0))$;

    $\delta \leftarrow \delta/2$;

return (SAFE, $\mathcal{R}$);

Outline

Local Error Bounds

$$E(t) = \begin{cases} d_0, & \text{if } t = 0, \\ E(t_i) + (t - t_i)e_{i+1}, & \text{if } t \in [t_i, t_{i+1}]. \end{cases}$$

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Validation Property :

$$\xi_{\mathbf{x}_0}(t) \in \mathcal{B}_{E(t)}\left(\frac{(t - t_i)\mathbf{y}_i + (t_{i+1} - t)\mathbf{y}_{i+1}}{t_{i+1} - t_i}\right), \text{ for each } t \in [t_i, t_{i+1}].$$

Simulation Algorithm

Algorithm 2: Simulation: a validated DDE solver producing rigorous bounds

input : The initial set $\mathcal{B}_\delta(\mathbf{x}_0)$, dynamics $\boldsymbol{f}(\mathbf{x}, \mathbf{u})$, delay term r, stepsize τ, time bound T.

output: A triple $\langle \mathbf{t}, \mathbf{y}, \mathbf{d} \rangle$, where the components represent lists, with the same length, respectively for the time points, numerical approximations (possibly multi-dimensional), and the rigorous local error bounds.

/* initializing the lists, whose indices start from -1 */

$\mathbf{t} \leftarrow [\![-\tau, 0]\!]$; $\mathbf{y} \leftarrow [\![\mathbf{x}_0, \mathbf{x}_0]\!]$; $\mathbf{d} \leftarrow [\![0, \delta]\!]$;

/* r has to be divisible by τ (in FP numbers) */

$n \leftarrow 0$; $m \leftarrow r/\tau$;

while $\mathbf{t}_n < T$ **do**

$\quad t_{n+1} \leftarrow \mathbf{t}_n + \tau$;

$\quad$ /* approximating y_{n+1} using *forward Euler method* */

$\quad y_{n+1} \leftarrow \mathbf{y}_n + \boldsymbol{f}(\mathbf{y}_n, \mathbf{y}_{n-m}) * \tau$;

$\quad$ /* computing error slope by constrained optimization, where σ is a positive slack constant */

$\quad e_n \leftarrow$ **Find** minimum e s.t.

$$\begin{cases} \|\boldsymbol{f}(\mathbf{x} + t * \mathbf{f}, \mathbf{u} + t * \mathbf{g}) - \boldsymbol{f}(\mathbf{y}_n, \mathbf{y}_{n-m})\| \leq e - \sigma, \text{ for} \\ \forall t \in [0, \tau] \\ \forall \mathbf{x} \in \mathcal{B}_{\mathbf{d}_n}(\mathbf{y}_n) \\ \forall \mathbf{u} \in \mathcal{B}_{\mathbf{d}_{n-m}}(\mathbf{y}_{n-m}) \\ \forall \mathbf{f} \in \mathcal{B}_e(\boldsymbol{f}(\mathbf{y}_n, \mathbf{y}_{n-m})) \\ \forall \mathbf{g} \in \mathcal{B}_{e_{n-m}}(\boldsymbol{f}(\mathbf{y}_{n-m}, \mathbf{y}_{n-2m})); \end{cases}$$

$\quad d_{n+1} \leftarrow \mathbf{d}_n + \tau e_n$;

$\quad$ /* updating the lists by appending the extrapolation */

$\quad \mathbf{t} \leftarrow [\![\mathbf{t}, t_{n+1}]\!]$; $\mathbf{y} \leftarrow [\![\mathbf{y}, y_{n+1}]\!]$; $\mathbf{d} \leftarrow [\![\mathbf{d}, d_{n+1}]\!]$;

$\quad n \leftarrow n + 1$;

return $\langle \mathbf{t}, \mathbf{y}, \mathbf{d} \rangle$;

Solving the Optimization by HySAT-II

$$\text{find } \min\{e \geq 0 \mid \forall x : \phi(x, e) \implies \psi(x, e)\}$$

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$$\Downarrow$$

$$\text{find } \max\{e \geq 0 \mid \exists x : \phi(x, e) \wedge \neg\psi(x, e)\}$$

Simulation Algorithm

Theorem (Correctness)

Suppose the maximum index of the lists is N, then $\forall t \in [0, T]$ *and* $\forall \mathbf{x} \in B_\delta(\mathbf{x}_0)$,

$$\xi_{\mathbf{x}}(t) \subseteq \bigcup_{n=0}^{N-1} conv(\mathcal{B}_{\mathbf{d}_n}(\mathbf{y}_n) \cup \mathcal{B}_{\mathbf{d}_{n+1}}(\mathbf{y}_{n+1})).$$

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Theorem (Completeness)

Suppose the function $\boldsymbol{f}$ is continuously differentiable in both arguments and the dynamical system is solvable for time interval $[0, T]$, then for any $\varepsilon > 0$, there exists δ, τ and σ such that the optimization problem has a solution e_n for all $n \leq \frac{T}{\tau}$, and moreover $\mathbf{d}_n \leq \varepsilon$.

Simulation Algorithm

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Further extension to simulations with variable stepsize.

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Delayed Logistic Equation

$$\dot{N}(t) = N(t)[1 - N(t - r)]$$

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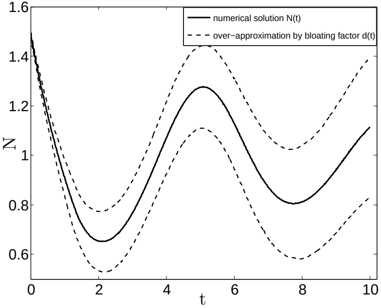

Figure : $\mathcal{X}_0 = \mathcal{B}_{0.01}(1.49)$, $r = 1.3$, $\tau_0 = 0.01$, $T = 10$s.

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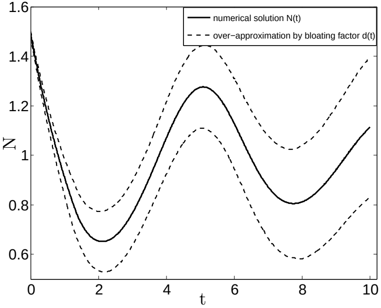

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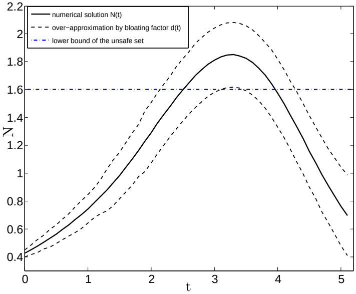
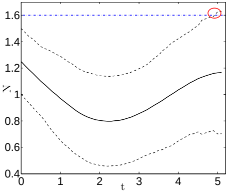
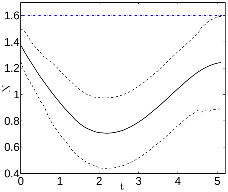


Figure : Over-approximation rigorously proving unsafe, with $r = 1.7$, $\mathcal{X}_0 = \mathcal{B}_{0.025}(0.425)$, $\tau_0 = 0.1$, $T = 5$s, $\mathcal{U} = \{N | N > 1.6\}$.

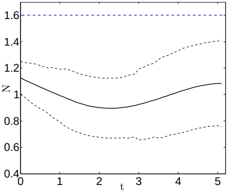
Delayed Logistic Equation



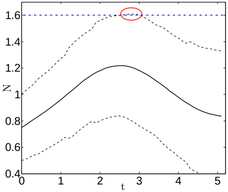
(a) An initial over-approximaion of trajectories starting from $\mathcal{B}_{0.225}(1.25)$. It overlaps with the unsafe set (s. circle). Initial set is consequently split (cf. Figs. 3b, 3c).



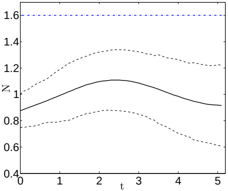
(b) All trajectories starting from $\mathcal{B}_{0.125}(1.375)$ are proven safe within the time bound, as the over-approximation does not intersect with the unsafe set.



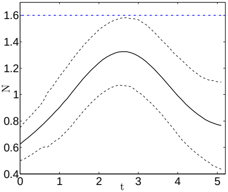
(c) Initial state set $\mathcal{B}_{0.125}(1.125)$ is verified to be safe as well.



(d) $\mathcal{B}_{0.25}(0.75)$ yields overlap w. unsafe; the ball is partitioned again (Figs. 3e, 3f).



(e) All trajectories originating from $\mathcal{B}_{0.125}(0.875)$ are provably safe.



(f) All trajectories originating from $\mathcal{B}_{0.125}(0.625)$ are provably safe as well.

Fig. 3: The logistic system is proven safe through 6 rounds of simulation with base stepsize $\tau_0 = 0.1$. Delay $r = 1.3$, initial state set $\mathcal{X}_0 = \{N | N \in [0.5, 1.5]\}$, time bound $T = 5$s, unsafe set $\{N | N > 1.6\}$.

Delayed Microbial Growth

$$\begin{cases} \dot{S}(t) = 1 - S(t) - f(S(t))x(t) \\ \dot{x}(t) = \mathrm{e}^{-r} f(S(t-r))x(t-r) - x(t) \end{cases}$$

Delayed Microbial Growth

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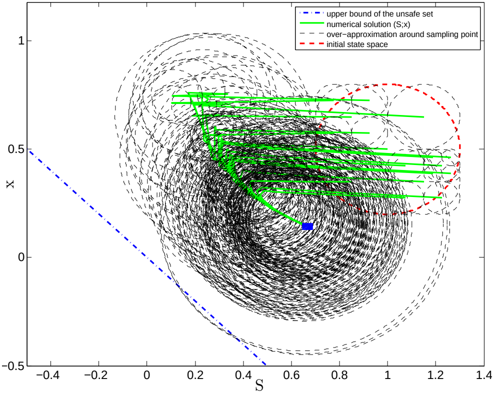

Figure : The microbial system is proven safe by 17 rounds of simulation with $\tau_0 = 0.45$. Here, $f(S) = 2eS/(1+S)$, $r = 0.9$, $\mathcal{X}_0 = \mathcal{B}_{0.3}((1;0.5))$, $\mathcal{U} = \{(S;x) | S + x < 0\}$, $T = 8$s.

Outline

Concluding Remarks

- A validated numerical solver for delay differential equations.
- A sound and robustly complete algorithm for automated formal verification of time-bounded reachability properties of a class of systems that feature delayed differential dynamics governed by DDEs with multiple delays.
- A prototypical implementation of the simulator, by which we have successfully demonstrated the method on several benchmark systems involving delayed differential dynamics.
- Forthcoming research : higher-order *Runge-Kutta methods* ; unbounded verification by Taylor-enclosures ; conformance testing.