Interpolant Synthesis for Quadratic Polynomial Inequalities and Combination with *EUF*

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   - Generalization of Motzkin's transposition theorem
   - Concave quadratic polynomials

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   - $\mathbf{SOS}$ ($\mathbf{NSOSC}$ not satisifed) : equalities from expressions in a sum of squares being equal to 0.

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3. Combination with uninterpreted function symbols (*EUF*)
   - similar to the linear case
4. Concluding remarks

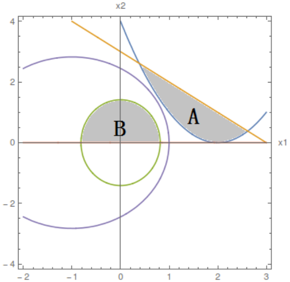
Overview of the idea

Example (running example)

Consider two formulas A and B with $A \wedge B \models \perp$, where

$$A := -{x_1}^2 + 4x_1 + x_2 - 4 \geq 0 \wedge -x_1 - x_2 + 3 - y^2 > 0,$$
$$B := -3{x_1}^2 - {x_2}^2 + 1 \geq 0 \wedge x_2 - z^2 \geq 0$$

We aim to generate an interpolant I for A and B, on the common variables (x_1 and x_2), such that $A \models I$ and $I \wedge B \models \perp$.



An intuitive description of a candidate interpolant is as the purple curve in the above right figure, which separates A and B in the panel of x_1 and x_2.

Overview of the idea

- A polynomial time algorithm for generating interpolants from mutually contradictory conjunctions of concave quadratic polynomial inequalities over the reals :

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  - If no nonpositive constant combination of nonstrict inequalities is a sum of squares polynomial, an interpolant a la McMillan can be generated essentially using the linearization of quadratic polynomials.
  - Otherwise, linear equalities relating variables are deduced, resulting to interpolation subproblems with fewer variables on which the algorithm is recursively applied.

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- A polynomial time algorithm for generating interpolants from mutually contradictory conjunctions of concave quadratic polynomial inequalities over the reals :
  - If no nonpositive constant combination of nonstrict inequalities is a sum of squares polynomial, an interpolant a la McMillan can be generated essentially using the linearization of quadratic polynomials.
  - Otherwise, linear equalities relating variables are deduced, resulting to interpolation subproblems with fewer variables on which the algorithm is recursively applied.
- An algorithm for generating interpolants for the combination of quantifier-free theory of concave quadratic polynomial inequalities and equality theory over uninterpreted function symbols (*EUF*).

Preliminaries

Theorem (Motzkin's transposition theorem)

Let A and B be matrices and let $\vec{\alpha}$ and $\vec{\beta}$ be column vectors. Then there exists a vector $\mathbf{x}$ with $A\mathbf{x} - \vec{\alpha} \geq 0$ and $B\mathbf{x} - \vec{\beta} > 0$, iff for all row vectors $\mathbf{y}, \mathbf{z} \geq 0$:

$$(i) \text{ if } \mathbf{y}A + \mathbf{z}B = 0 \text{ then } \mathbf{y}\vec{\alpha} + \mathbf{z}\vec{\beta} \leq 0;$$
$$(ii) \text{ if } \mathbf{y}A + \mathbf{z}B = 0 \text{ and } \mathbf{z} \neq 0 \text{ then } \mathbf{y}\vec{\alpha} + \mathbf{z}\vec{\beta} < 0.$$

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(ii)\ &\text{if } \mathbf{y}A + \mathbf{z}B = 0 \text{ and } \mathbf{z} \neq 0 \text{ then } \mathbf{y}\vec{\alpha} + \mathbf{z}\vec{\beta} < 0.
\end{aligned}
$$

Corollary

Let $A \in \mathbb{R}^{r \times n}$ and $B \in \mathbb{R}^{s \times n}$ be matrices and $\vec{\alpha} \in \mathbb{R}^r$ and $\vec{\beta} \in \mathbb{R}^s$ be column vectors, where $A_i, i = 1, \ldots, r$ is the ith row of A and $B_j, j = 1, \ldots, s$ is the jth row of B. There does not exist a vector $\mathbf{x}$ with $A\mathbf{x} - \vec{\alpha} \geq 0$ and $B\mathbf{x} - \vec{\beta} > 0$, iff there exist real numbers $\lambda_1, \ldots, \lambda_r \geq 0$ and $\eta_0, \eta_1, \ldots, \eta_s \geq 0$ such that

$$
\sum_{i=1}^{r} \lambda_i (A_i \mathbf{x} - \alpha_i) + \sum_{j=1}^{s} \eta_j (B_j \mathbf{x} - \beta_j) + \eta_0 \equiv 0 \;\; \textit{with} \sum_{j=0}^{s} \eta_j = 1. \tag{1}
$$

Concave quadratic polynomials

Definition (Concave Quadratic)

A polynomial $f \in \mathbb{R}[\mathbf{x}]$ is called *concave quadratic (CQ)* if the following two conditions hold :

- f has total degree at most 2, i.e., it has the form $f = \mathbf{x}^T A \mathbf{x} + 2\vec{\alpha}^T \mathbf{x} + a$, where A is a real symmetric matrix, $\vec{\alpha}$ is a column vector and $a \in \mathbb{R}$;
- the matrix A is negative semi-definite, written as $A \preceq 0$.

Example

Take $f = -3{x_1}^2 - {x_2}^2 + 1$ in the running example, which is from the ellipsoid domain and can be expressed as

$$f = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 1.$$

The corresponding $A = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} \preceq 0$. Thus, f is CQ.

Concave quadratic polynomials

- If $f \in \mathbb{R}[\mathbf{x}]$ is linear, then f is CQ because its total degree is 1 and the corresponding A is 0 which is of course negative semi-definite.

Concave quadratic polynomials

- If $f \in \mathbb{R}[\mathbf{x}]$ is linear, then f is CQ because its total degree is 1 and the corresponding A is 0 which is of course negative semi-definite.
- A quadratic polynomial $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + 2\vec{\alpha}^T\mathbf{x} + a$ can also be represented as an inner product of matrices, i.e., $\left\langle P, \begin{pmatrix} 1 & \mathbf{x}^T \\ \mathbf{x} & \mathbf{x}\mathbf{x}^T \end{pmatrix} \right\rangle$, where $P = \begin{pmatrix} a & \alpha^T \\ \alpha & A \end{pmatrix}$.

Linearization of CQ polynomials

Definition (Linearization)

Given a quadratic polynomial $f(\mathbf{x}) = \left\langle P, \begin{pmatrix} 1 & \mathbf{x}^T \\ \mathbf{x} & \mathbf{x}\mathbf{x}^T \end{pmatrix} \right\rangle$, its *linearization* is defined as $f(\mathbf{x}) = \left\langle P, \begin{pmatrix} 1 & \mathbf{x}^T \\ \mathbf{x} & \vec{X} \end{pmatrix} \right\rangle$, where $\begin{pmatrix} 1 & \mathbf{x}^T \\ \mathbf{x} & \vec{X} \end{pmatrix} \succeq 0$.

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let

$$K \quad \hat{=} \quad \{\mathbf{x} \in \mathbb{R}^n \mid f_1(\mathbf{x}) \geq 0, \ldots, f_r(\mathbf{x}) \geq 0, g_1(\mathbf{x}) > 0, \ldots, g_s(\mathbf{x}) > 0\}, \tag{2}$$

$$K_1 \quad \hat{=} \quad \{\mathbf{x} \mid \begin{pmatrix} 1 & \mathbf{x}^T \\ \mathbf{x} & \vec{X} \end{pmatrix} \succeq 0, \ \wedge_{i=1}^{r} \left\langle P_i, \begin{pmatrix} 1 & \mathbf{x}^T \\ \mathbf{x} & \vec{X} \end{pmatrix} \right\rangle \geq 0,$$

$$\wedge_{j=1}^{s} \left\langle Q_j, \begin{pmatrix} 1 & \mathbf{x}^T \\ \mathbf{x} & \vec{X} \end{pmatrix} \right\rangle > 0, \text{ for some } \vec{X}\}, \tag{3}$$

Linearization of CQ polynomials

Theorem

Let $f_1, \ldots, f_r$ and $g_1, \ldots, g_s$ be CQ polynomials, K and K_1 as above, then $K = K_1$.

Therefore, when f_is and g_js are CQ, the CQ polynomial inequalities can be transformed equivalently to a set of linear inequality constraints and a positive semi-definite constraint.

Problem formulation

Problem 1

Given two formulas ϕ and ψ on n variables with $\phi \wedge \psi \models \bot$, where

$$
\begin{aligned}
\phi &= f_1 \geq 0 \wedge \ldots \wedge f_{r_1} \geq 0 \wedge g_1 > 0 \wedge \ldots \wedge g_{s_1} > 0, \\
\psi &= f_{r_1+1} \geq 0 \wedge \ldots \wedge f_r \geq 0 \wedge g_{s_1+1} > 0 \wedge \ldots \wedge g_s > 0,
\end{aligned}
$$

in which $f_1, \ldots, f_r, g_1, \ldots, g_s$ are all CQ, develop an algorithm to generate a (reverse) Craig interpolant I for ϕ and ψ, on the common variables of ϕ and ψ, such that $\phi \models I$ and $I \wedge \psi \models \bot$.

$\mathbf{x} = (x_1, \ldots, x_d)$, $\mathbf{y} = (y_1, \ldots, y_u)$ and $\mathbf{z} = (z_1, \ldots, z_v)$, where $d + u + v = n$.

NSOSC Condition

Definition (NSOSC)

Formulas ϕ and ψ in Problem 1, satisfy the non-existence of an SOS polynomial condition (NSOSC) iff there do not exist $\delta_1 \geq 0, \ldots, \delta_r \geq 0$, s.t. $-(\delta_1 f_1 + \ldots + \delta_r f_r)$ is a non-zero SOS.

Example

Formulas A and B in the running example do not satisfy NSOSC, since there exist $\delta_1 = 1, \delta_2 = 1, \delta_3 = 1$, s.t.

$$
\begin{aligned}
& -(\delta_1(-{x_1}^2 + 4x_1 + x_2 - 4) + \delta_2(-3{x_1}^2 - {x_2}^2 + 1) + \delta_3(x_2 - z^2)) \\
& = (2x_1 - 1)^2 + (x_2 - 1)^2 + z^2
\end{aligned}
$$

is a non-zero SOS.

Generalization of Motzkin's theorem

Theorem (Generalization of Motzkin's theorem)

Let $f_1, \ldots, f_r, g_1, \ldots, g_s$ be CQ polynomials whose conjunction is unsatisfiable. If the condition $\mathbf{NSOSC}$ holds, then there exist $\lambda_i \geq 0$ ($i = 1, \cdots, r$), $\eta_j \geq 0$ ($j = 0, 1, \cdots, s$) and a quadratic SOS polynomial h of the form $(l_1)^2 + \ldots + (l_k)^2$ where l_i are linear expressions in x, y, z. , s.t.

$$\sum_{i=1}^{r} \lambda_i f_i + \sum_{j=1}^{s} \eta_j g_j + \eta_0 + h \equiv 0, \tag{4}$$

$$\eta_0 + \eta_1 + \ldots + \eta_s = 1. \tag{5}$$

Using this generalization, an interpolant for ϕ and ψ is generated from the SOS polynomial h by splitting it into two SOS polynomials.

When NSOSC is satisfied

Theorem

Let ϕ and ψ as defined in Problem 1 with $\phi \wedge \varphi \models \perp$, which satisfy NSOSC. Then there exist $\lambda_i \geq 0$ ($i = 1, \cdots, r$), $\eta_j \geq 0$ ($j = 0, 1, \cdots, s$) and two quadratic SOS polynomial $h_1 \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$ and $h_2 \in \mathbb{R}[\mathbf{x}, \mathbf{z}]$ s.t.

$$\sum_{i=1}^{r} \lambda_i f_i + \sum_{j=1}^{s} \eta_j g_j + \eta_0 + h_1 + h_2 \equiv 0, \tag{6}$$

$$\eta_0 + \eta_1 + \ldots + \eta_s = 1. \tag{7}$$

Let $I = \sum_{i=1}^{r_1} \lambda_i f_i + \sum_{j=1}^{s_1} \eta_j g_j + \eta_0 + h_1 \in \mathbb{R}[\mathbf{x}]$. Then, if $\sum_{j=0}^{s_1} \eta_j > 0$, then $I > 0$ is an interpolant ; otherwise $I \geq 0$ is an interpolant.

Computing interpolant using semi-definite programming

Let $W = \begin{pmatrix} 1 & \mathbf{x}^T & \mathbf{y}^T & \mathbf{z}^T \\ \mathbf{x} & \mathbf{x}\mathbf{x}^T & \mathbf{x}\mathbf{y}^T & \mathbf{x}\mathbf{z}^T \\ \mathbf{y} & \mathbf{y}\mathbf{x}^T & \mathbf{y}\mathbf{y}^T & \mathbf{y}\mathbf{z}^T \\ \mathbf{z} & \mathbf{z}\mathbf{x}^T & \mathbf{z}\mathbf{y}^T & \mathbf{z}\mathbf{z}^T \end{pmatrix}$, $f_i = \langle P_i, W \rangle, \;\; g_j = \langle Q_j, W \rangle$, where P_i and Q_j are $(n+1) \times (n+1)$ matrices, $h_1 = \langle M, W \rangle, \;\; h_2 = \langle \hat{M}, W \rangle$, and $M = (M_{ij})_{4\times 4}$, $\hat{M} = (\hat{M}_{ij})_{4\times 4}$ with appropriate dimensions, e.g., $M_{12} \in \mathbb{R}^{1\times d}$ and $\hat{M}_{34} \in \mathbb{R}^{u\times v}$.

Then, with $\mathbf{NSOSC}$, computing the interpolant is reduced to the following $\mathbf{SDP}$ feasibility problem :

Computing interpolant using semi-definite programming

Find : $\lambda_1, \ldots, \lambda_r, \eta_1, \ldots, \eta_s \in \mathbb{R}$, $M, \hat{M} \in \mathbb{R}^{(n+1)\times(n+1)}$ subject to

$$\begin{cases} \sum_{i=1}^{r} \lambda_i P_i + \sum_{j=1}^{s} \eta_j Q_j + \eta_0 E_{1,1} + M + \hat{M} = 0, \sum_{j=1}^{s} \eta_j = 1, \\ M_{41} = (M_{14})^T = 0, M_{42} = (M_{24})^T = 0, M_{43} = (M_{34})^T = 0, M_{44} = 0, \\ \hat{M}_{31} = (\hat{M}_{13})^T = 0, \hat{M}_{32} = (\hat{M}_{23})^T = 0, \hat{M}_{33} = 0, \hat{M}_{34} = (\hat{M}_{43})^T = 0, \\ M \succeq 0, \hat{M} \succeq 0, \lambda_i \geq 0, \eta_j \geq 0, \text{ for } i = 1, \ldots, r, j = 1, \ldots, s, \end{cases}$$

where $E_{(1,1)}$ is a $(n+1) \times (n+1)$ matrix, whose all other entries are 0 except for $(1,1)$ entry being 1.

Generating interpolants when NSOSC holds

Algorithm 1: IGFCH

input : Two formulas ϕ, ψ with **NSOSC** and $\phi \wedge \psi \models \perp$, where
$\phi = f_1 \geq 0 \wedge \ldots \wedge f_{r_1} \geq 0 \wedge g_1 > 0 \wedge \ldots \wedge g_{s_1} > 0$,
$\psi = f_{r_1+1} \geq 0 \wedge \ldots \wedge f_r \geq 0 \wedge g_{s_1+1} > 0 \wedge \ldots \wedge g_s > 0$,
$f_1, \ldots, f_r, g_1, \ldots, g_s$ are all concave quadratic polynomials,
$f_1, \ldots, f_{r_1}, g_1, \ldots, g_{s_1} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$, $f_{r_1+1}, \ldots, f_r, g_{s_1+1}, \ldots, g_s \in \mathbb{R}[\mathbf{x}, \mathbf{z}]$

output: A formula I to be an interpolant for ϕ and ψ

Find $\lambda_1, \ldots, \lambda_r \geq 0, , \eta_0, \eta_1, \ldots, \eta_s \geq 0, h_1 \in \mathbb{R}[\mathbf{x}, \mathbf{y}], h_2 \in \mathbb{R}[\mathbf{x}, \mathbf{z}]$ by SDP s.t.

$$\sum_{i=1}^{r} \lambda_i f_i + \sum_{j=1}^{s} \eta_j g_j + \eta_0 + h_1 + h_2 \equiv 0,$$

$$\eta_0 + \eta_1 + \ldots + \eta_s = 1,$$

h_1, h_2 are SOS polynomials;

/* This is essentially a **SDP** problem, see Section 4.2 */

$f := \sum_{i=1}^{r_1} \lambda_i f_i + \sum_{j=1}^{s_1} \eta_j g_j + \eta_0 + h_1$;

if $\sum_{j=0}^{s_1} \eta_j > 0$ **then** $I := (f > 0)$; **else** $I := (f \geq 0)$;

return I

When NSOSC is not satisfied

If ϕ and ψ do not satisfy $\mathbf{NSOSC}$, i.e., an SOS polynomial $h(\mathbf{x}, \mathbf{y}, \mathbf{z}) = -(\sum_{i=1}^{r} \lambda_i f_i)$ can be computed which can be split into two SOS polynomials $h_1(\mathbf{x}, \mathbf{y})$ and $h_2(\mathbf{x}, \mathbf{z})$ as discussed previously. Then an SOS polynomial $f(\mathbf{x})$ such that $\phi \models f(\mathbf{x}) \geq 0$ and $\psi \models -f(\mathbf{x}) \geq 0$ can be constructed as

$$f(\mathbf{x}) = (\sum_{i=1}^{r_1} \delta_i f_i) + h_1 = -(\sum_{i=r_1+1}^{r} \delta_i f_i) - h_2, \delta_i \geq 0.$$

Lemma

If Problem 1 does not satisfy $\mathbf{NSOSC}$*, there exists* $f \in \mathbb{R}[\mathbf{x}]$*, s.t.* $\phi \Leftrightarrow \phi_1 \vee \phi_2$ *and* $\psi \Leftrightarrow \psi_1 \vee \psi_2$*, where,*

$$\phi_1 = (f > 0 \wedge \phi), \phi_2 = (f = 0 \wedge \phi), \psi_1 = (-f > 0 \wedge \psi), \psi_2 = (f = 0 \wedge \psi). \quad (8)$$

When NSOSC is not satisfied

Using the previous lemma, an interpolant I for ϕ and ψ can be constructed from an interpolant $I_{2,2}$ for ϕ_2 and ψ_2.

Theorem

With $\phi, \psi, \phi_1, \phi_2, \psi_1, \psi_2$ as in previous Lemma, from an interpolant $I_{2,2}$ for ϕ_2 and ψ_2, $I := (f > 0) \vee (f \geq 0 \wedge I_{2,2})$ is an interpolant for ϕ and ψ.

When NSOSC is not satisfied

Using the previous lemma, an interpolant I for ϕ and ψ can be constructed from an interpolant $I_{2,2}$ for ϕ_2 and ψ_2.

Theorem

With $\phi, \psi, \phi_1, \phi_2, \psi_1, \psi_2$ as in previous Lemma, from an interpolant $I_{2,2}$ for ϕ_2 and ψ_2, $I := (f > 0) \vee (f \geq 0 \wedge I_{2,2})$ is an interpolant for ϕ and ψ.

If h and hence h_1, h_2 have a positive constant $a_{n+1} > 0$, then f cannot be 0, implying that ϕ_2, ψ_2 are $\perp$. We thus have :

Theorem

With $\phi, \psi, \phi_1, \phi_2, \psi_1, \psi_2$ as in previous Lemma and h has $a_{n+1} > 0$, $f > 0$ is an interpolant for ϕ and ψ.

In case h does not have a constant, i.e., $a_{n+1} = 0$, elimination of variables can be recursively performed to terminate the algorithm.

Generating interpolants for CQI

Algorithm 2: IGFQC

input : Two formulas ϕ, ψ with $\phi \wedge \psi \models \bot$, where
$\phi = f_1 \geq 0 \wedge \ldots \wedge f_{r_1} \geq 0 \wedge g_1 > 0 \wedge \ldots \wedge g_{s_1} > 0$,
$\psi = f_{r_1+1} \geq 0 \wedge \ldots \wedge f_r \geq 0 \wedge g_{s_1+1} > 0 \wedge \ldots \wedge g_s > 0$,
$f_1, \ldots, f_r, g_1, \ldots, g_s$ are all CQ polynomials,
$f_1, \ldots, f_{r_1}, g_1, \ldots, g_{s_1} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$, and $f_{r_1+1}, \ldots, f_r, g_{s_1+1}, \ldots, g_s \in \mathbb{R}[\mathbf{x}, \mathbf{z}]$

output: A formula I to be an interpolant for ϕ and ψ

if *Var*$(\phi) \subseteq$ *Var*(ψ) **then** $I := \phi$; **return** I;
Find $\delta_1, \ldots, \delta_r \geq 0, h \in \mathbb{R}[\mathbf{x}, \mathbf{y}, \mathbf{z}]$ by SDP s.t. $\sum_{i=1}^{r} \delta_i f_i + h \equiv 0$ and h is SOS;
/* Check the condition **NSOSC** */
if *no solution* **then** $I :=$ **IGFCH**(ϕ, ψ); **return** I;
/* **NSOSC** holds */
Construct $h_1 \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$ and $h_2 \in \mathbb{R}[\mathbf{x}, \mathbf{z}]$ with the forms (H1) and (H2);
$f := \sum_{i=1}^{r_1} \delta_i f_i + h_1 = -\sum_{i=r_1+1}^{r} \delta_i f_i - h_2$;
Construct ϕ' and ψ' using Theorem 6 and Theorem 7 by eliminating variables due to $h_1 = h_2 = 0$;
$I' =$ **IGFQC**(ϕ', ψ');
$I := (f > 0) \vee (f \geq 0 \wedge I')$;
return I

Generating interpolants for CQI

Example

Recall the running example where

$$\begin{aligned} h &= (2x_1 - 1)^2 + (x_2 - 1)^2 + z^2 \\ &= \underbrace{\frac{1}{2}((2x_1 - 1)^2 + (x_2 - 1)^2)}_{h_1} + \underbrace{\frac{1}{2}((2x_1 - 1)^2 + (x_2 - 1)^2) + z^2}_{h_2} \end{aligned}$$

$$\begin{aligned} f &= \delta_1(-{x_1}^2 + 4x_1 + x_2 - 4) + h_1 \\ &= -3 + 2x_1 + {x_1}^2 + \frac{1}{2}{x_2}^2 \end{aligned}$$

We construct A' from A by setting $x_1 = \frac{1}{2}, x_2 = 1$ derived from $h_1 = 0$; similarly B' is constructed by setting $x_1 = \frac{1}{2}, x_2 = 1, z = 0$ in B as derived from $h_2 = 0$. It follows that, $A' := B' := \perp$ Thus, $I(A', B') := (0 > 0)$ is an interpolant for (A', B').

An interpolant for A and B is thus $(f(x) > 0) \vee (f(x) = 0 \wedge I(A', B'))$, i.e.

$$-3 + 2x_1 + {x_1}^2 + \frac{1}{2}{x_2}^2 > 0.$$

which corresponds to the purple curve mentioned previously.

Combination with EUF

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- $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$: a finite set of uninterpreted function symbols in *EUF*;
- $\Omega_{12} = \Omega_1 \cup \Omega_2, \Omega_{13} = \Omega_1 \cup \Omega_3$;

Combination with EUF

- $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$: a finite set of uninterpreted function symbols in *EUF*;
- $\Omega_{12} = \Omega_1 \cup \Omega_2$, $\Omega_{13} = \Omega_1 \cup \Omega_3$;
- $\mathbb{R}[\mathbf{x}, \mathbf{y}, \mathbf{z}]^{\Omega}$: the extension of $\mathbb{R}[\mathbf{x}, \mathbf{y}, \mathbf{z}]$ in which polynomials can have terms built using function symbols in Ω and variables in $\mathbf{x}, \mathbf{y}, \mathbf{z}$.

Combination with EUF

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Problem 2

Suppose two formulas ϕ and ψ with $\phi \wedge \psi \models \bot$, where

$$\phi = f_1 \geq 0 \wedge \ldots \wedge f_{r_1} \geq 0 \wedge g_1 > 0 \wedge \ldots \wedge g_{s_1} > 0,$$
$$\psi = f_{r_1+1} \geq 0 \wedge \ldots \wedge f_r \geq 0 \wedge g_{s_1+1} > 0 \wedge \ldots \wedge g_s > 0,$$

where $f_1, \ldots, f_r, g_1, \ldots, g_s$ are all CQ polynomials, $f_1, \ldots, f_{r_1}, g_1, \ldots, g_{s_1} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^{\Omega_{12}}$, $f_{r_1+1}, \ldots, f_r, g_{s_1+1}, \ldots, g_s \in \mathbb{R}[\mathbf{x}, \mathbf{z}]^{\Omega_{13}}$, the goal is to generate an interpolant I for ϕ and ψ, expressed using the common symbols $\mathbf{x}, \Omega_1$, i.e., I includes only polynomials in $\mathbb{R}[\mathbf{x}]^{\Omega_1}$.

Sketch of the idea (Algorithm $\mathbf{IGFQCE}$)

1. Flatten and purify the formulas ϕ and ψ as $\overline{\phi}$ and $\overline{\psi}$ by introducing fresh variables for each term with uninterpreted symbols as well as for the terms with uninterpreted symbols.

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   $$N = \{\bigwedge_{k=1}^{n} c_k = b_k \rightarrow c = b \mid \omega(c_1, \dots, c_n) = c \in D, \omega(b_1, \dots, b_n) = b \in D\},$$

   where D consists of unit clauses of the form $\omega(c_1, \dots, c_n) = c$, with $c_1, \dots, c_n$ be variables and $\omega \in \Omega$.

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where D consists of unit clauses of the form $\omega(c_1, \ldots, c_n) = c$, with $c_1, \ldots, c_n$ be variables and $\omega \in \Omega$.
3. Partition N into N_ϕ, N_ψ, and N_{mix} with all symbols in N_ϕ, N_ψ appearing in $\overline{\phi}$, $\overline{\psi}$, respectively, and N_{mix} consisting of symbols from both $\overline{\phi}, \overline{\psi}$.
$$\phi \wedge \psi \models \bot \text{ iff } \overline{\phi} \wedge \overline{\psi} \wedge D \models \bot \text{ iff } (\overline{\phi} \wedge N_\phi) \wedge (\overline{\psi} \wedge N_\psi) \wedge N_{\text{mix}} \models \bot. \quad (9)$$

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4. Generate interpolant : Notice that $(\overline{\phi} \wedge N_\phi) \wedge (\overline{\psi} \wedge N_\psi) \wedge N_{\text{mix}} \models \bot$ has no uninterpreted function symbols. If N_{mix} can be replaced by N_{sep}^{ϕ} and N_{sep}^{ψ} as in [Rybalchenko & Sofronie-Stokkermans 10] using separating terms, then $\mathbf{IGFQC}$ can be applied. An interpolant generated for this problem can be used to generate an interpolant for ϕ, ψ after uniformly replacing all new symbols by their corresponding expressions from D.

An illustrating example

Example

$$\begin{aligned}\phi :=& (f_1 = -(y_1 - x_1 + 1)^2 - x_1 + x_2 \geq 0) \wedge (y_2 = \alpha(y_1) + 1) \\ & \wedge (g_1 = -x_1^2 - x_2^2 - y_2^2 + 1 > 0), \\ \psi :=& (f_2 = -(z_1 - x_2 + 1)^2 + x_1 - x_2 \geq 0) \wedge (z_2 = \alpha(z_1) - 1) \\ & \wedge (g_2 = -x_1^2 - x_2^2 - z_2^2 + 1 > 0).\end{aligned}$$

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1. Flattening and purification gives

$$\overline{\phi} := (f_1 \geq 0 \land y_2 = y + 1 \land g_1 > 0), \quad \overline{\psi} := (f_2 \geq 0 \land z_2 = z - 1 \land g_2 > 0).$$

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   where $D = \{y = \alpha(y_1), z = \alpha(z_1)\}$, $N = (y_1 = z_1 \rightarrow y = z)$.

2. **NSOSC** is not satisfied, since $h = -f_1 - f_2 = (y_1 - x_1 + 1)^2 + (z_1 - x_2 + 1)^2$ is an SOS. $h_1 = (y_1 - x_1 + 1)^2$, $h_2 = (z_1 - x_2 + 1)^2$. This gives

$$f := f_1 + h_1 = -f_2 - h_2 = -x_1 + x_2.$$

An illustrating example

3 An interpolant for ϕ, ψ is an interpolant of $((\phi \wedge f > 0) \vee (\phi \wedge f = 0))$ and $((\psi \wedge -f > 0) \vee (\phi \wedge f = 0))$ which simplifies to : $(f > 0) \vee (f \geq 0 \wedge I_2)$ where I_2 is an interpolant for $\phi \wedge f = 0$ and $\psi \wedge f = 0$. Substituting $\phi \wedge f = 0 \models y_1 = x_1 - 1$ and $\psi \wedge f = 0 \models z_1 = x_2 - 1$ into $\overline{\phi}$ and $\overline{\psi}$, we get

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4. Recursively call $\mathbf{IGFQCE}$ until $\mathbf{NSOSC}$ is satisfied. $y_1 = z_1$ is deduced from linear inequalities in $\overline{\phi'}$ and $\overline{\psi'}$, and separating terms for y_1, z_1 are constructed :

$$\overline{\phi'} \models x_1 - 1 \leq y_1 \leq x_2 - 1, \quad \overline{\psi'} \models x_2 - 1 \leq z_1 \leq x_1 - 1.$$

Let $t = \alpha(x_2 - 1)$, then separate $y_1 = z_1 \rightarrow y = z$ into two parts :

$$y_1 = t^+ \rightarrow y = t, \quad t^+ = z_1 \rightarrow t = z.$$

Add them to $\overline{\phi'}$ and $\overline{\psi'}$ respectively, we have

$$\overline{\phi'}_1 := -x_1 + x_2 \geq 0 \wedge y_2 = y + 1 \wedge g_1 > 0 \wedge y_1 = x_1 - 1 \wedge y_1 = x_2 - 1 \rightarrow y = t,$$
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An illustrating example

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Since $\overline{\phi'}_3 = \textit{false}$, then $\overline{\phi'}_1 = \overline{\phi'}_2 \vee \overline{\phi'}_4$. Then find interpolant $I(\overline{\phi'}_2, \overline{\psi'}_1)$ and $I(\overline{\phi'}_4, \overline{\psi'}_1)$.

An illustrating example

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5. Finally we conclude that $I(\overline{\phi'}_2, \overline{\psi'}_1) \vee I(\overline{\phi'}_4, \overline{\psi'}_1)$ is an interpolant.

Implementation

We have implemented the presented algorithms in *Mathematica* to synthesize interpolation for concave quadratic polynomial inequalities as well as their combination with *EUF*. To deal with SOS solving and semi-definite programming, the Matlab-based optimization tool *Yalmip* and the SDP solver *SDPT3* are invoked for assistant solving.

Evaluation results

Example	Type	Time (sec)			
		CLP-Prover	Foci	CSIsat	Our Approach
Exp.1	NLA	--	--	--	0.003
Exp.2	NLA+*EUF*	--	--	--	0.036
Exp.3	NLA	--	--	--	0.014
Exp.4	NLA	--	--	--	0.003
Exp.5	LA	0.023	×	0.003	0.003
Exp.6	LA+*EUF*	0.025	0.006	0.007	0.003
Exp.7	Ellipsoid	--	--	--	0.002
Exp.8	Ellipsoid	--	--	--	0.002
Exp.9	Octagon	0.059	×	0.004	0.004
Exp.10	Octagon	0.065	×	0.004	0.004

-- means interpolant generation fails, and × specifies particularly wrong answers (satisfiable).

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- Dai et al. [L. Dai, B. Xia & N. Zhan 13] provided an approach to constructing non-linear interpolants based on *semi-definite programming*.

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