Line Segment Sampling with Blue-Noise Properties : Supplementary Material

This appendix contains preliminaries on blue-noise point sampling [Cook 1986] and the power spectrum analysis for uniform jittering in Section 3.3.

A Uniform sampling

Uniformly distributed point samples provide low discrepancy by which signals with frequency content below the Nyquist limit ν_N can be perfectly reconstructed. However, functions with frequency above ν_N will be translated to low frequency and generate aliasing artifacts. The left column of Fig. 1 shows the corresponding process which will be described in the following.

Taking $f(x)$ for example, the interval p between two adjacent samples determines the Nyquist limit $\nu_N = \frac{1}{2p}$. The uniform samples are represented by a shah function whose frequency content is a scaled shah function with an inverse interval as in Fig. 1 (c):

$$\text{III}_p(x) = \sum_{k=-\infty}^{\infty} \delta(x - kp) \tag{1}$$

$$\mathcal{F}(\text{III}_p(x)) = \frac{1}{p}\text{III}_{\frac{1}{p}}(\omega), \tag{2}$$

where $\delta(x)$ is the impulse function, p is the interval between two adjacent samples, $\mathcal{F}$ represents the Fourier transform operator. The operation of sampling is to uniformly duplicate the function in the frequency domain $F(\omega) = \mathcal{F}(f(x))$ as in Fig. 1 (a) with every interval of $\frac{1}{p}$ as in Fig. 1 (e):

$$f(x)\,\text{III}_p(x) = \sum_{k=-\infty}^{\infty} f(kp)\,\delta(x - kp) \tag{3}$$

$$F(\omega) \otimes \frac{1}{p}\text{III}_{\frac{1}{p}}(\omega) = \frac{1}{p}\sum_{k=-\infty}^{\infty} F\left(\omega - \frac{k}{p}\right). \tag{4}$$

If $F(\omega)$ is non-zero only within $(-\nu_N, \nu_N)$, the function can be perfectly reconstructed by the normalized box function of Fig. 1 (b) in the frequency domain:

$$F(\omega) = \left(F(\omega) \otimes \frac{1}{p}\text{III}_{\frac{1}{p}}(\omega)\right)\Pi_{\frac{1}{p}}(\omega) \tag{5}$$

$$\Pi_{\frac{1}{p}}(\omega) = \begin{cases} p, & \omega \in \left(-\frac{1}{2p}, \frac{1}{2p}\right) \\ 0, & else. \end{cases} \tag{6}$$

And the corresponding convolution in the spatial domain is

$$f(x) = (f(x)\,\text{III}_p(x)) \otimes \text{sinc}\left(\frac{x}{p}\right) \tag{7}$$

$$\mathcal{F}\left(\text{sinc}\left(\frac{x}{p}\right)\right) = \Pi_{\frac{1}{p}}(\omega) \tag{8}$$

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}. \tag{9}$$

But usually Eq. (5) and Eq. (7) do not hold because the function is not frequency limited or the sampling rate is not high enough as in Fig. 1 (a). Provided that the frequency content is not constantly zero for $\omega \in \left(\frac{k'}{p} - \frac{1}{2p}, \frac{k'}{p} + \frac{1}{2p}\right), k' \neq 0$, uniform sampling will duplicate the high frequency content within the Nyquist limit:

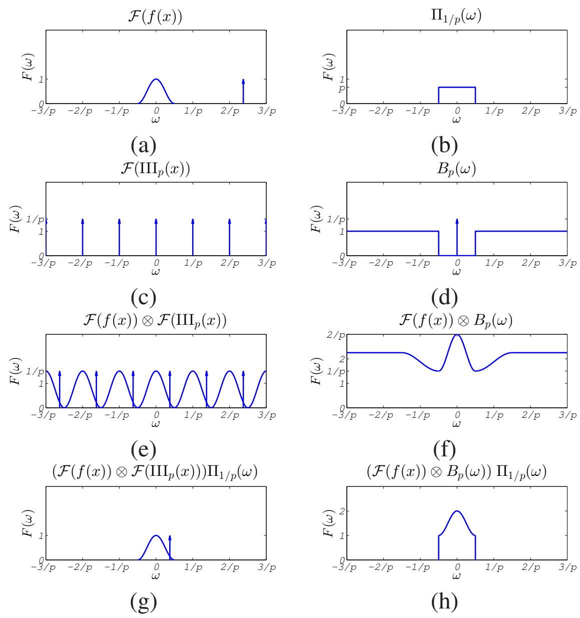

Figure 1: *Reconstruction of a function (a) with non-zero frequency above the Nyquist limit $\nu_N = \frac{1}{2p}$ with the kernel of a box function (b) $\Pi_p(\omega)$. If it is uniformly sampled with shah function $\text{III}_p(x)$ (c), the high frequencies will be duplicated (e) and result in low frequency aliasing (g). The blue-noise sampling $B_p(\omega)$ (d) will smoothly distribute the high frequency impulse to the whole frequency domain (f) and convert the low frequency aliasing to low frequency noise (h).*

$$F_A(\omega) = F(\omega) + F\left(\omega + \frac{k'}{p}\right), \omega \in \left(-\frac{1}{2p}, \frac{1}{2p}\right). \tag{10}$$

Consequently F_A is a kind of reconstruction with aliasing artifacts, because each impluse within the range of $\left(\frac{k'}{p} - \frac{1}{2p}, \frac{k'}{p} + \frac{1}{2p}\right)$ will be moved to a specific low frequency $\omega - \frac{k'}{p}$ below the Nyquist limit as shown in Fig. 1 (g).

B Blue-noise sampling

Different from uniform sampling, blue-noise sampling does not have any impulses in high frequencies. That is the reason why blue-noise sampling transforms low frequency aliasing to low frequency noise. The right column of Fig. 1 shows an example. Noise is preferred over aliasing in most cases.

The frequency content B_p of ideal blue-noise samples $b(x)$ is composed of an impulse on the original point and Heaviside step functions outside the Nyquist limit as Fig. 1 (d)

$$B_p(\omega) = \frac{1}{p}\delta(\omega) + \text{H}\left(\omega - \frac{1}{2p}\right) + \text{H}\left(\frac{1}{2p} - \omega\right) \tag{11}$$

$$\text{H}(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0. \end{cases} \tag{12}$$

Different from Eq. (10), all impulses in $\left(\frac{k'}{p} - \frac{1}{2p}, \frac{k'}{p} + \frac{1}{2p}\right)$ will be evenly distributed within the Nyquist limit as in Fig. 1 (h):

$$F_N\left(\omega\right) = F\left(\omega\right) + p\int_{-\frac{1}{2p}}^{\frac{1}{2p}} F\left(\omega + k'p\right)\, \mathrm{d}\omega, \omega \in \left(-\frac{1}{2p}, \frac{1}{2p}\right). \tag{13}$$

The low frequency aliasing in Eq. (10) is transformed into low frequency noise in Eq. (13).

The generalized blue-noise properties of Eq. (11) in nD space is

$$B\left(\boldsymbol{\omega}\right) = \frac{V_n}{\left(2p\right)^n}\delta\left(\boldsymbol{\omega}\right) + \mathrm{H}\left(\left|\boldsymbol{\omega}\right| - \frac{1}{2p}\right), \tag{14}$$

where V_n is the volume of a unit sphere in nD space.

C Power spectrum of uniform jittering

The bounding of the power spectrum in Section 3.3 is for general jittering. Usually the jittered direction is uniformly distributed, i.e., $\rho\left(\mathbf{d}^{\|}\right)$ is a constant. As it is difficult to get an analytical derivation of Eq. (10) in the main text, we plot the weight and corresponding power spectrum in Fig. 2. The frequency band covers more directions after uniform jittering but is smoothed out with increasing distance from the origin. Greater jittering and greater l produce a greater power spectrum fall-off.

References

COOK, R. L. 1986. Stochastic sampling in computer graphics. *ACM Trans. Graph. 5*, 1 (Jan.), 51–72.

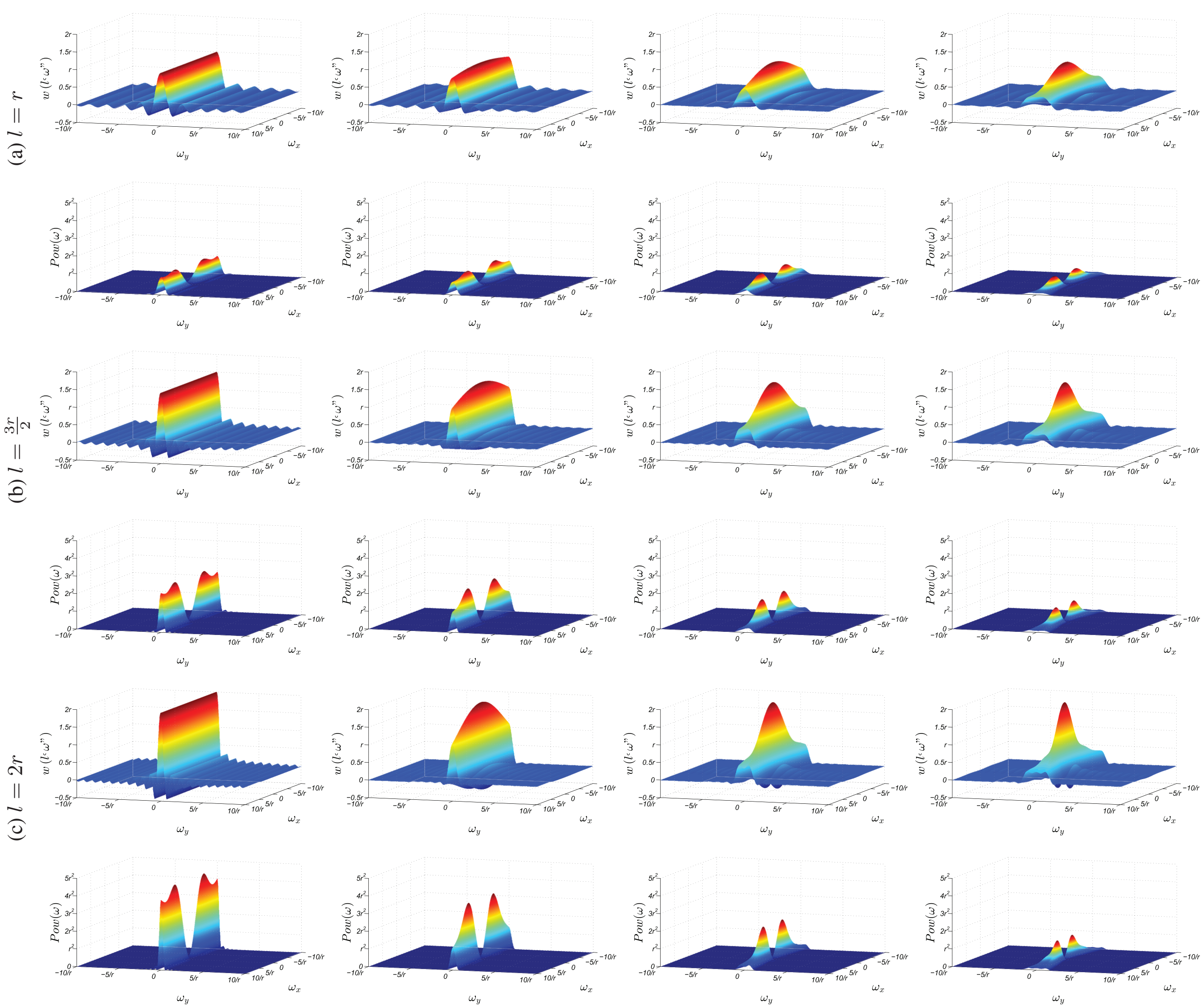

Figure 2: *Comparisons of the frequency content of line segment samples with jittering of constant pdf* $\rho(\boldsymbol{\omega})$*. Within each group, the top row is the weight and the bottom row is the power spectrum. The ranges of jittering are* 0°*,* $\pm 3^\circ$*,* $\pm 6^\circ$ *and* $\pm 9^\circ$ *from left to right.*