Hierarchical Diffusion Curves for Accurate Automatic Image Vectorization

1 Integration over a Rectangle

Motivated by Sun et al. [2012], in order to more efficiently reconstruct anti-aliased results, we have derived closed-form analytic solutions to the image reconstruction integral \( u(x) \) (in Eq. (2) of the paper) over a rectangular region \( R \), as opposed to simply evaluating it e.g. at a pixel center point \( x \):

\[
\phi(R) = \int_R u(x) \, dx.
\] (1)

The integral in Eq. (1) of \( u(x) \) over a rectangular region \( R \) can be expressed in terms of integrations of the Green’s function kernels \( G^L(x, x') \) over \( R \) = \( \{ x \in (x_0, x_1), y \in (y_0, y_1) \} \):

\[
\phi(R) = \int_R \left( \frac{\partial^2 (x, x')}{\partial n(x')} G^L(x, x') - u(x') G^L_n(x, x') \right) dx'\right) dx
\nonumber
+ \int_R \left( \frac{\partial^2 (x, x')}{\partial n(x')} G^B(x, x') - v(x') G^B_n(x, x') \right) dx'\right) dx.
\] (2)

where \( G^L_n(x, x') = \frac{\partial G^L(x, x')}{\partial n(x)} \), \( G^B_n(x, x') = \frac{\partial G^B(x, x')}{\partial n(x)} \).

As derived in Sun et al. [2012], closed-form integrals for \( F^L(x, x') = \int_R G^L(x, x') \, dx \) and \( F^B(x, x') = \int_R G^B(x, x') \, dx \) exist for this Green’s function over a rectangular region \( R \).

We derive new closed-form integrals \( F^B(x, x') = \int_R G^B(x, x') \, dx \) for the bilaplacian term \( G^B \):

\[
F^B(x, x') = \sum_{i,j \in \{0,1\}} (-1)^{i+j} H^B_{i,j}(\hat{x}, \hat{y})
\] (3)

and \( F^B_n(x, x') = \int_R G^B_n(x, x') \, dx \) for the bilaplacian normal term \( G^B_n \):

\[
F^B_n(x, x') = \sum_{i,j \in \{0,1\}} (-1)^{i+j} H^B_{i,j}(\hat{x}, \hat{y}, n_x, n_y)
\] (4)

where \( x = (x, y), x' = (x', y'), n(x') = (n_x, n_y), \) and \( (\hat{x}, \hat{y}) = x - x' \). Here, we define \( H^B_{i,j}(\hat{x}, \hat{y}) \) in Eq. (3) as

\[
H^B_{i,j}(\hat{x}, \hat{y}) = \int \int G^B(x, x') \, dx\nonumber
= \frac{1}{8\pi} \int \left( \int (\hat{x}^2 + \hat{y}^2) \left( \ln \left( \frac{1}{\sqrt{\hat{x}^2 + \hat{y}^2}} + 1 \right) \right) \, d\hat{x} \right) \, d\hat{y}
= \frac{1}{144\pi} \int \left( 8\hat{x}^3 + 30\hat{x}\hat{y}^2 - 12\hat{y}^3 \tan \left( \frac{\hat{x}}{\hat{y}} \right) \right) \, d\hat{y}
- \frac{1}{144\pi} \int \left( 3(\hat{x}^3 + 3\hat{x}\hat{y}^2) \ln (\hat{x}^2 + \hat{y}^2) \right) \, d\hat{y}
= \frac{1}{48\pi} \left( \hat{x}^4 - \hat{y}^4 \right) \tan \left( \frac{\hat{x}}{\hat{y}} \right)
+ \frac{1}{48\pi} \hat{x}\hat{y} (\hat{x}^2 + \hat{y}^2) \left( 11 - 3 \ln (\hat{x}^2 + \hat{y}^2) \right)
\] (5)

Similarly, \( H^B_{i,j}(\hat{x}, \hat{y}, n_x, n_y) \) in Eq. (4) is defined as:

\[
H^B_{i,j}(\hat{x}, \hat{y}, n_x, n_y) = \int \int G^B_n(x, x') \, dx
= \frac{1}{8\pi} \int \left( \int (\hat{x}n_x + \hat{y}n_y) \left( -1 + \ln (\hat{x}^2 + \hat{y}^2) \right) \, d\hat{x} \right) \, d\hat{y}
= \frac{1}{16\pi} \int \left( 2\hat{x} (\hat{x}n_x + 3\hat{y}n_y) + 4\hat{y}^2 n_y \tan \left( \frac{\hat{y}}{\hat{x}} \right) \right) \, d\hat{y}
- \frac{1}{16\pi} \int \left( (2\hat{y}n_y + (\hat{x}^2 + \hat{y}^2) n_x) \ln (\hat{x}^2 + \hat{y}^2) \right) \, d\hat{y}
= \frac{1}{48\pi} \left( 10\hat{y} (\hat{x}n_x + \hat{y}n_y) - 4\hat{y}^3 n_y \tan \left( \frac{\hat{y}}{\hat{x}} \right) - 4\hat{x}^3 n_x \tan \left( \frac{\hat{y}}{\hat{x}} \right) \right)
- \frac{1}{48\pi} \left( \hat{x}^3 n_y + 3\hat{x}\hat{y} n_x + 3\hat{x}^2\hat{y} n_y + \hat{y}^3 n_x \right) \ln (\hat{x}^2 + \hat{y}^2)
\] (6)

References