Application of pumping lemma

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1 The problem

Prove that $L = \{a^i b^j c^i d^j \mid i, j \geq 1\}$ is not a CFL.

2 The solution

To the contrary, suppose that L is a CFL. Then there is n satisfying the conditions specified in the pumping lemma.

Consider $z = a^n b^n c^n d^n$. Then $z = uvwxy$ such that $|vx| \geq 1$ and $|vwx| \leq n$. There are the following situations.

- $vwx = a^r$ or $vwx = b^r$ or $vwx = c^r$ or $vwx = d^r$ for some $1 \leq r \leq n$.
  Consider $z' = uv^2wx^2y$. According to the pumping lemma, $z' \in L$. On the other hand, if $vwx = a^r$, then z' contains more a's than c's, therefore, $z' \notin L$, a contradiction, similarly for $vwx = b^r, c^r, d^r$.

- $vwx = a^r b^s$ or $vwx = b^r c^s$ or $vwx = c^r d^s$ for $r, s \geq 1$.
  Consider $z' = uv^2wx^2y$. According to the pumping lemma, $z' \in L$. On the other hand, if $vwx = a^r b^s$, then z' either contains more a's than c's, or more b's than d's, therefore, $z' \notin L$, a contradiction, similarly for $vwx = b^r c^s, c^r d^s$.

  We conclude that L is not a CFL.