Homework

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Denote $|w|_a$ as the number of occurrences of a in w. Let $\Sigma = \{a, b\}$ and $L = \{w | \ |w|_a = |w|_b \text{ and } w \in \Sigma^*\}$.

Claim. L is not regular.

Proof. Suppose L is regular. Then there is a DFA $D = (Q, \Sigma, \delta, q_0, F)$ such that $L = L(D)$. Let $m = |Q| + 1$ and $w = a^m b^m$. Clearly, $w \in L$. Hence there is a run of D on w, say $q_0 a q_1 \ldots a q_m b q_{m+1} \ldots b q_{2m}$. For $w \in L$, $q_{2m} \in F$. Because $m = |Q| + 1$, there must be r, s: $0 \le r < s \le m$ such that $q_r = q_s$. Hence $q_0 a q_1 \ldots a q_r a q_{s+1} \ldots a q_m b q_{m+1} \ldots b q_{2m}$ is also a run of D. Since $q_{2m} \in F$, $a^{m-(s-r)} b^m \in L$, which is a contradiction. Therefore L is not regular.

Problem 1. L is not a VPL.

Proof. We can partition all the partitions of Σ into following two classes:

(1)$\Sigma^c = \emptyset$ and $\Sigma^r = \emptyset$; $\Sigma^c = \{a\}$ and $\Sigma^r = \emptyset$; $\Sigma^c = \{b\}$ and $\Sigma^r = \emptyset$; $\Sigma^c = \{a, b\}$ and $\Sigma^r = \emptyset$; $\Sigma^c = \emptyset$ and $\Sigma^r = \{a\}$; $\Sigma^c = \emptyset$ and $\Sigma^r = \{b\}$; $\Sigma^c = \emptyset$ and $\Sigma^r = \{a, b\}$;

(2)$\Sigma^c = \{a\}$ and $\Sigma^r = \{b\}$; $\Sigma^c = \{b\}$ and $\Sigma^r = \{a\}$;

Visibly pushdown automata with respect to partitions in class (1) do not have *calls* or do not have *returns*. And the visibly pushdown automata with respect to partitions in class (2) have *calls* and *returns*. The proofs for the partitions in the same class are alike. We only prove the following two partitions: $\Sigma^c = \{a\}$ and $\Sigma^r = \emptyset$; $\Sigma^c = \{a\}$ and $\Sigma^r = \{b\}$;

We fist consider the partition $\Sigma^c = \{a\}$ and $\Sigma^r = \emptyset$. Suppose there is a VPA $M = (Q, Q_{in}, \Gamma, \delta, Q_F)$ such that $L(M) = L$. We can define an NFA $N = (Q \cup \{q_0\}, \Sigma, \delta_1, q_0, Q'_F)$ as follows.

If $Q_{in} \cap Q_F \neq \emptyset$, then $Q'_F = Q_F \cup \{q_0\}$. Otherwise, $Q'_F = Q_F$.

The transition function δ_1 is defined as follows:

(1) For any $(q_1, a, q_2, \gamma) \in \delta$, we define $\delta_1(q_1, a) = q_2$.

(2) For any $(q_1, b, q_2) \in \delta$, we define $\delta_1(q_1, b) = q_2$.

(3) For any $q \in Q_{in}$, if $(q, a, q_1, \gamma) \in \delta$, we define $\delta_1(q_0, a) = q_1$. And if $(q, b, q_1) \in \delta$, we define $\delta_1(q_0, b) = q_1$.

We will prove that $L(N) = L(M)$.

$L(N) \supseteq L(M)$: For any word $w = a_1 a_2 \ldots a_n \in L(M)$, there is an accepting run $\rho = (q'_0, \sigma_0), (q'_1, \sigma_1), \ldots, (q'_n, \sigma_n)$ of M on w, where $q'_0 \in Q_{in}$ and $q'_n \in Q_F$. Then from the definition of N, we know that $q_0 q'_1 \ldots, q'_n$ is also an accepting run of N over w. So $w \in L(N)$.

$L(N) \subseteq L(M)$: For any word $w = a_1 a_2 \dots a_n \in L(N)$, there is an accepting run $\rho = q_0 q_1 \dots, q_n$ of N over w. If $q_n = q_0$, then from the definition of N, we have $w = \varepsilon$. Since $q_0 \in Q'_F$, there is a state $q'_0 \in Q_{in}$ such that $q'_0 \in Q_F$. We construct the following run of M on w: $\rho = (q'_0, \perp)$. Since $q'_0 \in Q_F$, $w \in L(M)$. In the following, we suppose $q_n \neq q_0$. Because $\delta_1(q_0, a_1) = q_1$, there must be $q'_0 \in Q_{in}$ such that $(q'_0, a, q_1, \gamma) \in \delta$ for some $\gamma \in \Gamma$ or $(q'_0, a, q_1) \in \delta$. Then we can construct the following run of M on w: $\rho = (q'_0, \sigma_0), (q_1, \sigma_1), \dots, (q_n, \sigma_n)$, where the following holds:
(1) $\sigma_0 = \perp$.
(2) **[Push]** If $a_i = a$, then from the definition of N we know there is some $\gamma \in \Gamma$ such that $(q_{i-1}, a, q_i, \gamma) \in \delta$. Let $\sigma_i = \gamma \cdot \sigma_{i-1}$.
(3) **[Local]** If $a_i = b$, then from the definition of N we know there is a transition $(q_{i-1}, b, q_i) \in \delta$. Let $\sigma_i = \sigma_{i-1}$.

Because $q_n \in Q'_F$, $q_n \in Q_F$. Hence ρ is an accepting run of M on w. That is $w \in L(M)$. So $L(M) = L(N)$. However, N is an NFA, which contradicts with Claim 1. Therefore $L(M) \neq L$.

Next, we consider the partition $\Sigma^c = \{a\}$ and $\Sigma^r = \{b\}$. Suppose there is a VPA $M = (Q, Q_{in}, \Gamma, \delta, Q_F)$ such that $L(M) = L$. Let $m = |Q| + 1$ and $w = b^m a^m$. So $w \in L$. Then there is an accepting run $\rho = (q_0, \sigma_0), (q_1, \sigma_1), \dots, (q_{2m}, \sigma_{2m})$ of M on w. Denote ρ_1 as $(q_0, \sigma_0), (q_1, \sigma_1), \dots, (q_m, \sigma_m)$. Then ρ_1 is a run of M on b^m. Because b is a *return*, $\sigma_0 = \sigma_1 = \dots = \sigma_m = \perp$. Since $m = |Q| + 1$, there are r, s: $0 \leq r < s \leq m$ such that $q_r = q_s$. Then $\rho_2 = (q_0, \sigma_0), (q_1, \sigma_1), \dots, (q_r, \sigma_r)(q_{s+1}, \sigma_{s+1}), (q_{s+2}, \sigma_{s+2}), \dots, (q_m, \sigma_m)$ is a run of M on $b^{m-(s-r)}$. Hence $\rho' = (q_0, \sigma_0), (q_1, \sigma_1), \dots, (q_r, \sigma_r)(q_{s+1}, \sigma_{s+1}), (q_{s+2}, \sigma_{s+2}), \dots, (q_m, \sigma_m), \dots, (q_{2m}, \sigma_{2m})$ is a run of M on $b^{m-(s-r)} a^m$. Because $q_{2m} \in Q_F$, ρ' is an accepting run. Therefore $b^{m-(s-r)} a^m \in L(M)$. But $b^{m-(s-r)} a^m \notin L$. So $L(M) \neq L$.

The other cases can be proved likely. Then L is not a VPL.

Claim. L is a context-free language.

Proof. We construct the following context-free grammar G:

$S \rightarrow \varepsilon | aSbS | bSaS$

It is easy to see that $L(G) \subseteq L$. Next, we prove that $L \subseteq L(G)$.

For any $w \in L$, we prove $w \in L(G)$ by induction on the length of w. From the definition of L, we know that $|w| \equiv 0(\text{mod } 2)$.

The cases for $w = \varepsilon$, $w = ab$ and $w = ba$ are straightforward.

Suppose for any word w' such that $w' \in L$ and $|w'| \leq 2k$, we have $w' \in L(G)$.

Let $w \in L$ and $|w| = 2(k+1)$. Suppose $w = aw_1$. Since $w \in L$, w_1 can be decomposed as $w_2 b w_3$, where $w_2 \in L$ and $w_3 \in L$. Because $|w_2| \leq 2k$ and $|w_3| \leq 2k$, from the inductive hypothesis, we have $S \stackrel{*}{\Rightarrow} w_2$ and $S \stackrel{*}{\Rightarrow} w_3$. Therefore $S \Rightarrow aSbS \stackrel{*}{\Rightarrow} aw_2bw_3$. That is $S \stackrel{*}{\Rightarrow} w$. Hence $w \in L(G)$. The case that $w = bw_1$ can be proved likely. So $L \subseteq L(G)$.