Automata theory and its applications

Lecture 3-4: Chomsky hierarchy-Context sensitive and free languages

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October 24, 2012

Outline

Context-sensitive grammar

Context-sensitive grammar (CSG)

A formal grammar $G = (\mathcal{N}, \Sigma, \mathcal{P}, S)$ of the production rules of the form

$$\alpha A \beta \to \alpha \gamma \beta \text{ such that } \gamma \neq \varepsilon.$$

Note that for every CSG G, $\varepsilon \notin L(G)$.

Noncontracting grammar (NCG)

A formal grammar $G = (\mathcal{N}, \Sigma, \mathcal{P}, S)$ of the production rules of the form

$$\alpha \to \beta \text{ such that } |\alpha| \leqslant |\beta|.$$

CSG $\equiv$ NCG

Left as homework.

1. Warm-up exercise: First transform the following NCG into a CSG,

$$L = \{a^n b^n c^n \mid n \geqslant 1\} : S \to aSBc \mid abc,\ cB \to Bc,\ bB \to bb.$$

2. Prove that each NCG can be transformed into an equivalent CSG.

Linear-bounded automata (LBA)

Intuitively, a LBA is a (single-tape) nondeterministic TM using linear space.

Formally, a LBA is a nondeterministic TM s.t.

- Its input alphabet includes two special symbols # and \$, the left and right endmarkers,
- The LBA has no moves left from # or right from \$, nor may print another symbol over # or \$.

  The language accepted by a linear-bounded TM M,

$$\{w \mid w \in (\Sigma - \{\#, \$\})^*, q_0 \# w \$ \vdash_M^* \alpha q \beta \text{ for some } q \in F\}$$

From LBA to CSG

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$ be a LBA. The CSG G

- first nondet. generates a finite word

$$\left[\begin{array}{c} a_1 \\ q_0\#a_1 \end{array}\right]\left[\begin{array}{c} a_2 \\ a_2 \end{array}\right]\cdots\left[\begin{array}{c} a_{n-1} \\ a_{n-1} \end{array}\right]\left[\begin{array}{c} a_n \\ a_n\$ \end{array}\right],$$

  with the intention that $a_1 \ldots a_n$ is the input of M.

- then G simulates the computation of M over $a_1 \ldots a_n$, by rewriting the second components of the generated word.

From LBA to CSG

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$ be a LBA. The CSG G

- first nondet. generates a finite word

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Formally, $G = (\mathcal{N}, \Sigma, \mathcal{P}, S)$ is defined as follows.

- $S = A_1$,
- $\mathcal{P}$ includes the following rules,
  - $A_1 \to [a, q_0\#a]A_2$, $A_1 \to [a, q_0\#a\$]$, $A_2 \to [a, a]A_2$, $A_2 \to [a, a\$]$.
  - If $\delta(q, \#) = (p, \#, R)$, then $[a, q\#\alpha] \to [a, \#p\alpha]$.
  - If $\delta(q, X) = (p, Y, R)$ for $X, Y \in \Sigma\backslash\{\#, \$\}$, then
    for each $a, b \in \Sigma\backslash\{\#, \$\}$, $\beta \in \Sigma \cup \#\Sigma \cup \Sigma\$ \cup \#\Sigma\$$,
    $[a, qX][b, \beta] \to [a, Y][b, p\beta]$, $[a, \#qX][b, \beta] \to [a, \#Y][b, p\beta]$,
    $[a, \#qX\$)] \to [a, \#Yp\$]$.
  - Symmetrically for the transition $\delta(q, \$) = (p, \$, L)$ and $\delta(q, X) = (p, Y, L)$.
  - For each $a, b \in \Sigma\backslash\{\#, \$\}$, $q \in F$, and $\alpha_1\alpha_2, \alpha \in \Sigma \cup \#\Sigma \cup \Sigma\$ \cup \#\Sigma\$$,
    $[a, \alpha_1 q\alpha_2] \to a$, $[a, \alpha]b \to ab$, $b[a, \alpha] \to ba$.

From CSG to LBA

Recalling: From Type-0 grammar to TM

Let $G = (\mathcal{N}, \Sigma, \mathcal{P}, S)$ be a Type-0 grammar.
Construct a nondet. TM M to recognize the language $L(G)$.

- M has two tapes.
- The input of M (say w) is in tape 1.
- M simulates the derivation relation of G in tape 2 by repeating the following procedure.
  1. It nondet. chooses a position i in tape 2 and a production rule $\alpha \to \beta$.

     If α appears from position i in tape 2, then α is replaced by β in tape 2. Some shifting over of the symbols on tape 2 should be done if $|\alpha| \neq |\beta|$.

  2. M compares the sequence of symbols in tape 2 with the sequence in tape 1, to see whether w has been generated by G. If so, then M accepts.

From CSG to LBA

Recalling: From Type-0 grammar to TM

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From CSG to LBA

G is a CSG $\Rightarrow$ G is noncontracting ($\alpha \to \beta$ s.t. $|\alpha| \leqslant |\beta|$) $\Rightarrow$
The space of tape 2 used by M is bounded by $|w|$.
Therefore, M can be simulated by

a single-tape nondeterministic TM M' using linear space.

Properties of CSL: A bird's-eye view

Closure properties

Theorem. CSL are closed under all Boolean operations.

Decision problems

- The nonemptiness problem for CSG is undecidable.
- The membership problem for CSG is PSPACE-complete.

Outline

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Definition

A CFG is a formal grammar $(\mathcal{N}, \Sigma, \mathcal{P}, S)$ such that

each production rule in $\mathcal{P}$ *is of the form* $A \to \alpha$ *with* $A \in \mathcal{N}$ *and* $\alpha \in (\mathcal{N} \cup \Sigma)^*$.

Example: CFG for arithmetic expressions,

$$E \to E + E \quad E \to E * E$$
$$E \to (E) \qquad E \to id$$

Derivation:

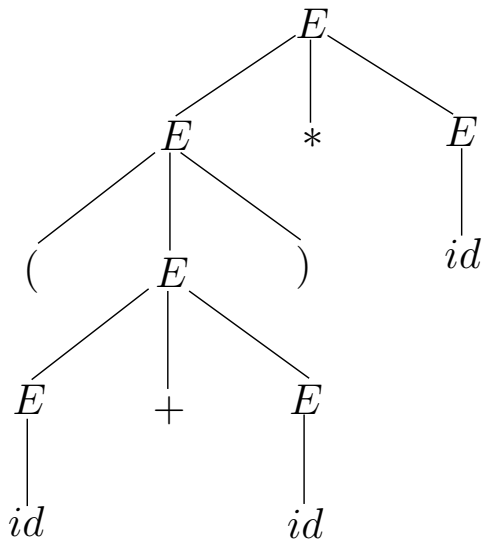
$$E \Rightarrow E * E \Rightarrow (E) * E \Rightarrow (E + E) * E \Rightarrow (id + id) * id.$$

Derivation trees

A derivation tree for a CFG $G = (\mathcal{N}, \Sigma, \mathcal{P}, S)$: A labeled tree s.t.

- every vertex has a label from $\mathcal{N} \cup \Sigma \cup \{\varepsilon\}$,
- the root is labeled by S,
- every internal vertex is labeled by symbols from $\mathcal{N}$,
- for every vertex labeled by $A \in \mathcal{N}$,
  if its children are labeled by $X_1, \ldots, X_k$ (from left to right),
  then $A \to X_1 \ldots X_k$ is a production rule of G.

Example: Derivation tree for $(id + id) * id$



Leftmost derivation

Leftmost derivation:

Each step in the derivation is applied to the leftmost nonterminal.

Roughly speaking, leftmost derivation corresponds to

a preorder DFS traversal of the derivation tree.

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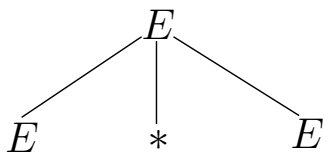

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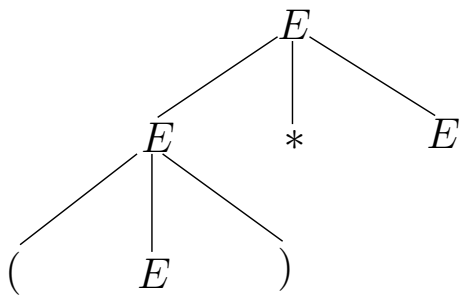

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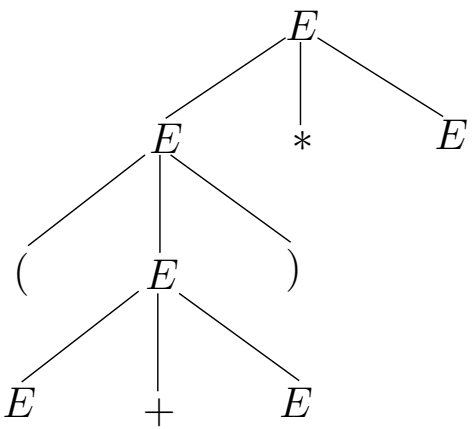

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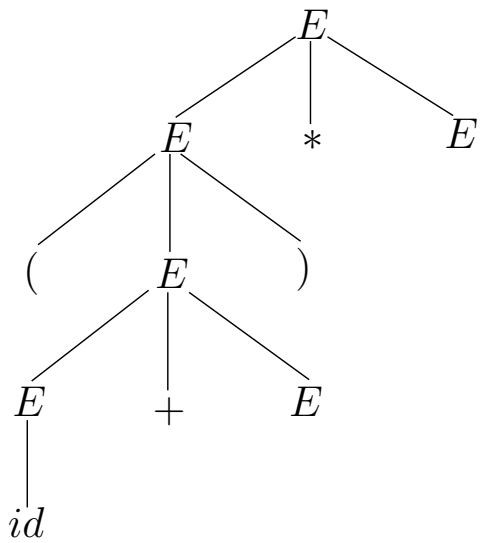

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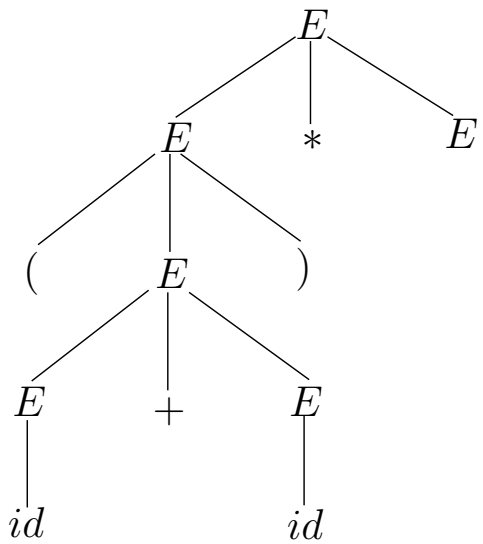

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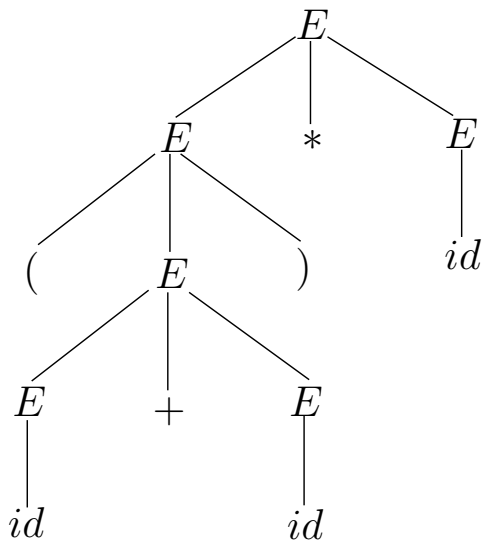


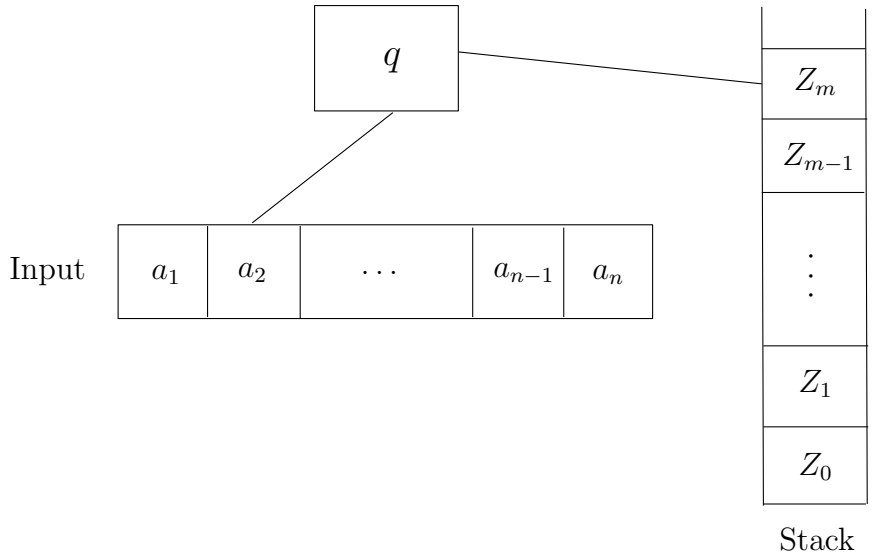
Outline

Definition

Intuitively, a pushdown automaton is a TM with the infinite tape organized as a stack.

Formally, a PDA M is a tuple $(Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$, where

- Q: Finite set of states,
- Σ: The input alphabet,
- Γ: The stack alphabet,
- $q_0 \in Q$: The initial state,
- $Z_0 \in \Gamma$: The start symbol,
- $F \subseteq Q$: The set of final states,
- δ: A mapping from $Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma$ to finite subsets of $Q \times \Gamma^*$.



Instantaneous configuration (IC)

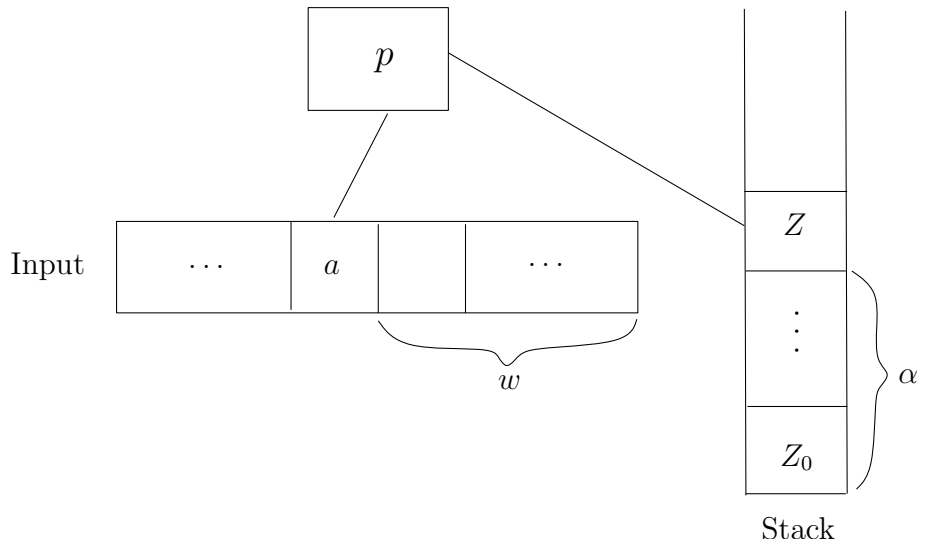
IC and $\vdash_M$

An instantaneous configuration (IC) is a tuple (q, w, γ) such that

- $q \in Q$,
- $w \in \Sigma^*$ (the suffix of the input not read yet),
- $\gamma \in \Gamma^*$ (the content of the stack, with the topmost symbol leftmost).

The $\vdash_M$ relation between ICs is defined as follows:

If $(p, \beta) \in \delta(q, a, Z)$ *(where* $a \in \Sigma \cup \{\varepsilon\}$*), then* $(q, aw, Z\alpha) \vdash_M (p, w, \beta\alpha)$.



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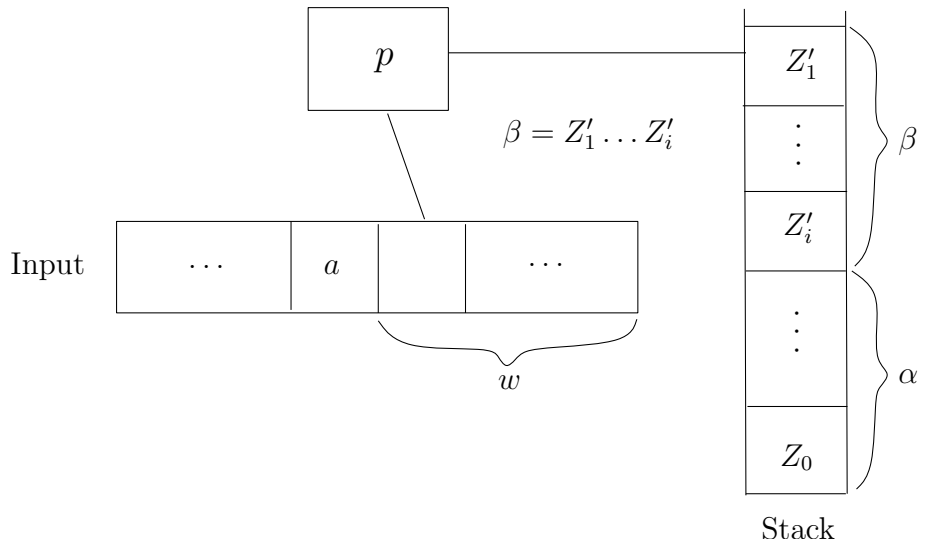

Accepted languages

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a PDA.

- Define $L(M)$, the language accepted by final state, as follows,

$$\{w \in \Sigma^* \mid (q_0, w, Z_0) \vdash_M^* (p, \varepsilon, \gamma) \text{ for } p \in F, \gamma \in \Gamma^*\}.$$

- Define $N(M)$, the language accepted by empty stack, as follows,

$$\{w \in \Sigma^* \mid (q_0, w, Z_0) \vdash_M^* (p, \varepsilon, \varepsilon) \text{ for } p \in Q\}.$$

Example: PDA $M = (\{q_0, q_1\}, \{0, 1\}, \{A, B, Z_0\}, \delta, q_0, Z_0, \varnothing)$ accepting $\{ww^R \mid w \in \{0, 1\}^*\}$ by empty stack.

$\delta(q_0, 0, Z_0) = (q_0, AZ_0)$	$\delta(q_0, 1, Z_0) = (q_0, BZ_0)$
$\delta(q_0, 0, A) = (q_0, AA)$	$\delta(q_0, 1, A) = (q_0, BA)$
$\delta(q_0, 0, B) = (q_0, AB)$	$\delta(q_0, 1, B) = (q_0, BB)$
$\delta(q_0, 0, A) = (q_1, \varepsilon)$	$\delta(q_0, 1, B) = (q_1, \varepsilon)$
$\delta(q_1, 0, A) = (q_1, \varepsilon)$	$\delta(q_1, 1, B) = (q_1, \varepsilon)$
$\delta(q_1, \varepsilon, Z_0) = (q_1, \varepsilon)$	$\delta(q_0, \varepsilon, Z_0) = (q_0, \varepsilon)$

Equivalence of the two accepting means

From final state to empty stack

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a PDA.
The goal: Define a PDA M' such that $N(M') = L(M)$.
The intuition:

> *if M accepts an input by a final state q, then from q, M' enters q_e, empties the stack, and accepts.*

In addition, M' includes a new start symbol Z_0' to deal with the following situation,

> *M reads all the input, enters a non-final state, and the stack is empty.*

$M' = (Q \cup \{q_0', q_e\}, \Sigma, \Gamma \cup \{Z_0'\}, \delta', q_0', Z_0', \varnothing)$ is defined as follows.
δ' includes all the elements of δ and the transitions

- $\delta'(q_0', \varepsilon, Z_0') = \{(q_0, Z_0 Z_0')\}$,
- $\delta'(q, \varepsilon, X) = \delta(q, \varepsilon, X) \cup \{(q_e, X)\}$ for $q \in F$ and $X \in \Gamma$,
- $\delta'(q_e, \varepsilon, X) = \{(q_e, \varepsilon)\}$ for every $X \in \Gamma \cup \{Z_0'\}$.

Equivalence of the two accepting means

From empty stack to final state

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, \varnothing)$ be a PDA.
The goal: Define a PDA M' such that $L(M') = N(M)$.
The intuition of M':

> *Add a new start symbol Z_0' to M, if Z_0' becomes the topmost symbol of the stack, then M' enters a final state.*

$M' = (Q \cup \{q_0', q_f\}, \Sigma, \Gamma \cup \{Z_0'\}, \delta', q_0', Z_0', \{q_f\})$ is defined as follows.
δ' includes all the elements of δ and the transitions

- $\delta'(q_0', \varepsilon, Z_0') = \{(q_0, Z_0 Z_0')\}$,
- $(q_f, \varepsilon) \in \delta'(q, \varepsilon, Z_0')$ for every $q \in Q$.

Outline

From CFG to PDA

The intuition

Use the stack to store the words reached in the leftmost derivation.

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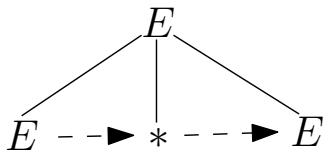

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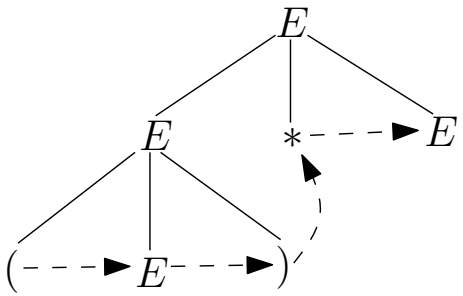

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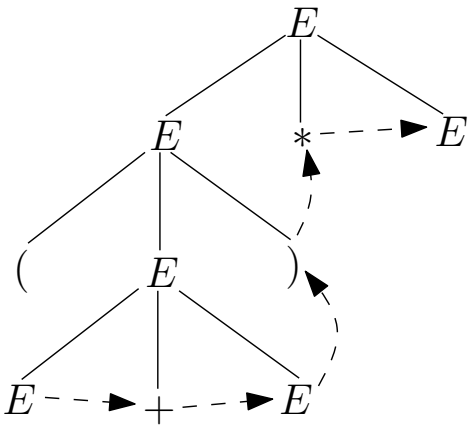

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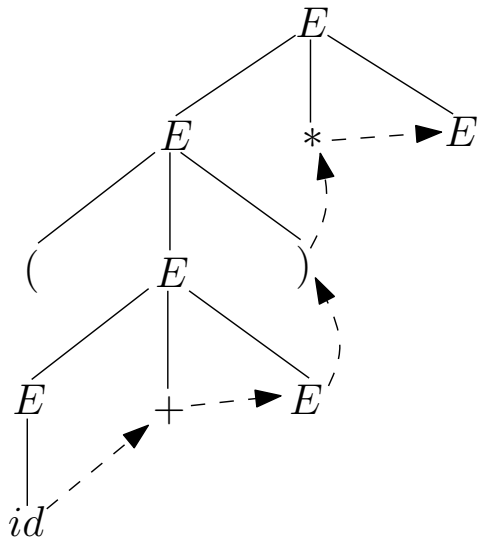

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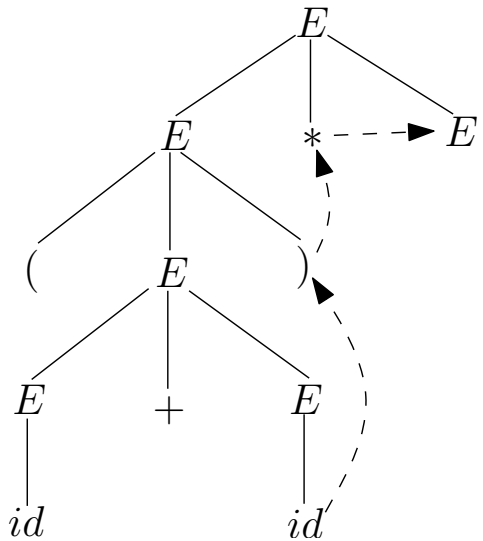

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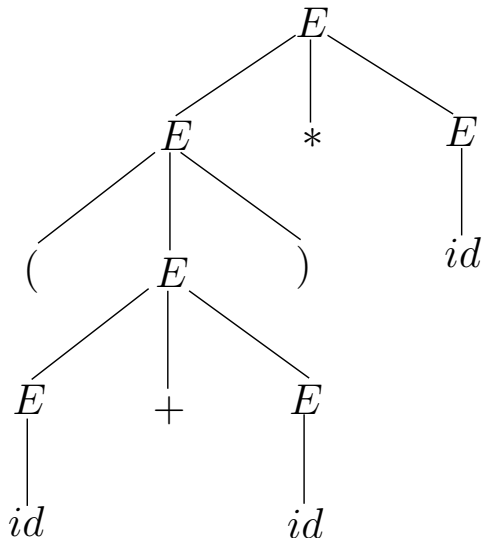

From CFG to PDA: continued

The construction

Let $G = (\mathcal{N}, \Sigma, \mathcal{P}, S)$ be a CFG.
Define a PDA $M = (\{q\}, \Sigma, \Sigma \cup \mathcal{N}, \delta, q, S, \varnothing)$ as follows.

- $\delta(q, \varepsilon, A) = \{(q, \alpha)\}$ for each production rule $A \to \alpha$,
- $\delta(q, a, a) = \{(q, \varepsilon)\}$ for every $a \in \Sigma$.

The correctness of the construction $N(M) = L(G)$

Claim.

$\forall x \in \Sigma^*, \alpha \in (\mathcal{N} \cup \Sigma)^*$,
$S \Rightarrow^* x\alpha$ *by leftmost derivation in* G *iff* $(q, x, S) \vdash_M^* (q, \varepsilon, \alpha)$.

Proof of the claim.
Induction on the derivation or computation steps.

From PDA to CFG

An illustration of the intuition:

Analyze the computation tree of PDA M for $\{ww^R \mid w \in \{0,1\}^*\}$ over the input 0110.

$$\delta(q_0, 0, Z_0) = (q_0, AZ_0)$$

$$\delta(q_0, 1, A) = (q_0, BA)$$

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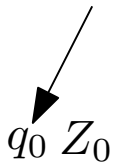

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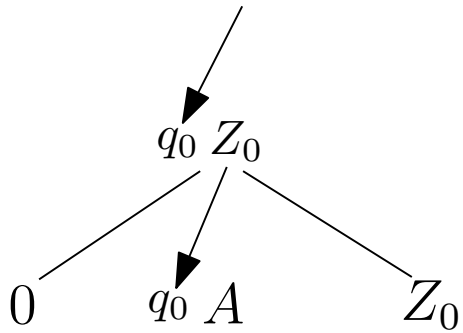

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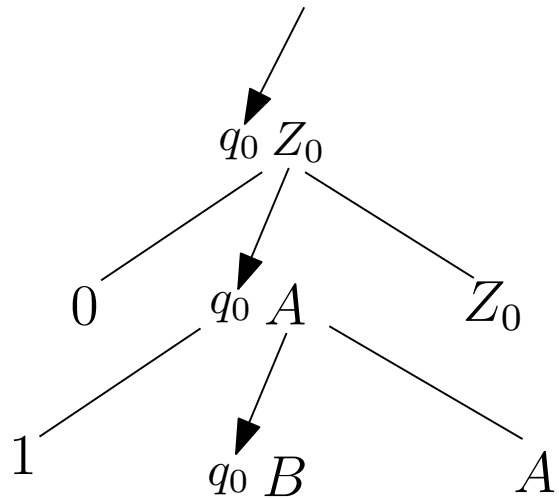

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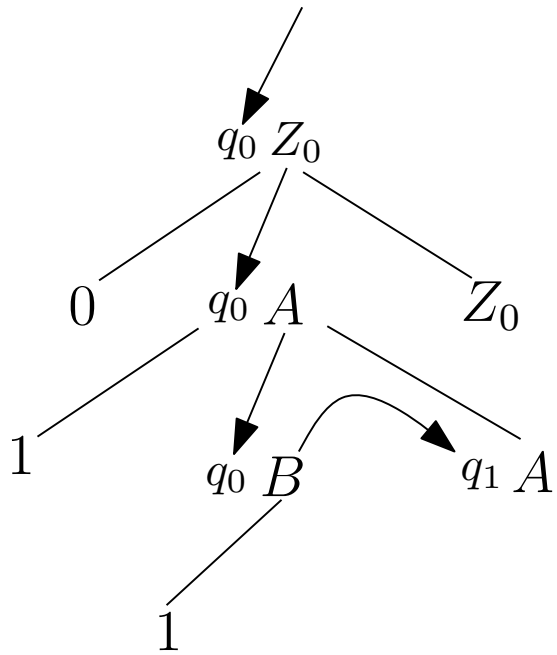

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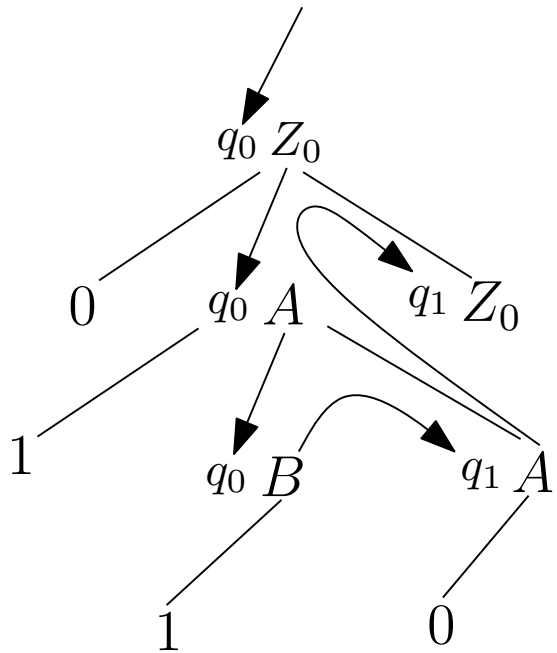

From PDA to CFG

An illustration of the intuition:

Analyze the computation tree of PDA M for $\{ww^R \mid w \in \{0,1\}^*\}$ over the input 0110.

$$\delta(q_0, 0, Z_0) = (q_0, AZ_0)$$

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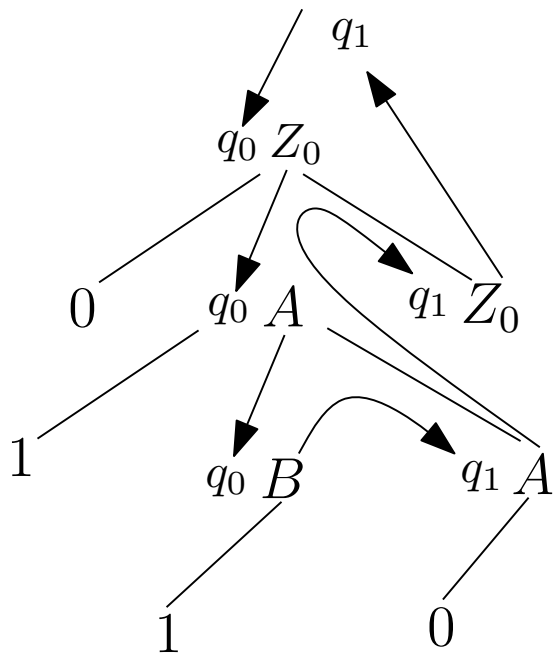

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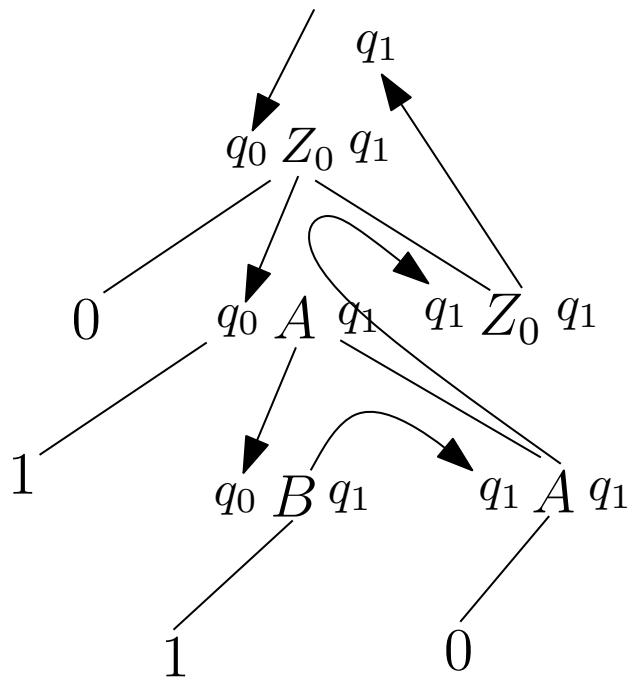

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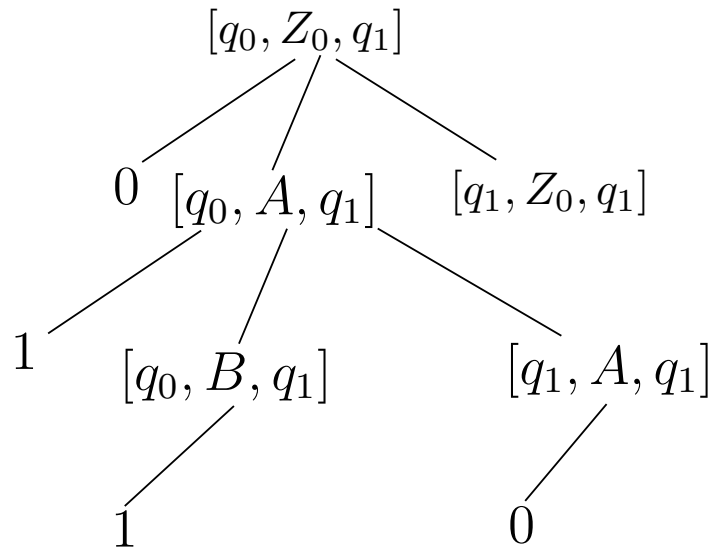


From PDA to CFG: continued

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, \varnothing)$ be a PDA.
The goal: Define a CFG $G = (\mathcal{N}, \Sigma, \mathcal{P}, S)$ such that $L(G) = N(M)$.

- $\mathcal{N} = Q \times \Gamma \times Q \cup \{S\}$.
- $\mathcal{P}$ is defined as follows.
  - $S \to [q_0, Z_0, q]$ for every $q \in Q$.
  - If $(p, B_1 \dots B_k) \in \delta(q, a, X)$ (where $a \in \Sigma \cup \{\varepsilon\}$), then $[q, X, q_{k+1}] \to a[q_1, B_1, q_2] \dots [q_k, B_k, q_{k+1}]$ such that $q_1 = p$.

The correctness of the construction

Claim. $[q, A, p] \Rightarrow^* x$ iff $(q, x, A) \vdash_M^* (p, \varepsilon, \varepsilon)$.

Outline

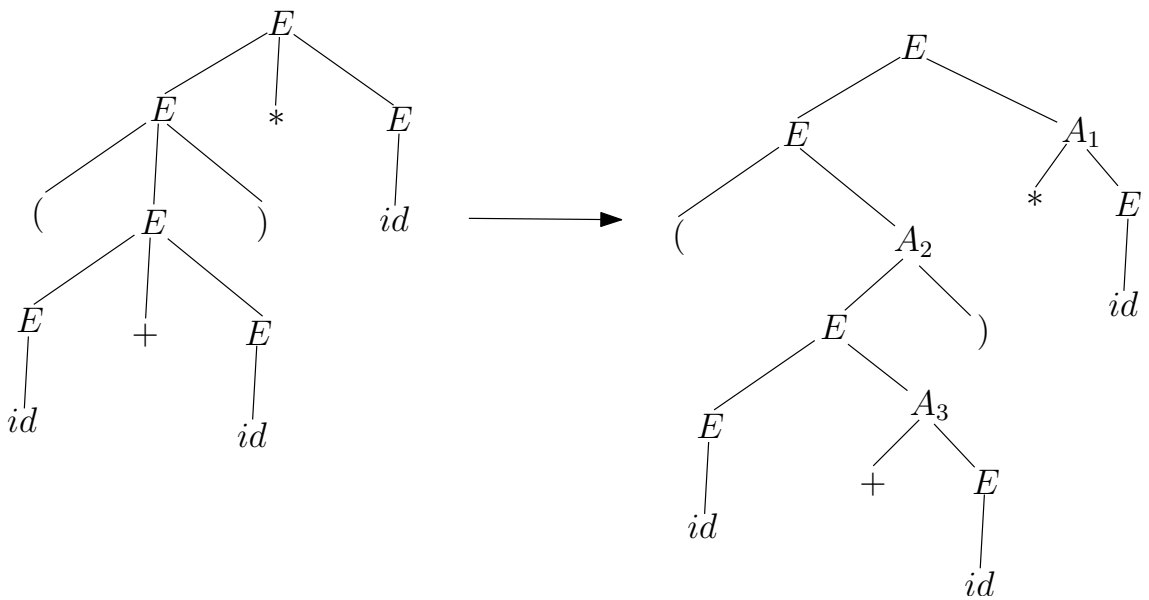
Outline

Binary normal form (2NF)

Theorem. For every CFG G, a CFG G' can be computed in linear time satisfying that

- $L(G) = L(G')$,
- G' contains only rules of the form $A \to \alpha$ such that $|\alpha| \leqslant 2$.

The intuition.



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The intuition.

$$E \to E + E \quad E \to E * E$$
$$E \to (E) \quad E \to id$$

$$E \to EA_3 \quad A_3 \to +E \quad E \to EA_1 \quad A_1 \to *E$$
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Formally, a 2NF can be obtained from a CFG by the following procedure.

Replace every rule $A \to X_1 \dots X_k$ *(where* $k \geqslant 3$*) by the following rules*

$$A \to X_1C_1, C_1 \to X_2C_2, \dots, C_{k-2} \to X_{k-1}X_k$$

where $C_1, \dots, C_{k-2}$ are new nonterminal symbols.

Outline

Pumping lemma

Pumping lemma. For every CFL L, there is a natural number $n \geqslant 1$ such that for every string $z \in L$ with $|z| \geqslant n$, z can be decomposed into $uvwxy$ satisfying that

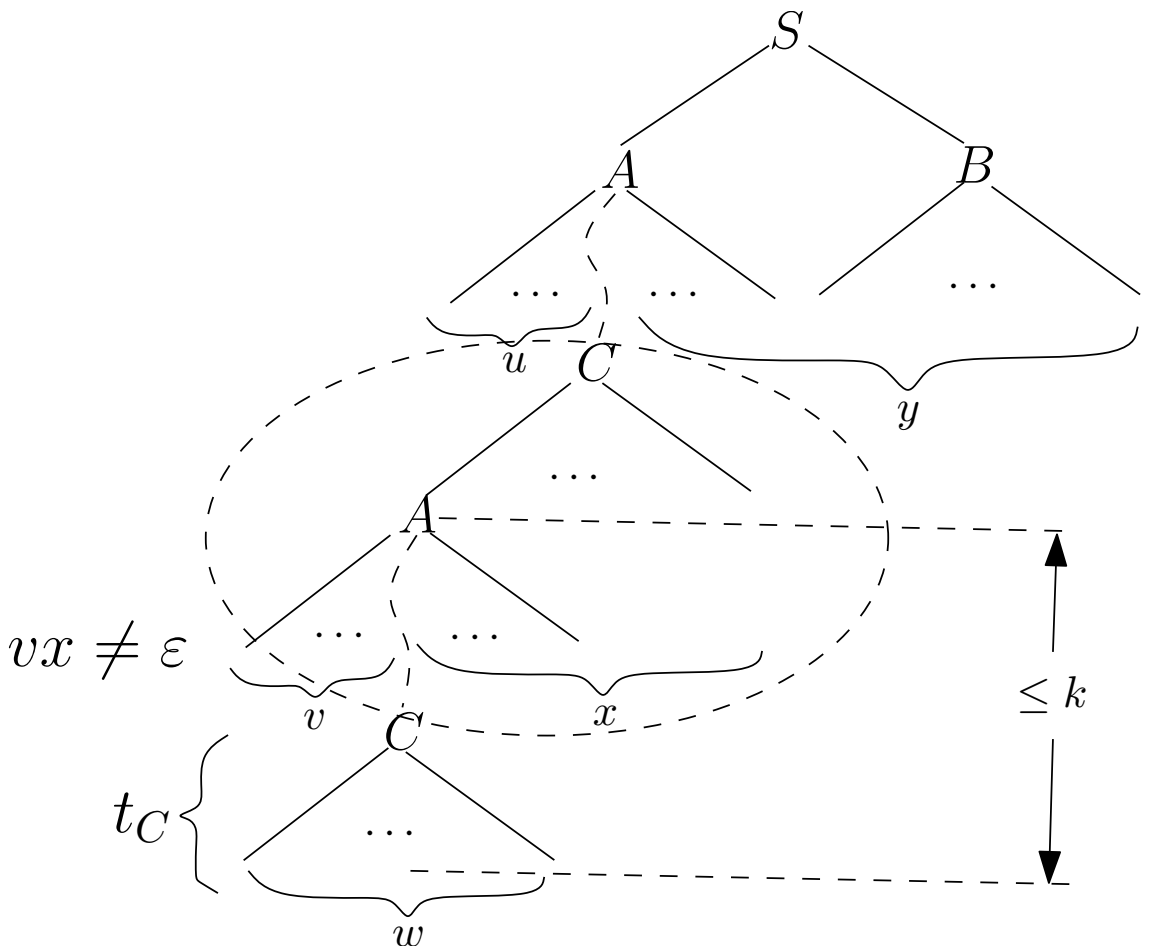
- $|vx| \geqslant 1$,
- $|vwx| \leqslant n$,
- for every $i \in \mathbb{N}$, $uv^iwx^iy \in L$.

Let L be a CFL and $G = (\mathcal{N}, \Sigma, \mathcal{P}, S)$ be a CFG in 2NF s.t. $L = L(G)$. Let k be the number of nonterminals in G.

Pumping lemma: continued

The idea of the proof.

Analyzing the structure of the derivation trees of CFGs.

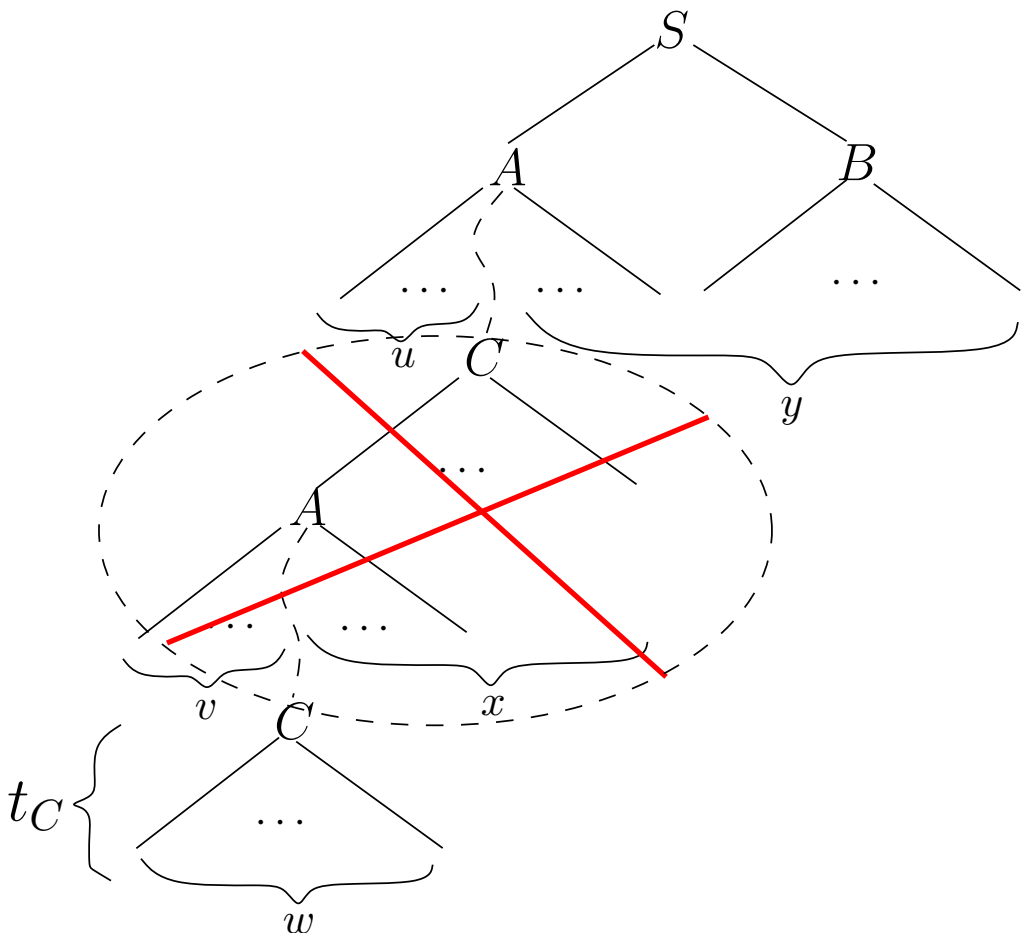


Pumping lemma: continued

The idea of the proof.

Analyzing the structure of the derivation trees of CFGs.

If $vx = \varepsilon$, then consider the reduced derivation tree instead



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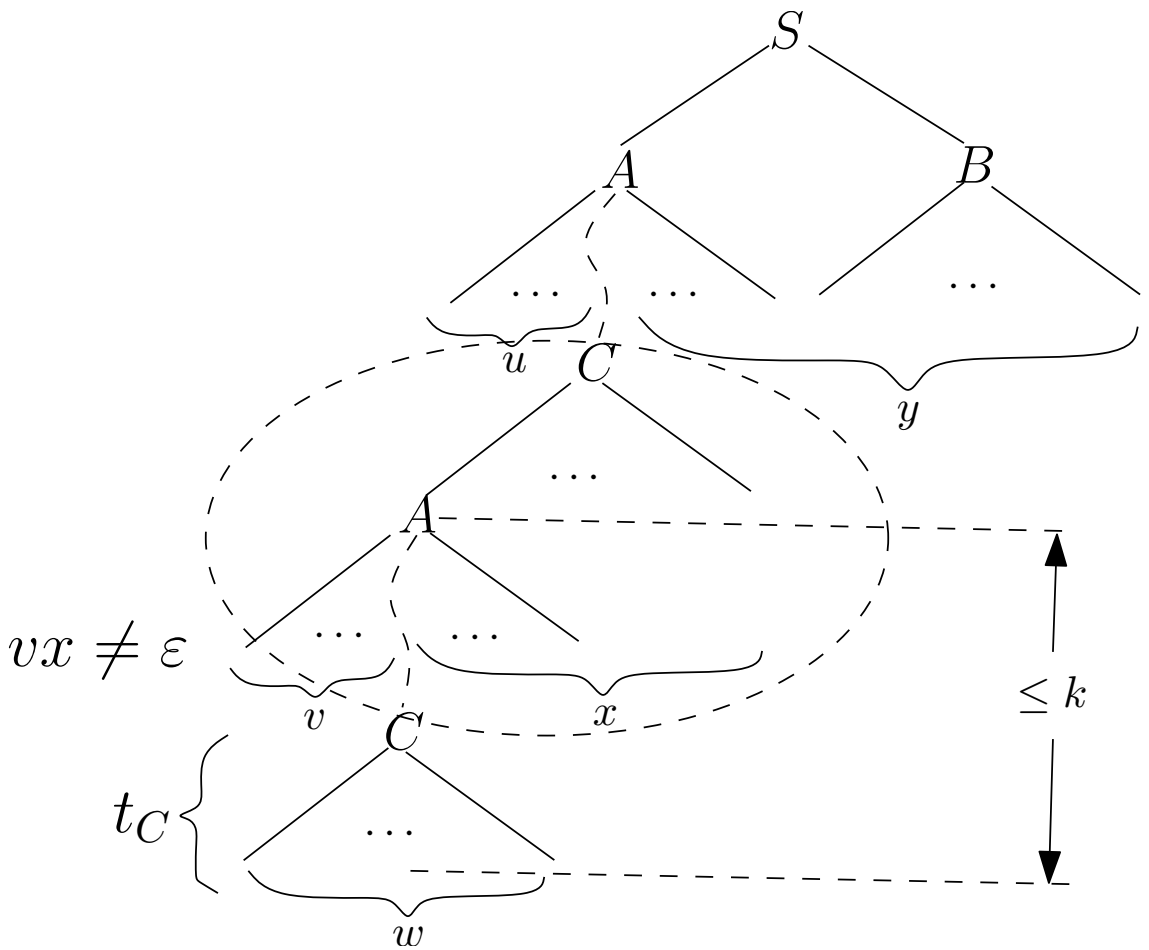

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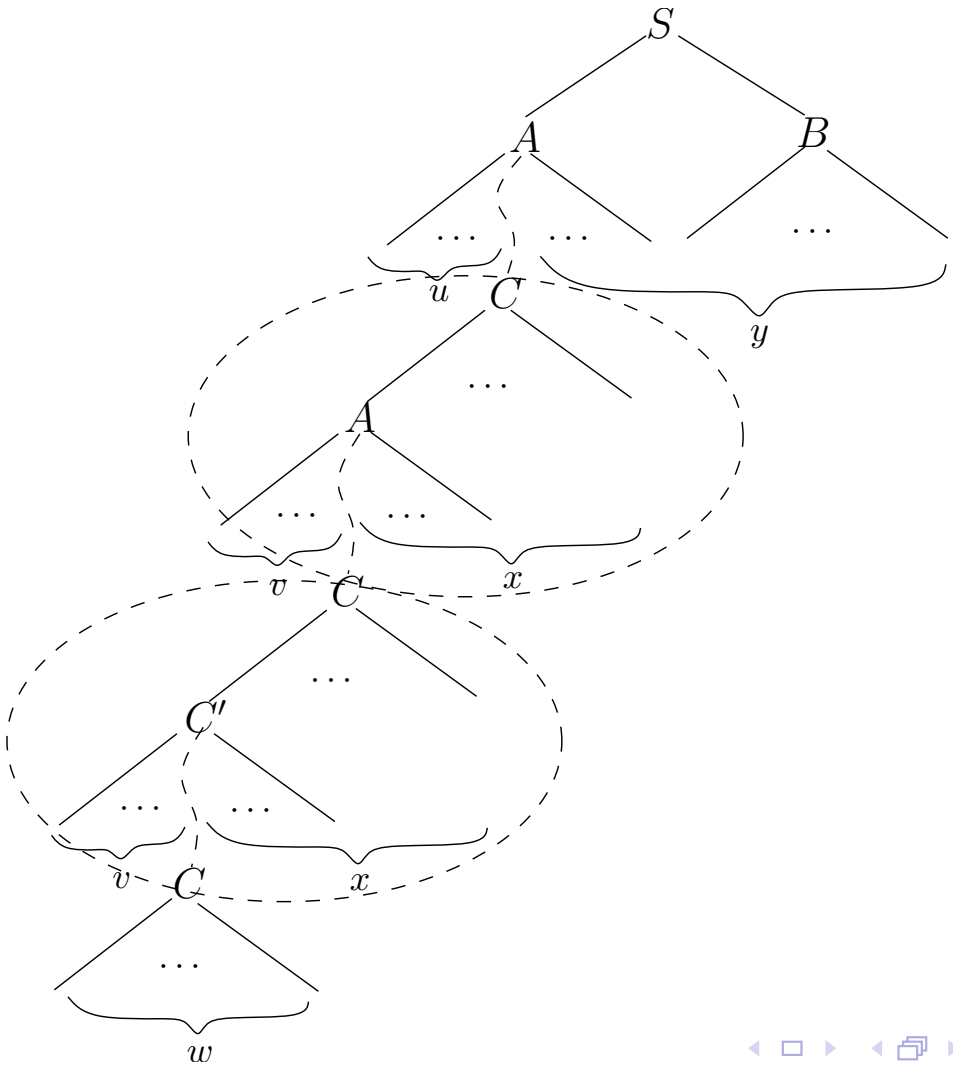

Pumping lemma: continued

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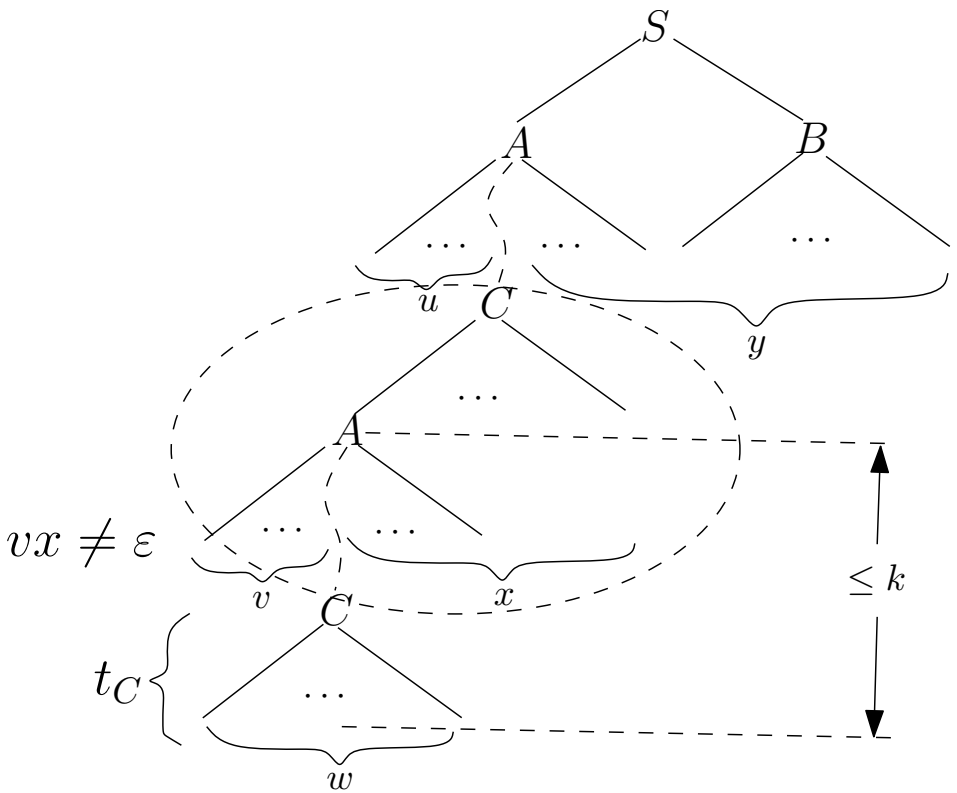

$|vwx|$: # leaves of a binary tree of depth $\leqslant k+1$ (depth 0 for root): $\leqslant 2^{k+1}$.

Set $n := 2^{k+1}$.

Applications of pumping lemma

Proposition. $L = \{a^i b^i c^i \mid i \geqslant 1\}$ is not a CFL.

To the contrary, suppose that L is a CFL.

Then there is a CFG G such that $L = L(G)$.

So, there is $n \geqslant 1$ satisfying the conditions in the pumping lemma.

Consider $z = a^n b^n c^n$.

Then $z = uvwxy$ s.t. $|vx| \geqslant 1$ and $|vwx| \leqslant n$. $\exists$ the following situations

- $vwx = a^i$ or $vwx = b^i$ or $vwx = c^i$ for some $i : 1 \leqslant i \leqslant n$.

  Consequences: $uv^2wx^2y \notin L$, *a contradiction.*

- $vwx = a^i b^j$ or $vwx = b^i c^j$ for some $i, j : 1 \leqslant i, j, i + j \leqslant n$.

  Consequences: $uv^2wx^2y \notin L$, *a contradiction.*

Corollary. $CFL \subset CSL$.

Homework.

Prove that $\{a^i b^j c^i d^j \mid i, j \geqslant 1\}$ is not a CFL, using pumping lemma.

Outline

Closure properties

Theorem. The class of CFLs is

- closed under union,
- not closed under intersection,
- closed under intersection with regular languages,
- not closed under complementation.

Union:

Suppose $G_1 = (\mathcal{N}_1, \Sigma, \mathcal{P}_1, S_1)$ and $G_2 = (\mathcal{N}_2, \Sigma, \mathcal{P}_2, S_2)$.

Then G: Add the rules $S \to S_1, S \to S_2$.

It is easy to see that $L(G) = L(G_1) \cup L(G_2)$.

Closure properties

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Intersection:

$L_1 = \{a^i b^i c^j \mid i, j \geqslant 1\}$ and $L_2 = \{a^j b^i c^i \mid i, j \geqslant 1\}$ are both CFL.

On the other hand, $L_1 \cap L_2 = \{a^i b^i c^i \mid i \geqslant 1\}$ is not a CFL.

Closure properties

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Intersection with regular languages:

Let

- $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a PDA for L_1,
- $\mathcal{A} = (Q', \Sigma, \delta', q_0', F')$ be a FSA for L_2.

Then $L_1 \cap L_2 = L(M'')$ such that

$$M'' = (Q \times Q', \Sigma, \Gamma, \delta'', (q_0, q_0'), Z_0, F \times F'),$$

where

- $\delta''((q, q'), a, X) = \{((p, p'), \gamma) \mid (p, \gamma) \in \delta(q, a, X), \delta'(q', a) = p'\}$ for $a \in \Sigma$,
- $\delta''((q, q'), \varepsilon, X) = \{((p, q'), \gamma) \mid (p, \gamma) \in \delta(q, \varepsilon, X)\}$.

Closure properties

Theorem. The class of CFLs is

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Complementation:

If the class of CFLs is closed under complementation, then

from the fact that $L_1 \cap L_2 = \overline{(\overline{L_1} \cup \overline{L_2})}$,

the class of CFLs is closed under intersection, a contradiction.

Outline

Outline

Emptiness

Let G be a CFG.

- $T_0(G) = \Sigma \cup \{A \mid \exists \alpha \in \Sigma^*.A \to \alpha\}$.
- $T_{i+1}(G) = T_i(G) \cup \{A \mid A \to X_1 \dots X_k, \forall j : 1 \leqslant j \leqslant k.X_j \in T_i(G)\}$.

Define $T(G) = T_i(G)$ such that $T_i(G) = T_{i+1}(G)$.

Claim. $L(G) \neq \varnothing$ iff $S \in T(G)$.

Theorem. The emptiness of CFG can be solved in PTIME.

Outline

Universality

Universality problem (A special case of inclusion problem):

Given a CFG G, whether $L(G) = \Sigma^*$.

Theorem. The universality problem of CFG is undecidable.

Proof idea: Reduction from emptiness problem of TM.

Lemma. Emptiness problem of TM is undecidable.

Reduction from halting problem:
From $\langle M, w\rangle$, construct M' as follows,

M': *Over an input $x \in \{0,1\}^*$, simulate the computation of M on w. If the computation halts, then accepts.*

The following fact holds.

- If the computation of M over w halts, then $L(M') = \{0,1\}^*$,
- otherwise, $L(M') = \varnothing$.

$$\langle M, w\rangle \in L_h \text{ iff}$$
$$\text{the computation of } M \text{ over } w \text{ halts iff}$$
$$L(M') \neq \varnothing.$$

Universality

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Proof idea: Reduction from emptiness problem of TM.

Valid computation of a TM M

A valid computation of $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$ is

$$C_1\$C_2^R\$C_3\$C_4^R\$\ldots\$C_n$$

such that

- for every i, C_i is an IC of M,
- C_1 is of the form q_0x for some $x \in \Sigma^*$,
- for every i, $C_i \vdash_M C_{i+1}$,
- $C_n = \alpha q \beta$ such that $q \in F$, $\alpha, \beta \in \Gamma^*$.

The set of invalid computations is the complement of the valid computations wrt. $(Q \cup \Gamma \cup \{\$\})^*$.

Universality: Continued

Theorem. The set of invalid computations of a TM M is a CFL.
Let L_I be the set of invalid computations of M.

A word $w \in (Q \cup \Gamma \cup \$)^*$ belongs to L_I iff one of the following conditions hold,

1. w is not of the form $C_1\$C_2\$C_3 \ldots \$C_n$ such that each $C_i \in \Gamma^* Q \Gamma^*$,
2. C_1 is not initial, i.e. is not of the form $q_0\Sigma^*$,
3. there is an odd i such that $C_i \nvdash_M C_{i+1}^R$,
4. there is an even i such that $C_i^R \nvdash_M C_{i+1}$,
5. C_n is not final, namely, $C_n \notin \Gamma^* F \Gamma^*$.

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Claim. $L_I = (Q \cup \Gamma \cup \$)^*$ *iff* $L(M) = \varnothing$.

Outline

Cocke-Younger-Kasami (CYK) algorithm

Membership problem:

Given a CFG $G = (\mathcal{N}, \Sigma, \mathcal{P}, S)$ *and a word* $w \in \Sigma^*$, *is* $w \in L(G)$*?*

Theorem. The membership problem for CFG G can be solved in PTIME.

W.l.o.g. assume that G is in 2NF.
Let $n = |w| \geqslant 1$.

The intuition:

Bottom-up computation of the derivation tree through dynamic programming.

The CFG G

$$\begin{array}{cccc} E \to EA_3 & A_3 \to +E & E \to EA_1 & A_1 \to *E \\ E \to (A_2 & A_2 \to E) & E \to aI & E \to bI \\ I \to 0I & I \to 1I & I \to \varepsilon & \end{array}$$

Question: $(a + b0) * a1 \in L(G)$?

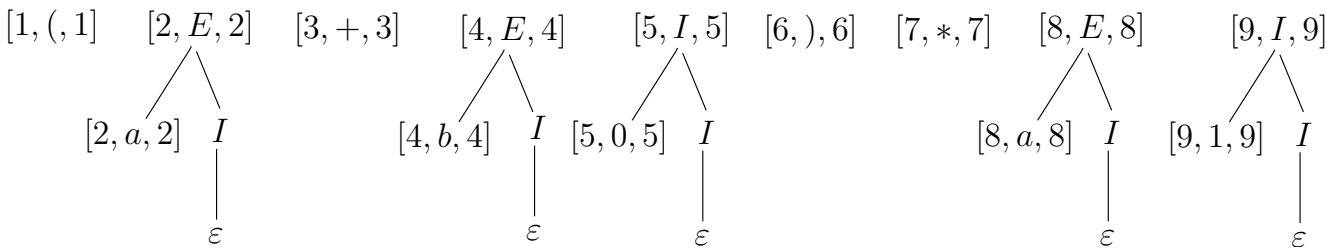
Cocke-Younger-Kasami (CYK) algorithm: continued

The intuition:

The CFG G:

$$E \to EA_3 \quad A_3 \to +E \quad E \to EA_1 \quad A_1 \to *E \quad E \to (A_2 \quad A_2 \to E)$$
$$E \to aI \quad E \to bI \quad I \to 0I \quad I \to 1I \quad I \to \varepsilon$$

(	a	$+$	b	0	)	$*$	a	1
1	2	3	4	5	6	7	8	9



Cocke-Younger-Kasami (CYK) algorithm: continued

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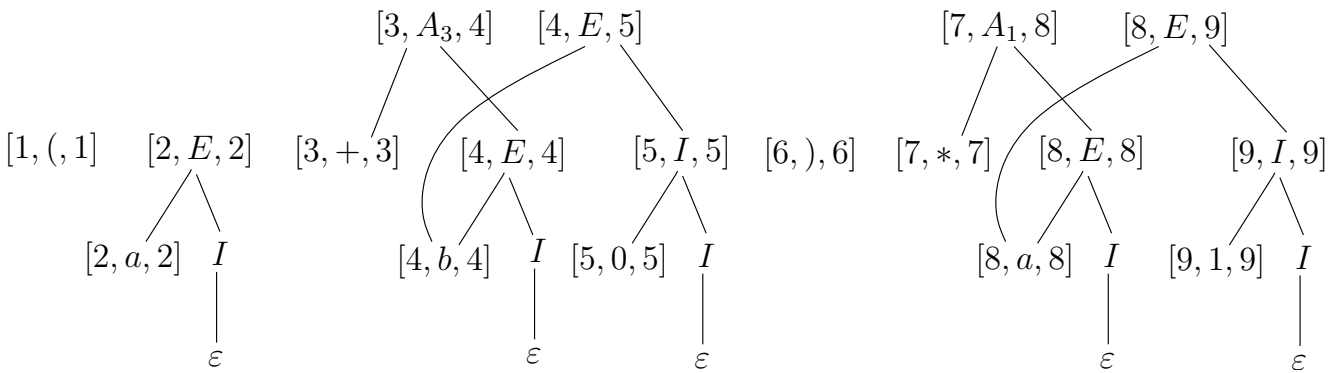

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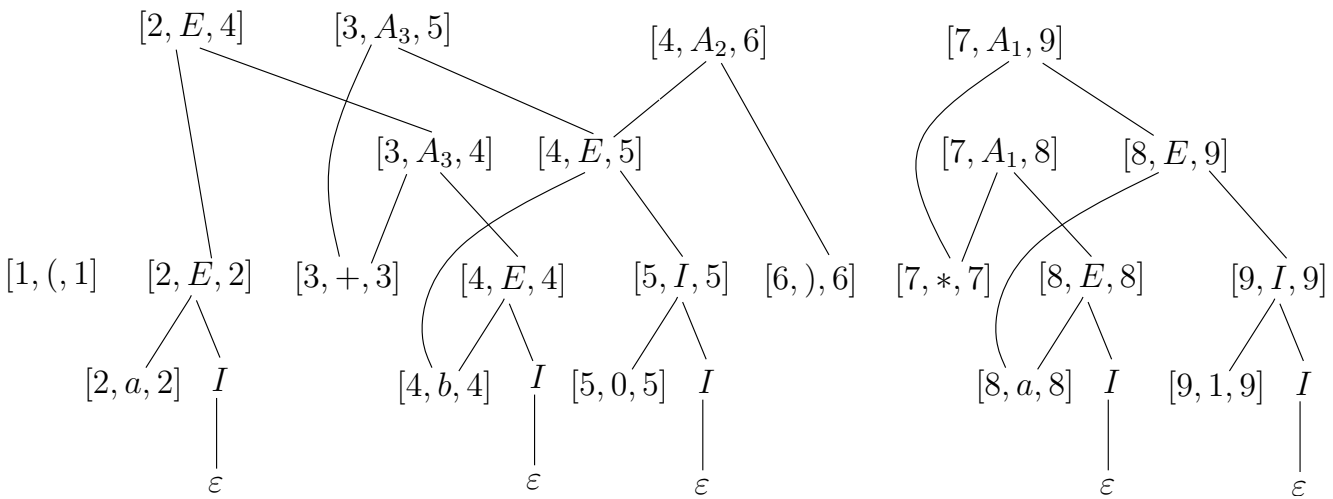

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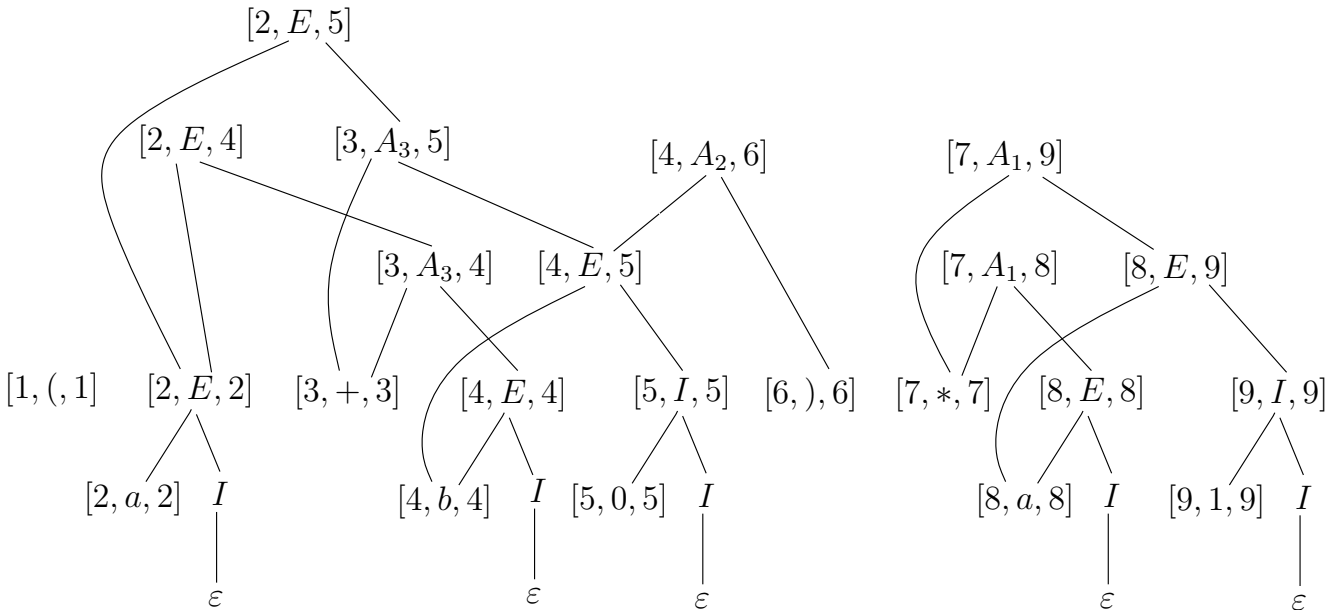

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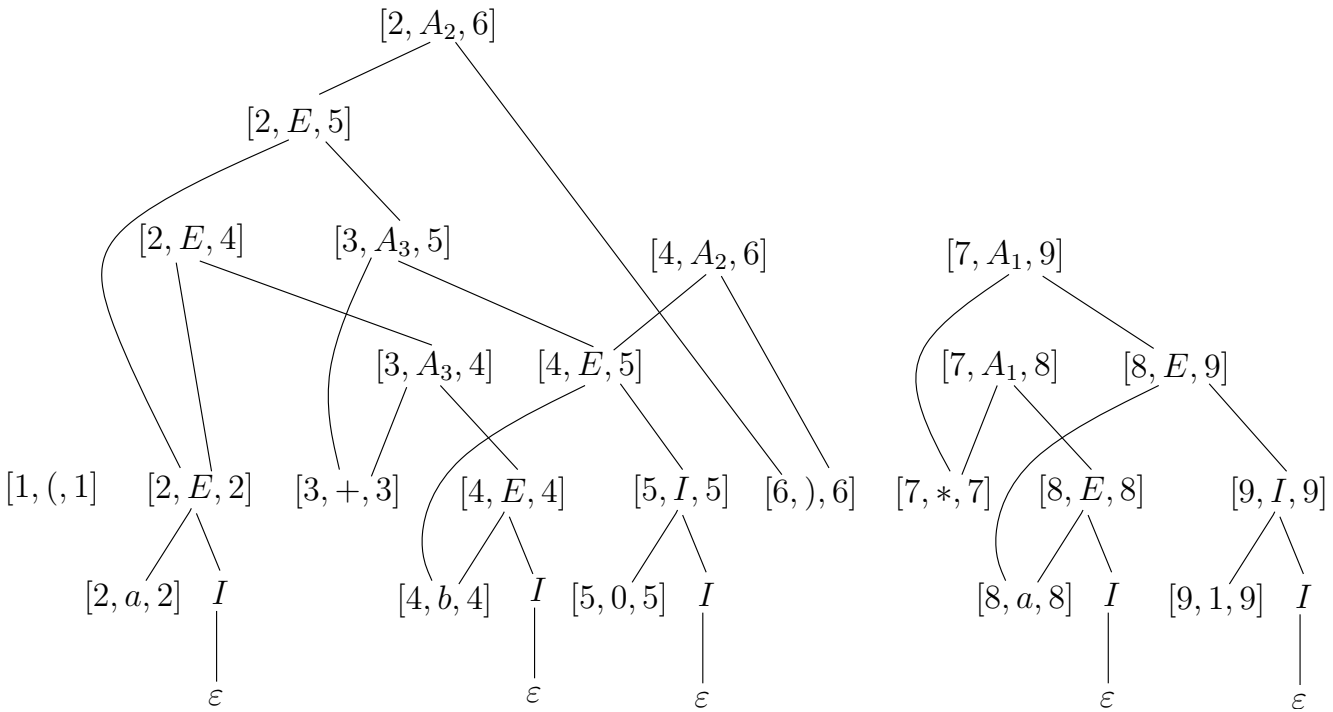

Cocke-Younger-Kasami (CYK) algorithm: continued

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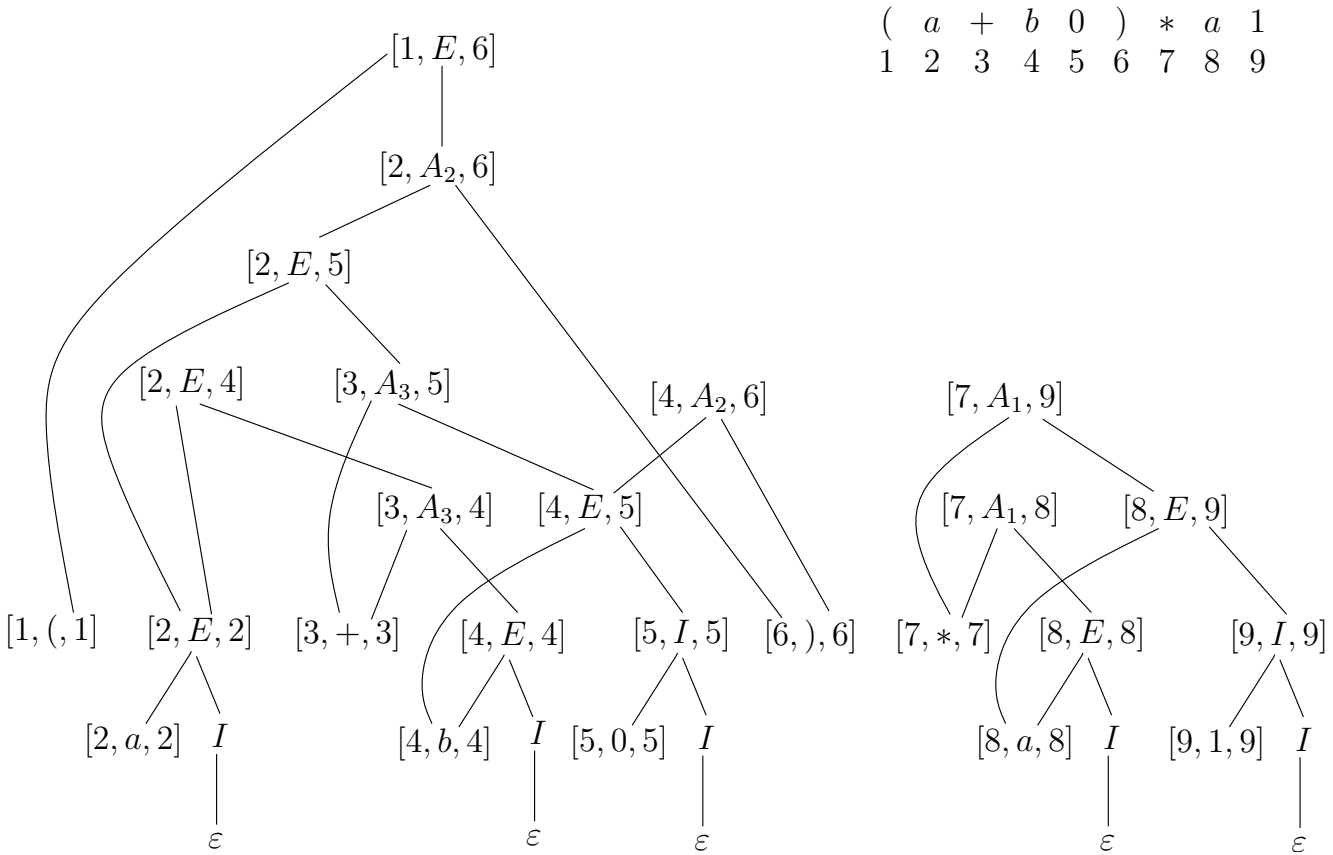

Cocke-Younger-Kasami (CYK) algorithm: continued

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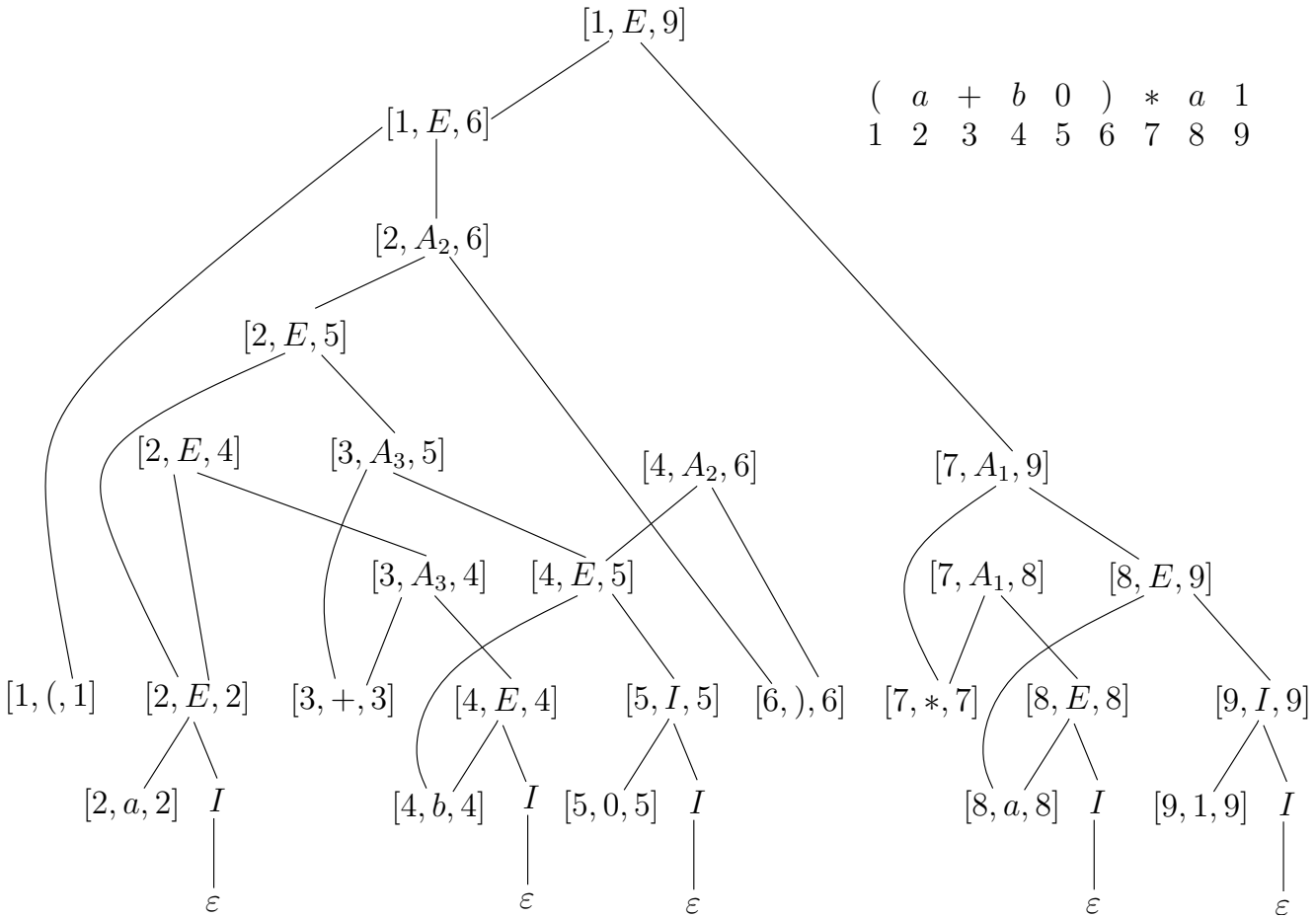
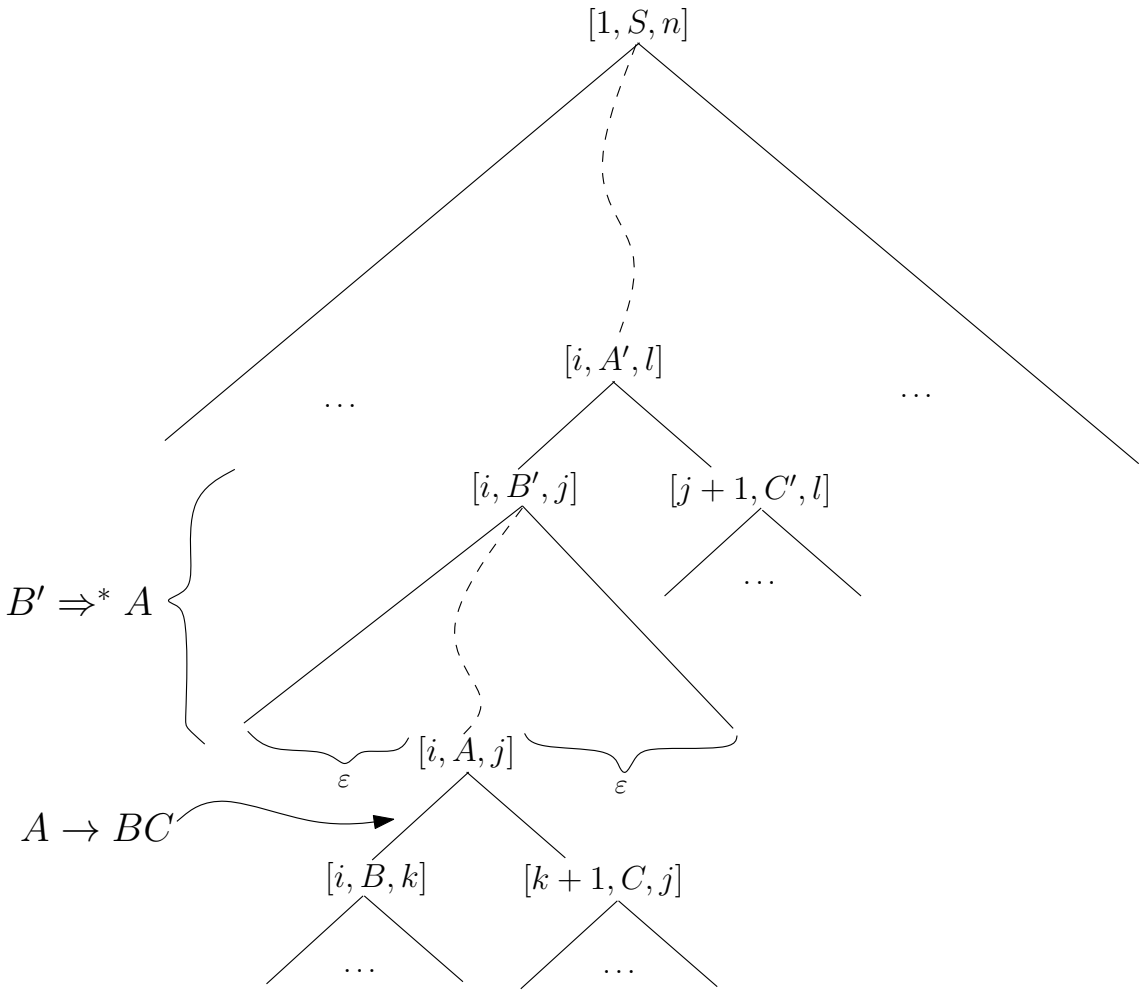


Cocke-Younger-Kasami (CYK) algorithm: continued

The intuition: Suppose G is a CFG in 2NF and $w \in \Sigma^*$.



Cocke-Younger-Kasami (CYK) algorithm: continued

Nullable nonterminals $N(G)$

Nonterminals A s.t. $A \Rightarrow^* \varepsilon$.
The set of nullable nonterminals, $N(G)$, is computed as follows.

- $N_0(G) = \{A \mid A \to \varepsilon\}$,
- $N_{i+1}(G) =$
  $N_i(G) \cup \{A \mid \exists B \in N_i(G).A \to B\} \cup \{A \mid \exists B, C \in N_i(G).A \to BC\}$.

Example: CFG G

$$\begin{array}{cccc} E \to EA_3 & A_3 \to +E & E \to EA_1 & A_1 \to *E \\ E \to (A_2 & A_2 \to E) & E \to aI & E \to bI \\ I \to 0I & I \to 1I & I \to \varepsilon & \end{array}$$

$N(G) = \{I\}$.

Cocke-Younger-Kasami (CYK) algorithm: continued

Unit derivation relation $U(G)$

The unit derivation relation of G, $U(G)$, is defined as

$$\{(A, \gamma) \mid A \Rightarrow^* \gamma, \gamma \in \mathcal{N} \cup \Sigma\}.$$

$U(G)$ is computed as follows.

- $U_0(G) = \left\{ (A, \gamma) \,\middle|\, \begin{array}{c} \gamma \in \mathcal{N} \cup \Sigma, \text{ either } A \to \gamma, \\ \text{or } \exists B \in N(G). A \to B\gamma \vee A \to \gamma B \end{array} \right\}$,
- $U_{i+1}(G) = U_i(G) \cup \left\{ (A, \gamma) \,\middle|\, \begin{array}{c} \gamma \in \mathcal{N} \cup \Sigma, \\ \exists B \in \mathcal{N}. (A, B) \in U_i(G), (B, \gamma) \in U_i(G) \end{array} \right\}$.

For a subset $\mathcal{T} \subseteq \mathcal{N} \cup \Sigma$, define

$$U(\mathcal{T}) = \mathcal{T} \cup \{A \mid \exists \gamma \in \mathcal{T}, (A, \gamma) \in U(G)\}.$$

Example: $U(G) = \{(I, 0), (I, 1), (E, a), (E, b)\}$.

Cocke-Younger-Kasami (CYK) algorithm: continued

Define $V_{i,j} := \{A \mid A \Rightarrow^* w[i,j]\}$ for $i, j : 1 \leqslant i \leqslant j \leqslant n$.
Then $V_{i,j}$'s are computed inductively as follows.

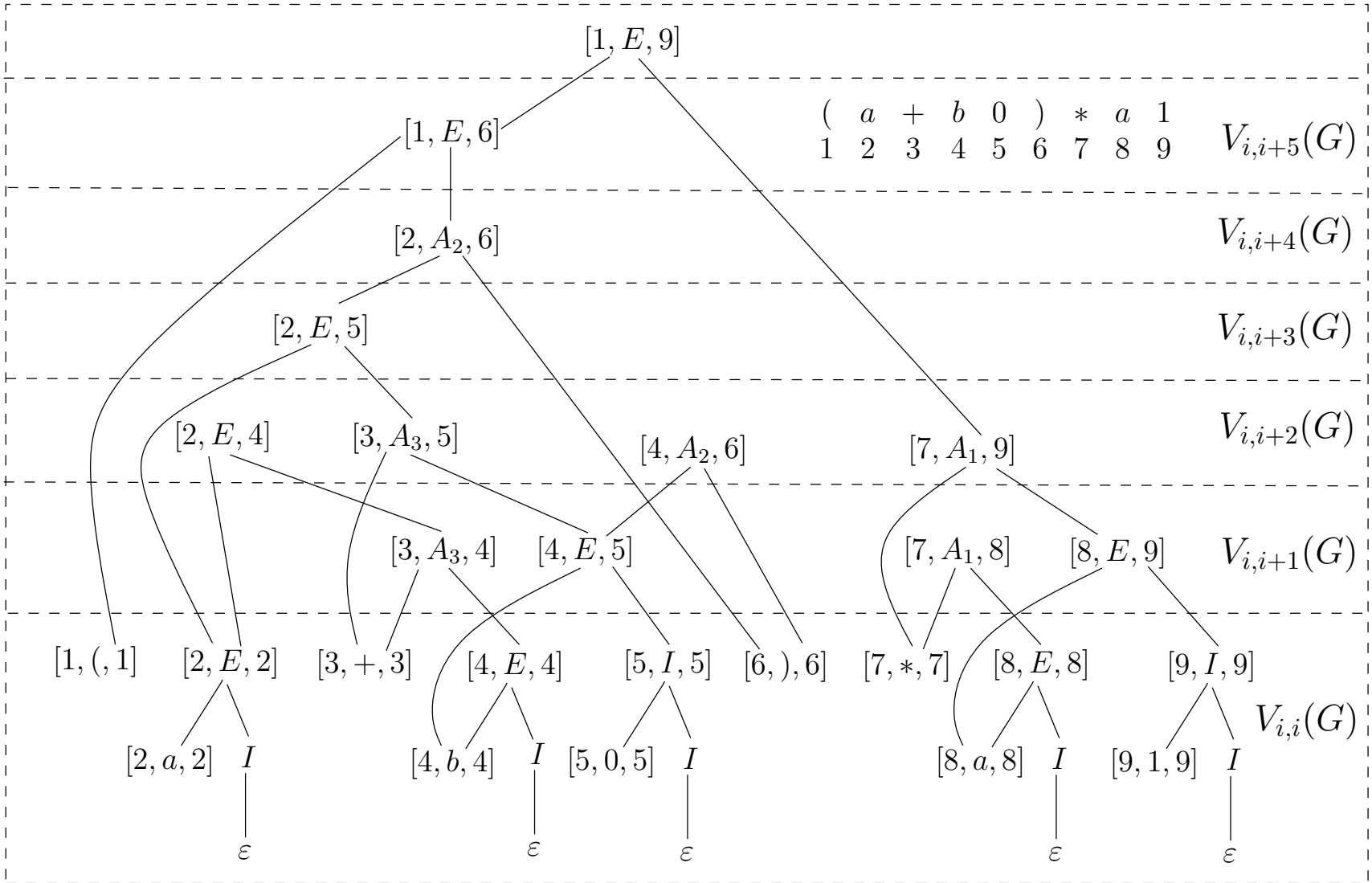
- $V_{i,i} = U(\{w_i\})$,
- For $i, j : 1 \leqslant i < j \leqslant n$,

$$V'_{i,j} = \{A \mid \exists k : i \leqslant k \leqslant j, B \in V_{i,k}, C \in V_{k+1,j}. \text{ s.t. } A \to BC\}.$$

- For $i, j : 1 \leqslant i < j \leqslant n$, $V_{i,j} = U(V'_{i,j})$.

Claim. $w \in L(G)$ *iff* $S \in V_{1,n}$.

Cocke-Younger-Kasami (CYK) algorithm: continued



Summary

1. Equivalence of CSG and LBA
2. Equivalence of CFG and pushdown automata
   - Context-free grammar (CFG)
   - Pushdown automata (PDA)
   - Equivalence of CFG and PDA
3. Properties of CFL
   - Normal form of CFG
   - Pumping lemma
   - Closure properties
4. Decision problems of CFG
   - Emptiness
   - Inclusion
   - Membership

Finite state automata