Automata theory and its applications

Lecture 15 -16: Automata over infinite (ranked) trees

Zhilin Wu

State Key Laboratory of Computer Science,
Institute of Software, Chinese Academy of Sciences

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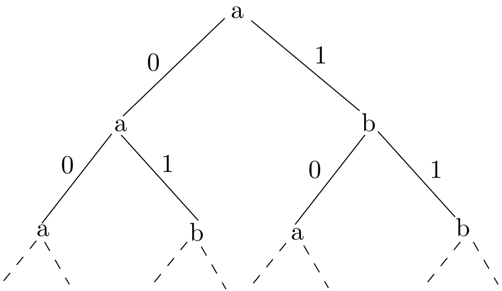
Infinite binary trees

A function $t : \{0,1\}^* \to \Sigma$,



$t(\varepsilon) = a$

$t(x0) = a$

$t(x1) = b$

Let T_Σ^ω denote the set of infinite binary trees over Σ.

Outline

Büchi, Muller, Rabin and parity tree automata

A Büchi tree automaton (BTA) $\mathcal{A}$ is a tuple $(Q, \Sigma, \delta, q_0, F)$ such that

- Q is the set of states,
- Σ is the finite alphabet,
- $\delta \subseteq Q \times \Sigma \times Q \times Q$,
- $q_0 \in Q$, $F \subseteq Q$.

A *run* of a BTA $\mathcal{A}$ over an infinite binary tree t:

An infinite binary tree $r : \{0,1\}^* \rightarrow Q$ s.t.

- $r(\varepsilon) = q_0$,
- for every $x \in \{0,1\}^*$, $(r(x), t(x), r(x0), r(x1)) \in \delta$.

A run r of $\mathcal{A}$ over t is *accepting* if $\forall$ path π in r, $\mathrm{Inf}(r|_\pi) \cap F \neq \emptyset$.

Büchi, Muller, Rabin and parity tree automata

A Muller tree automaton (MTA) $\mathcal{A}$ is a tuple $(Q, \Sigma, \delta, q_0, \mathcal{F})$ s.t.

- Q, Σ, δ, q_0 are the same as BTA,
- $\mathcal{F} \subseteq 2^Q$.

A run r of a MTA $\mathcal{A}$ over an infinite binary tree t is accepting if
$\forall$ path π in r, $\mathrm{Inf}(r|_\pi) \in \mathcal{F}$.

A Rabin tree automaton (RTA) $\mathcal{A}$ is a tuple $(Q, \Sigma, \delta, q_0, (U_i, V_i)_{1 \leqslant i \leqslant k})$ s.t.

- Q, Σ, δ, q_0 are the same as BTA,
- $\forall i : 1 \leqslant i \leqslant k.\ U_i, V_i \subseteq Q$.

A run r of a RTA $\mathcal{A}$ over an infinite binary tree t is accepting if
$\forall$ path π in r, $\exists i : 1 \leqslant i \leqslant k$, $\mathrm{Inf}(r|_\pi) \cap U_i = \varnothing$ and $\mathrm{Inf}(r|_\pi) \cap V_i \neq \varnothing$.

A Parity tree automaton (PTA) $\mathcal{A}$ is a tuple $(Q, \Sigma, \delta, q_0, c)$ s.t.

- Q, Σ, δ, q_0 are the same as BTA,
- $c : Q \to \{1, \ldots, k\}$.

A run r of a PTA $\mathcal{A}$ over an infinite binary tree t is accepting if
$\forall$ path π in r, $\min\{c(q) \mid q \in \mathrm{Inf}(r|_\pi)\}$ is even.

Outline

Expressibility

Proposition. MTA $\equiv$ RTA $\equiv$ PTA.

Proof.

PTA $\subseteq$ RTA $\subseteq$ MTA: By definition.
MTA $\subseteq$ PTA: Latest appearance record. $\square$

Expressibility

Proposition. MTA $\equiv$ RTA $\equiv$ PTA.

Proposition. MTA $>$ BTA.

Proof.

L: The set of trees s.t. along every path, a only occurs finitely many times.

Claim. L is expressible in MTA, but not in BTA.

L is defined by the MTA $\mathcal{A} = (\{q_0, q_1\}, \Sigma, \delta, q_0, \{\{q_1\}\})$, where
$\delta = \{(q_0, a, q_0, q_0), (q_0, b, q_1, q_1), (q_1, b, q_1, q_1), (q_1, a, q_0, q_0)\}$.

□

Expressibility

Proposition. MTA $\equiv$ RTA $\equiv$ PTA.

Proposition. MTA $>$ BTA.

Proof.

L: The set of trees s.t. along every path, a only occurs finitely many times.

Claim. L is expressible in MTA, but not in BTA.

To the contrary, suppose L is defined by a BTA $\mathcal{B} = (Q, \Sigma, \delta, q_0, F)$ of n states. Consider the infinite tree t

where a occurs exactly at the positions 1^+0, 1^+01^+0, $\ldots$, $(1^+0)^n$.

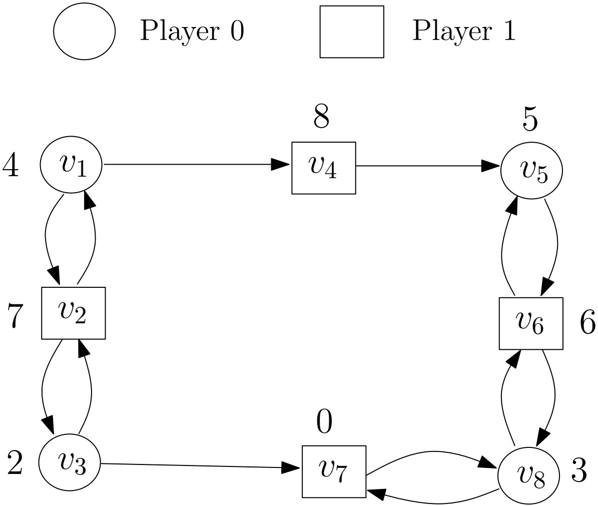
Evidently, t is accepted by $\mathcal{B}$, so there is an accepting run r of $\mathcal{B}$ over t. Then $\exists m_0, m_1, \ldots, m_n$ s.t. $r(1^{m_0}), r(1^{m_0}01^{m_1}), \ldots, r(1^{m_0}0\ldots01^{m_n}) \in F$. Therefore, $\exists i, j : i < j$ s.t. $r(1^{m_0}0\ldots01^{m_i}) = r(1^{m_0}0\ldots01^{m_j})$. Let t' be the tree obtained from t by

repeating the path from $1^{m_0}0\ldots01^{m_i}$ *to* $1^{m_0}0\ldots01^{m_j}$,
with subtrees of the nodes on the path copied.

Then t' is accepted by $\mathcal{B}$, but t' contains a path where a occurs infinitely often, a contradiction. □

Outline

Parity game: An example



Parity game

A *parity game* $\mathcal{G}$ consists of

- a *game graph* (possibly infinite) which is a bipartite graph $G = (V_0, V_1, E)$ s.t. $\forall v \in V_0 \cup V_1$, vE is nonempty and finite,
- a *colouring* function $c : V_0 \cup V_1 \to \mathbb{N}$.

Two *players*: Player 0 and 1 in $\mathcal{G}$, with V_0 and V_1 as resp. their *territory*.

A *Play* π is an infinite path $v_0 v_1 \dots$ in the graph $\mathcal{G}$.

Winning condition

Player 0 (resp. Player 1) *wins* a play π if

$$\min(\mathrm{Inf}(c(\pi))) \textit{ is even (resp. odd).}$$

Winning strategy

Conform to ...

Let $\sigma \in \{0,1\}$ and $f_\sigma : (V_0 \cup V_1)^* V_\sigma \to V_{1-\sigma}$ a partial function.
A prefix of a play $v_0 \dots v_n$ *conforms* to f_σ if

for every $i < n$ *s.t.* $v_i \in V_\sigma$, $v_{i+1} = f_\sigma(v_1 \dots v_i)$.

A play π conforms to f_σ if every prefix of π conforms to f_σ.

Strategy and winning strategy

A *strategy* of Player σ on $U \subseteq V_0 \cup V_1$ is
a partial function $f_\sigma : (V_0 \cup V_1)^* V_\sigma \to V_{1-\sigma}$ s.t.

$\forall$ *prefix of a play* $v_0 \dots v_n \in (V_0 \cup V_1)^* V_\sigma$ *starting from* U *and conforming to* f_σ, $f_\sigma(v_0 \dots v_n)$ *is defined.*

We can assume that the domain of f_σ is minimal wrt. the above condition.

A *winning strategy* of Player σ on U is a strategy f_σ of Player σ on U s.t.

Player σ *wins every play* π *starting from* U *and conforming to* f_σ.

Winning region

Proposition. If Player σ has a winning strategy on U_1 and U_2, then Player σ has a winning strategy on $U_1 \cup U_2$.

Proof.

Let $f_{\sigma,1}, f_{\sigma,2}$ be the winning strategy of Player σ on U_1 and U_2 respectively. Define a strategy f_σ for Player σ on U as follows:

$$f_\sigma(v_0 \dots v_n) = \begin{cases} f_{\sigma,1}(v_0 \dots v_n), & f_{\sigma,1}(v_0 \dots v_n) \text{ is defined} \\ f_{\sigma,2}(v_0 \dots v_n), & \text{otherwise} \end{cases}$$

f_σ is a winning strategy for Player σ on $U_1 \cup U_2$:
For every play $\pi = v_0 v_1 \dots$ conforming to f_σ and starting from $U_1 \cup U_2$,

- if π starts from a vertex in U_1, then $f_{\sigma,1}$ is used, Player σ wins,
- otherwise, $f_{\sigma,2}$ is used, Player σ wins.

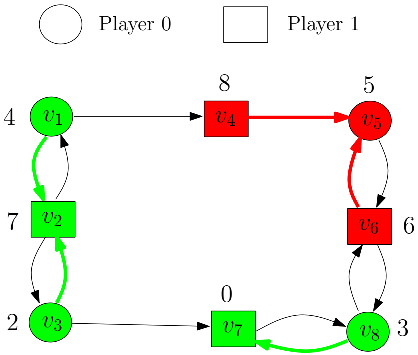
□

Winning region

Proposition. If Player σ has a winning strategy on U_1 and U_2, then Player σ has a winning strategy on $U_1 \cup U_2$.

Winning region of Player σ in $\mathcal{G}$ ($\text{Win}_\sigma(\mathcal{G})$)

The maximum set U s.t. Player σ has a winning strategy on U.



Determinacy

Theorem (Martin 1975). Every parity game is determined, i.e. $\text{Win}_0(\mathcal{G})$ and $\text{Win}_1(\mathcal{G})$ form a partition of $V_0 \cup V_1$.

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Theorem (Martin 1975). Every parity game is determined, i.e. $\text{Win}_0(\mathcal{G})$ and $\text{Win}_1(\mathcal{G})$ form a partition of $V_0 \cup V_1$.

Memoryless strategy for Player σ in $\mathcal{G}$ on U:

A partial function $f_\sigma : (V_0 \cup V_1)^* V_\sigma \to V_{1-\sigma}$ *s.t.*
$f_\sigma(v_0 \dots v_{n-1} v_n)$ *is independent of* $v_0 \dots v_{n-1}$,
that is, there is a partial function $g : V_\sigma \to V_{1-\sigma}$ *s.t.*
$\forall v_0 \dots v_n \in (V_0 \cup V_1)^* V_\sigma.\ f_\sigma(v_0 \dots v_{n-1} v_n) = g(v_n)$.

Theorem(Emerson & Jutla 1991, Mostowski 1991). Every parity game is memoryless determined, i.e.

Player 0 (resp. Player 1) has a memoryless winning strategy in $\mathcal{G}$ *on* $\text{Win}_0(\mathcal{G})$ *(resp.* $\text{Win}_1(\mathcal{G})$*).*

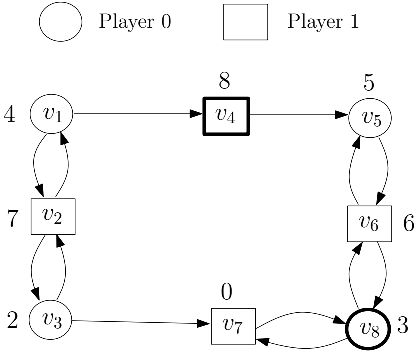
Memoryless determinacy: A proof

Reachability game: $\mathcal{G} = (G, U)$ s.t.

- $G = (V_0, V_1, E)$ is the same as that in parity games,
- $U \subseteq V_0 \cup V_1$: the set of destination vertices.

Two players: Player 0 and Player 1,

- the goal of Player 0 is to reach a destination,
- the goal of Player 1 is to prevent Player 0 to do so.



Memoryless determinacy: A proof

Attractor set ($Att_\sigma(G, U)$):

Player σ can force a visit to vertices in U in finitely many steps, no matter how Player $1 - \sigma$ plays.

$Att_\sigma(G, U) = \bigcup_{i \geqslant 0} U_i$, where U_i's are defined as follows,

- $U_0 = U$,
- $U_{i+1} = U_i \cup \{u \in V_\sigma \mid \exists v. v \in uE \wedge v \in U_i\} \cup \{u \in V_{1-\sigma} \mid uE \subseteq U_i\}$.

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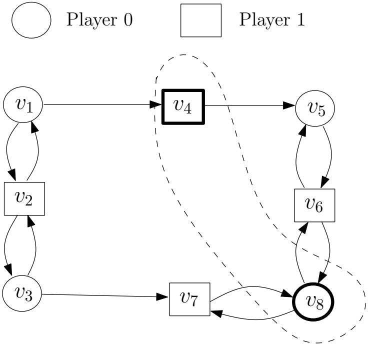

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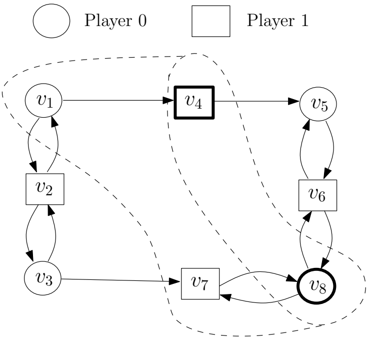

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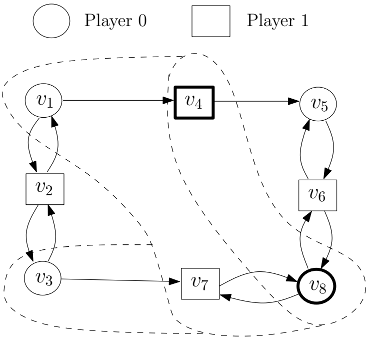

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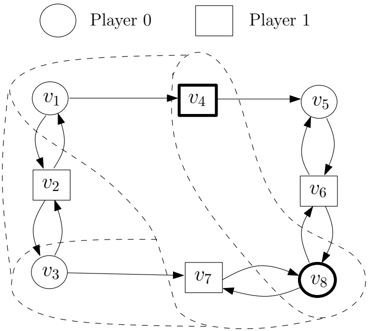

Memoryless determinacy: A proof

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In addition, a memoryless strategy for Player σ on $Att_\sigma(G, U)$ is obtained by choosing for every vertex in $(U_{i+1} \backslash U_i) \cap V_\sigma$ a successor in U_i.

A **Trap** for Player σ:

A set $Z \subseteq V_0 \cup V_1$ s.t. for every vetex $v \in V_\sigma \cap Z$, $vE \subseteq Z$.

Proposition. Let $Z = (V_0 \cup V_1) \backslash Att_\sigma(G, U)$. Then the following facts hold.

- Z is a trap for Player σ.
- For every vertex $v \in V_{1-\sigma} \cap Z$, $vE \cap Z \neq \emptyset$.

Corollary. Let $G' = (Z, E' = E \cap Z \times Z)$. Then G' is a game graph.

Memoryless determinacy: A proof

A notation(subgame):

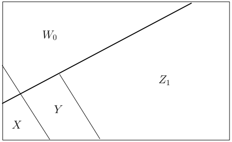
Let $\mathcal{G} = (G = (V_0, V_1, E), c)$ and $Z \subseteq V_0 \cup V_1$ s.t. $G[Z]$ is a game graph. Then $\mathcal{G}[Z]$ denotes the parity game $(G[Z], c|_Z)$.

The proof is by an induction on the number of colours in a parity game $\mathcal{G}$.

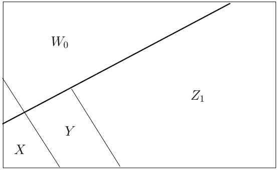
W.l.o.g. assume that $k = \min\{c(v) \mid v \in V_0 \cup V_1\}$ is odd.
Let $X = \{v \mid c(v) = k\}$.
Let W_0 be

> *the maximum set of vertices on which Player 0 has a memoryless winning strategy.*

In addition, let $Z = (V_0 \cup V_1) \backslash W_0$.
We show that Player 1 has a memoryless winning strategy on Z.
Let $Y = \mathrm{Attr}_1(G, X \backslash W_0)$ and $Z_1 = (V_0 \cup V_1) \backslash (W_0 \cup Y)$.



Memoryless determinacy: A proof



Fact. $G[Z_1]$ is a game graph.

- $Z = Y \cup Z_1$ is a trap for Player 0 in G + $Y = \text{Attr}_1(G, X \backslash W_0)$
  $\Longrightarrow \forall v \in V_0 \cap Z_1 . vE \cap Z_1 \neq \emptyset$.
- $\forall v \in V_1 \cap Z$, $vE \cap Z \neq \emptyset$ + Z_1 is a trap for Player 1 in $G[Z]$
  $\Longrightarrow \forall v \in V_1 \cap Z_1 . vE \cap Z_1 \neq \emptyset$.

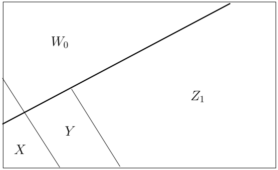
By induction hypothesis, $\mathcal{G}[Z_1]$ is memoryless determined.

Fact. $\text{Win}_1(\mathcal{G}[Z_1]) = Z_1$.

Player 0 has a memoryless winning strategy on $\emptyset \neq U \subseteq Z_1$ in $\mathcal{G}[Z_1]$
$\Rightarrow$ Player 0 has also one on U in $\mathcal{G}$.

- if during a play, Player 1 chooses to enter W_0, then Player 0 applies the strategy on W_0 in $\mathcal{G}$,
- otherwise, the play stays in Z_1, Player 0 applies the strategy on U in $\mathcal{G}[Z_1]$.

Memoryless determinacy: A proof



Memoryless winning strategy f of Player 1 on Z in $\mathcal{G}$:

- if during a play starting from a vertex in Z, the current vertex is in Z_1, then Player 1 applies the memoryless strategy of Player 1 in $\mathcal{G}[Z_1]$,
- if during a play starting from a vertex in Z, the current vertex is in Y, then Player 1 applies the memoryless strategy of the attractor set to force visiting $X \cap Z$.

For every play π starting from Z and conforming to f,

- if eventually, π stays in Z_1, then Player 1 wins,
- otherwise, π visits $X \cap Z$ infinitely often, the minimum color occurring in π is odd, Player 1 wins.

Outline

Union and intersection

Proposition. PTAs are closed under union and intersection.

Proof.

Union.

Suppose $\mathcal{A}_1 = (Q_1, \Sigma, \delta_1, q_{0,1}, c_1)$ and $\mathcal{A}_2 = (Q_2, \Sigma, \delta_2, q_{0,2}, c_2)$ are two PTAs.
Let $\mathcal{A} = (Q_1 \cup Q_2 \cup \{q_0\}, \Sigma, \delta, q_0, c)$ s.t.

$\delta = \delta_1 \cup \delta_2 \cup \{(q_0, q_{0,1}), (q_0, q_{0,2})\}$, and $c = c_1 \cup c_2 \cup \{q_0 \to 0\}$.

Then $\mathcal{A}$ defines $\mathcal{L}(\mathcal{A}_1) \cup \mathcal{L}(\mathcal{A}_2)$.

Intersection.

We prove instead that MTAs are closed under intersection.
Suppose $\mathcal{A}_1 = (Q_1, \Sigma, \delta_1, q_{0,1}, \mathcal{F}_1)$ and $\mathcal{A}_2 = (Q_2, \Sigma, \delta_2, q_{0,2}, \mathcal{F}_2)$ are two MTAs.
Then $\mathcal{A} = (Q_1 \times Q_2, \Sigma, \delta, (q_{0,1}, q_{0,2}), \mathcal{F})$ defines $\mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{A}_2)$, where

- δ: if $(q_1, \sigma, q_1') \in \delta_1, (q_2, \sigma, q_2') \in \delta_2$, then $((q_1, q_2), \sigma, (q_1', q_2')) \in \delta$,
- $\mathcal{F} = \{P \subseteq Q_1 \times Q_2 \mid \mathrm{Proj}_1(P) \in \mathcal{F}_1, \mathrm{Proj}_2(P) \in \mathcal{F}_2\}$.

□

Complementation

Game-theoretical view of PTA.

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, c)$ be a PTA and t be an infinite tree.
A run of $\mathcal{A}$ over t can be seen as a parity game $\mathcal{G}_{\mathcal{A},t} = (G, c')$

- Two players:
  - Player 0(Automaton):
    Guess a run of $\mathcal{A}$ over t and assert that the run is accepting,
  - Player 1(Pathfinder):
    Challenge Automaton by
    choosing a path and asserting that the path is not accepting.
- Game graph $G = (V_0, V_1, E)$:
  - $V_0 = \{0,1\}^* \times \Sigma \times Q$,
  - $V_1 = \{0,1\}^* \times \Sigma \times \delta$,
  - if $x \in \{0,1\}^*$, $t(x) = \sigma$, and $(q, \sigma, q_1, q_2) \in \delta$, then
    $(x, \sigma, q)E(x, \sigma, (q, \sigma, q_1, q_2))$,
    $(x, \sigma, (q, \sigma, q_1, q_2))E(x0, t(x0), q_1)$, $(x, \sigma, (q, \sigma, q_1, q_2))E(x1, t(x1), q_2)$.
- $c'((x, \sigma, q)) = c'(x, \sigma, (q, \sigma, q_1, q_2)) = c(q)$,

Proposition. $\mathcal{A}$ accepts t iff Automaton has a winning strategy in $\mathcal{G}_{\mathcal{A},t}$ starting from $(\varepsilon, t(\varepsilon), q_0)$.

Complementation

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, c)$ be a PTA and t be an infinite tree.
By memoryless determinacy of parity games,

$\mathcal{A}$ does not accept t
iff
Pathfinder has a memoryless winning strategy in $\mathcal{G}_{\mathcal{A},t}$ starting from $(\varepsilon, t(\varepsilon), q_0)$

Pathfinder's strategy:

A function $f : \{0,1\}^* \times \Sigma \times \delta \to \{0,1\}$.

$\forall x \in \{0,1\}^*$, let $f_x : \Sigma \times \delta \to \{0,1\}$ defined by $f_x(\sigma, \tau) = f(x, \sigma, \tau)$.

Let $I = \Sigma \times \delta \to \{0,1\}$, Pathfinder's strategy can be reformulated as:

An I-labeled infinite tree s, with each node x labeled by f_x.

Complementation

A *play* in $\mathcal{G}_{\mathcal{A},t}$ starting from $(\varepsilon, t(\varepsilon), q_0)$ can be described by

$$A\ sequence\ (\tau_0, \pi_0)(\tau_1, \pi_1) \cdots \in (\delta \times \{0,1\})^\omega\ s.t.$$

$\forall i$, let $\tau_i = (p_i, \sigma_i, p_{i,1}, p_{i,2})$, then

- $p_0 = q_0$,
- $\tau_0 \tau_1 \ldots$ is consecutive: $\forall i.\ p_{i+1} \in \{p_{i,1}, p_{i,2}\}$,
- $\tau_0 \tau_1 \ldots$ and $\pi_0 \pi_1 \ldots$ are compatible,
  $$\forall i.\ \pi_i = 0\ (resp.\ \pi_i = 1)\ iff\ p_{i+1} = p_{i,1}\ (resp.\ p_{i+1} = p_{i,2}),$$
- $\tau_0 \tau_1 \ldots$ respects $t|_{\pi_0 \pi_1 \ldots}$,
  $$\forall i.\ \sigma_i = t(\pi_0 \ldots \pi_{i-1}).$$

A reformulation:

Pathfinder has a memoryless winning strategy in $\mathcal{G}_{\mathcal{A},t}$ starting from $(\varepsilon, t(\varepsilon), q_0)$
iff
$\exists$ *an I-labeled tree s*
$\forall$ *plays* $(\tau_0, \pi_0)(\tau_1, \pi_1) \cdots \in (\delta \times \{0,1\})^\omega$ *conforming to s,*
the state-seq. determined by $\tau_0 \tau_1 \dots$ violates the min-parity cond.

$(\tau_0, \pi_0)(\tau_1, \pi_1) \cdots \in (\delta \times \{0,1\})^\omega$ conforms to s:

$s|_{\pi_0\pi_1\dots}$ *applies to* $t|_{\pi_0\pi_1\dots}$ *and $\tau_0\tau_1 \dots$ indeed produces $\pi_0\pi_1\dots$,*
more specifically, $\forall i.\ s(\pi_0 \dots \pi_{i-1})((t(\pi_0 \dots \pi_{i-1}), \tau_i)) = \pi_i$.

Complementation

$\exists$ *an* I*-labeled tree* s

$\forall$ *plays* $(\tau_0, \pi_0)(\tau_1, \pi_1) \cdots \in (\delta \times \{0,1\})^\omega$ *conforming to* s,

the state-seq. determined by $\tau_0 \tau_1 \ldots$ *violates the min-parity cond.*

iff

(1) $\exists$ *an* I*-labeled tree* s

(2) $\forall\ \pi \in \{0,1\}^\omega$,

(3) $\forall\ \tau_0 \tau_1 \cdots \in \delta^\omega$,

(4) *if* $(\tau_0, \pi_0)(\tau_1, \pi_1) \ldots$ *is a play in* $\mathcal{G}_{\mathcal{A},t}$ *and conforms to* s,

then the state-seq. determined by $\tau_0 \tau_1 \ldots$ *violates the min-parity cond.*

Complementation

(1) $\exists$ *an I-labeled tree s*
(2) $\forall\ \pi \in \{0,1\}^\omega$,
(3) $\forall\ \tau_0\tau_1 \cdots \in \delta^\omega$,
(4) *if* $(\tau_0, \pi_0)(\tau_1, \pi_1)\ldots$ *is a play in* $\mathcal{G}_{\mathcal{A},t}$ *and conforms to s,*
then the state-seq. determined by $\tau_0\tau_1\ldots$ *violates the min-parity cond.*

Condition (4): Seen as a property of $(I \times \Sigma \times \delta \times \{0,1\})$-labeled ω-words.
Let $(g_0, \sigma_0, \tau_0, \pi_0)(g_1, \sigma_1, \tau_1, \pi_1) \cdots \in (I \times \Sigma \times \delta \times \{0,1\})^\omega$, then

- $\forall i.\ \tau_i = (p_i, \sigma_i, p_{i,1}, p_{i,2})$ for some $p_i, p_{i,1}, p_{i,2}$,
- $p_0 = q_0$,
- $\tau_0\tau_1\ldots$ is consecutive: $\forall i.\ p_{i+1} \in \{p_{i,1}, p_{i,2}\}$,
- $\tau_0\tau_1\ldots$ and $\pi_0\pi_1\ldots$ are compatible:
  $\forall i.\ \pi_i = 0$ (resp. $\pi_i = 1$) iff $p_{i+1} = p_{i,1}$ (resp. $p_{i+1} = p_{i,2}$),
- $\forall i.\ g_i((\sigma_i, \tau_i)) = \pi_i$.

A (deterministic) PA $\mathcal{M}$ over $(I \times \Sigma \times \delta \times \{0,1\})$-labeled ω-words
can be constructed for Cond. (4).

Complementation

(1) $\exists$ *an* I*-labeled tree* s
(2) $\forall\ \pi \in \{0,1\}^\omega$,
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(4) *if* $(\tau_0,\pi_0)(\tau_1,\pi_1)\ldots$ *is a play in* $\mathcal{G}_{\mathcal{A},t}$ *and conforms to* s,
then the state-seq. determined by $\tau_0\tau_1\ldots$ *violates the min-parity cond.*

Condition (3).
A deterministic PA $\mathcal{M}'$ over $(I \times \Sigma \times \{0,1\})$-labeled ω-words for Cond. (3):

1. Complement $\mathcal{M}$,
2. Projection from $I \times \Sigma \times \delta \times \{0,1\}$ to $I \times \Sigma \times \{0,1\}$,
3. determinize and complement.

Size of $\mathcal{M}'$ (By Safra construction, not covered in this course):

- number of states: $2^{O(nk\log(nk))}$,
- number of colors: $O(nk)$,

where n and k are resp. the number of states and colors of $\mathcal{A}$.

Complementation

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(2) $\forall\ \pi \in \{0,1\}^\omega$,
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(4) *if* $(\tau_0, \pi_0)(\tau_1, \pi_1)\dots$ *is a play in* $\mathcal{G}_{\mathcal{A},t}$ *and conforms to* s*,*
then the state-seq. determined by $\tau_0\tau_1\dots$ *violates the min-parity cond.*

Condition (2).
Suppose $\mathcal{M}' = (Q', I \times \Sigma \times \{0,1\}, \delta', q'_0, c')$.
A det. PTA $\mathcal{C} = (Q', I \times \Sigma, \delta'', q'_0, c')$ over $(I \times \Sigma)$-labeled infinite trees for Cond. (2): $\delta''(q, (g, \sigma)) = (q_1, q_2)$ iff $\delta'(q, (g, \sigma, 0)) = q_1$, $\delta'(q, (g, \sigma, 1)) = q_2$.

Remark. Why $\mathcal{M}'$ should be deterministic in order to get the PTA $\mathcal{C}$?
A counter example:

Consider the NPA $(\{q_0, q_1\}, \{(a,0), (a,1), (b,0), (b,1)\}, \delta, q_0, c)$ s.t.

- $\delta = \{q_0 \xrightarrow{(a,i)} q_0, q_0 \xrightarrow{(a,i)} q_1, q_0 \xrightarrow{(b,i)} q_0, q_0 \xrightarrow{(b,i)} q_1, q_1 \xrightarrow{(b,i)} q_1\}$, where $i = 0, 1$,
- $c(q_0) = 1, c(q_1) = 2$.

Condition (1).
$\mathcal{C}$ is projected from $I \times \Sigma$ to Σ to get a PTA $\mathcal{B}$ over Σ-labeled infinite trees.

Outline

MSO over infinite binary trees

Syntax

$\varphi := P_\sigma(x) \mid X(x) \mid S_0(x,y) \mid S_1(x,y) \mid \neg\varphi_1 \mid \varphi_1 \vee \varphi_2 \mid \exists x \varphi_1(x) \mid \exists X \varphi_1(X)$

Semantics: Interpreted over infinite binary trees

- $\cdots$,
- $S_0(x,y)$ iff $y = x0$,
- $S_1(x,y)$ iff $y = x1$,
- $\cdots$.

Example:

- $x \leqslant y$:
  $\forall X(X(x) \wedge \forall z_1 \forall z_2 (X(z_1) \wedge (S_0(z_1,z_2) \vee S_1(z_1,z_2)) \rightarrow X(z_2)) \rightarrow X(y))$
- $\varphi_{path}(X)$:
  $$\exists x(\forall y(x \leqslant y) \wedge X(x)) \wedge \forall x(X(x) \rightarrow \exists y((S_0(x,y) \vee S_1(x,y)) \wedge X(y))) \\ \wedge \forall x \forall y(X(x) \wedge X(y) \rightarrow x \leqslant y \vee y \leqslant x).$$

Normal form for MSO over infinite binary trees:

$\varphi := P_\sigma \subseteq X \mid X \subseteq Y \mid \mathrm{Sing}(X) \mid S_0(X,Y) \mid S_1(X,Y) \mid \neg\varphi_1 \mid \varphi_1 \vee \varphi_2 \mid \exists X \varphi_1(X)$

PTA $\equiv$ MSO

Theorem. PTA $\equiv$ MSO.

Proof.

From MSO to PTA:
Similar to infinite word case,
using the normal form and the closure properties of PTA.

From PTA to MSO:
Describe a run of PTA over infinite binary trees by the MSO sentence,

$$\varphi := \exists q_1 \dots q_n (\varphi_{init} \wedge \varphi_{trans} \wedge \varphi_{acc}),$$

- $\varphi_{init} := \exists x (\forall y (x \leqslant y) \wedge q_0(x)),$
- $\varphi_{trans} := \forall x \forall y (S_0(x,y) \vee S_1(x,y) \rightarrow \bigvee_{(q,\sigma,q') \in \delta} q(x) \wedge P_\sigma(x) \wedge q'(y)),$
- $\varphi_{acc} := \forall X (\varphi_{path}(X) \rightarrow$

$$\exists x \left(X(x) \wedge \bigvee_{q : c(q) \text{ even}} \left(\begin{array}{c} \forall y (x \leqslant y \wedge X(y) \rightarrow \bigwedge_{q' : c(q') < c(q)} \neg q'(y)) \wedge \\ \forall y (x \leqslant y \wedge X(y) \rightarrow \exists z (y \leqslant z \wedge X(z) \wedge q(z))) \end{array} \right) \right)).$$

$\square$

Outline

Nonemptiness

Theorem. The nonemptiness of PTA is in NP $\cap$ co-NP.

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, c)$ be a PTA.
The nonemptiness of $\mathcal{A}$ is reduced to the parity game $\mathcal{G}_{\mathcal{A}} = (Q, \delta, E, c')$:

- For every $(q, \sigma, q_1, q_2) \in \delta$,
  - $(q, (q, \sigma, q_1, q_2)) \in E$,
  - $((q, \sigma, q_1, q_2), q_1), ((q, \sigma, q_1, q_2), q_2) \in E$.
- $c'(q) = c'((q, \sigma, q_1, q_2)) = c(q)$.

Lemma. Given a parity game $\mathcal{G} = (V_0, V_1, E, c)$ and $v \in V_0 \cup V_1$, deciding whether $v \in \text{Win}_0(\mathcal{G})$ or $v \in \text{Win}_1(\mathcal{G})$ is in NP $\cap$ co-NP.

Proof.

1. Guess a memoryless strategy $f : V_0 \to V_1$ for Player 0 on $\{v\}$.
2. Verify that f is winning for Player 0.

   Let $G_f = (V_0, V_1, E \cap \{(v, f(v)) \mid v \in V_0\})$.
   Decide whether there is a cycle whose min-parity is odd in G_f. □

Language inclusion

Theorem. The language inclusion problem of PTA is EXPTIME-c.

Let $\mathcal{A}_1 = (Q_1, \Sigma, \delta_1, q_{0,1}, c_1)$ and $\mathcal{A}_2 = (Q_2, \Sigma, \delta_2, q_{0,2}, c_2)$ be two PTAs.

Upper bound:

- construct a PTA $\mathcal{A}$ for $\mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\overline{\mathcal{A}_2})$ of $2^{O(nk)}$ states and $O(nk)$ colors, where
  - n: the maximum of the number of states of $\mathcal{A}_1$ and $\mathcal{A}_2$,
  - k: the maximum of the number of colors of $\mathcal{A}_1$ and $\mathcal{A}_2$.

  Reference. Christof. Löding, *Automata over infinite trees*, the handbook of the AutoMathA project.

- test the nonemptiness of $\mathcal{A}$,
  Complexity: $n^{O(k)}$, where n is the number of states and k is the number of colors. (Ref. Chapter 6, Automata, logics, and infinite games, LNCS 2500)

Lower bound:

Similar to the inclusion of Bottom-up tree automata over finite ranked trees,

Reduction from the nonemptiness of polynomial space alternating Turing machines.

Next lecture

Automata over unranked trees