Quantitative Reasoning of Opinions in Social Networks

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Abstract. In this paper, a mathematical framework to quantitatively reason about opinions in social networks is established. In the framework, a quantitative extension of signed graphs is chosen as the model of social networks, and opinions are defined as a collection of the answers of individuals in the network to a public question. In addition, a concept of social welfare for opinions is introduced to integrate the concept of opinions with social networks and facilitate the quantitative reasoning. Three relevant problems are considered in this framework and the corresponding results are obtained. The first problem is to investigate which opinions will maximize the social welfare for a given group of individuals. It is proved that for any given group of individuals in a harmonious social network where individuals are not enemies to each other, the individuals in the group are inclined to have the same opinion in order to maximize the social welfare for the whole group. Moreover, this result is extended to balanced social networks which are networks consisting of two antagonistic subnetworks. The second problem is to determine which choice of the pair of vertices of x, y will maximize or minimize the social welfare when their opinions are fixed a priori. For this problem, results are obtained for the special case that the social networks are trees. In particular, it is shown that in a harmonious tree network, the social welfare of the same opinion for x, y is minimized iff the distance between x, y in a related tree network is maximized. The last problem is to consider the evolution of social networks by an opinion-oriented updating rule. It is demonstrated that the networks whose biconnected scomponents are edges or cycles will eventually evolve into balanced social networks.

1 Introduction

Each of us lives in a huge social network, namely, the personal-relationship network where a link in the network represents whether one knows the other. On the other hand, with the invention of Internet, many virtual social networks

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have also been created, to name a few, World-Wide-Web, Facebook, and Email-address network. With the prosperity of these virtual networks, social networks are becoming a research focus of computer science (cf. the monograph [EK10]).

Social networks are usually modelled as graphs[4], where vertices represent individuals of the network, and edges denote a sort of relationships (e.g. friendship) between them.

While the graphs are useful for describing a *single* sort of relationships between the individuals in social networks, they are insufficient to model multiple sorts of relationships (e.g. friendship/antagonism) between them. To remedy this, Cartwright and Harary proposed in 1956 an extension of graphs, called signed graphs, where a "+/-" sign, which represents respectively the positive/negative relationship (e.g. "friendship/antagonism") between the individuals, is assigned to each edge in the graph (cf. [CH56]).

Opinion is a basic concept in social networks, an individual can express his/her opinion on public issues, or even opinion on the opinions of the other individuals. For instance, in Amazon.com, customers can write reviews (opinions) for books, they can also express their opinions on the reviews posted by the other customers, e.g. to answer the question "was this review helpful to you?". With the popularity of these opinion-rich online social networks, it becomes one of the vital issues for the social network community to lay down the mathematical foundations to facilitate the formal reasoning of opinions in social networks (cf. [PL08,BKO11]).

Our main goal in this paper is to establish a mathematical framework so that the opinions in social networks can be quantitatively reasoned. This brings the following two issues: Which social network model we choose and how we define opinions?

For the model of social networks, we choose a quantitative extension of signed graphs similar to that used in [KGG05,GGK05][5]: A social network in this paper is a weighted graph[6] $G = (V, E, \lambda)$, where (V, E) is a graph and $\lambda : E \to (0, 1) \cup (1, +\infty)$ denotes the strength of the relationship: Let $\{v, w\} \in E$,

- if $\lambda(\{v, w\}) > 1$, then v and w are friends and $\lambda(\{v, w\})$ denotes the strength of the friendship between them,
- if $0 < \lambda(\{v, w\}) < 1$, then v and w are enemies and $\lambda(\{v, w\})$ denotes the strength of the antagonism between them.

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[4] The graphs in this paper mean simple graphs, i.e. graphs without self-loops nor parallel edges.

[5] There is only a technical difference between our model and that in [KGG05,GGK05]: In [KGG05,GGK05], the weights are positive or negative real numbers, denoting respectively the friendship or antagonism, while in our model, the weights are real numbers in $(0, 1)$ or $(1, +\infty)$, which can be seen as the exponentiation of the weights in [KGG05,GGK05]: The weight λ of each edge in [KGG05,GGK05] is changed to 2^λ in our model.

[6] A weighted graph $G = (V, E, \lambda)$ can be seen as a complete weighted graph (V, E', λ'), with the unit weights included, where
- E' includes all the pairs $\{v, w\}$ for $v, w \in V : v \neq w$,
- For $\{v, w\} \in E'$, $\lambda'(\{v, w\}) = \lambda(\{v, w\})$ if $\{v, w\} \in E$, otherwise, $\lambda'(\{v, w\}) = 1$.

For the opinions, we choose a simple but meaningful definition: Let's imagine that there is a public issue (question), e.g. to raise the economic sanction on some country, on which the individuals in the network is required to express their "yes" or "no" opinions. The opinions of the individuals are modelled by an *opinion assignment function* β from V (the set of vertices in the network) to $\{0,1\}$: For each $v \in V$, $\beta(v) = 1$ (resp. $\beta(v) = 0$) means that v answers "yes" (resp. "no") to the public question. In addition, for a group of individuals $V' \subseteq V$, the opinions of V' can be modelled by a *partial* opinion assignment function $\beta' : V' \to \{0,1\}$.

Intuitively, if two individuals v, w in a social network are friends (resp. enemies), namely, $\lambda(\{v,w\}) > 1$ (resp. $0 < \lambda(\{v,w\}) < 1$), then they are inclined to have the same (resp. different) opinion on the public question.

To integrate the concept of opinions with social networks for quantitative reasoning, we introduce another concept, the *social welfare* of an opinion assignment β, denoted by $\Theta_G(\beta)$, as follows:

$$\Theta_G(\beta) = \prod_{\{v,w\} \in E} \theta_{\{v,w\}}(\beta(v), \beta(w)),$$

where

$$\theta_{\{v,w\}}(a,b) = \begin{cases} \lambda(\{v,w\}), & \text{if } a = b, \\ 1 & \text{otherwise.} \end{cases}$$

For the special case that the edge set of G is empty, $\Theta_G(\beta)$ is set to 1 for any β (Recall that a social network can be seen as a complete network with the unit weights included).

If the social network G is clear from the context, $\Theta_G(\beta)$ is usually abbreviated as $\Theta(\beta)$.

Intuitively, let $\{v,w\} \in E$, if v, w have the same opinion on the public question, then the edge $\{v,w\}$ contributes $\lambda(\{v,w\})$ to the social welfare, otherwise, its contribution is 1 and can be ignored. Note that the contribution $\lambda(\{v,w\})$ can be positive (> 1) or negative (< 1), depending on the positive or negative relationship between v, w.

We also talk about the social welfare for a set of opinion assignment functions as well as partial opinion assignment functions for a group of individuals.

- Let Γ be a set of opinion assignment functions, then the social welfare of Γ is $\sum_{\beta \in \Gamma} \Theta_G(\beta)$.
- Let $V' \subseteq V$ and $\beta' : V' \to \{0,1\}$, then the social welfare of β' is $\sum_{\beta \in \Gamma} \Theta_G(\beta)$, where Γ is the set of opinion assignment functions β such that $\beta|_{V'} = \beta'$, namely, β and β' agree on V'.

Here is an example to illustrate these concepts.

Suppose $G = (V, E, \lambda)$ is a cycle of three vertices, say $\{A,B\}\{B,C\}\{C,A\}$, and $\lambda(\{A,B\}) = 2$, $\lambda(\{B,C\}) = 1/2$, $\lambda(\{C,A\}) = 1/2$.

Let β be the opinion assignment function such that $\beta(A) = 1$, $\beta(B) = 1$, $\beta(C) = 0$. Then $\Theta(\beta)$, the social welfare of β, is $\lambda(\{A,B\}) \times 1 \times 1 = 2$.

Let $\beta' : \{A, B\} \to \{0, 1\}$ be the partial opinion assignment function such that $\beta'(A) = 1$ and $\beta'(B) = 1$. Then $\Theta(\beta') = \Theta(\beta_1) + \Theta(\beta_2)$, where $\beta_1|_{\{A,B\}} = \beta_2|_{\{A,B\}} = \beta'$, $\beta_1(C) = 1$, and $\beta_2(C) = 0$. Since $\Theta(\beta_1) = 2 \times 1/2 \times 1/2 = 1/2$ and $\Theta(\beta_2) = 2$, it follows that $\Theta(\beta') = 2\frac{1}{2}$.

Within this framework, we investigate in this paper the following three types of problems,

Problem I: Fixing a group of individuals
Given a group of individuals, which opinion assignment of this group maximizes the social welfare?

Problem II: Fixing the opinion of a pair of individuals
If we select a pair of individuals and force them to have the same or different opinion on the public question, what's the influence of the selection of the pair of individuals on the social welfare?

Problem III: Evolution of social networks
The evolution of social networks by the following updating rule:

> *In one updating step, the weight of each edge* $\{v, w\}$ *changes from* $\lambda(\{v, w\})$ *to* $\Theta(v = 1, w = 1)/\Theta(v = 1, w = 0)$.

The intuition of this updating rule is that the ratio of the social welfare of the agreement of v, w on the public question to that of the disagreement of v, w reflects the strength of their relationship.

Related work. With the popularity of opinion-rich resources such as online review sites and personal blogs, there is a huge body of work in the area of opinion mining and sentiment analysis, which deals with the techniques and approaches for opinion-oriented information seeking (cf. [PL08] for a survey). The most relevant work to our framework considered in this paper, as far as we know, is the recent active line of research to consider the processes how a group of people connected in a social network can arrive at a shared opinion through a form of repeated decentralized weighted averaging (cf. [DeG74,GJ10,AO11]). Moreover, in [BKO11], a game-theoretical framework was also proposed to consider the situation where a common opinion cannot be reached by the decentralized weighted averagings. While this body of work is in spirit similar to the problem III considered in this paper, there are essential technical differences: In their model, the opinions of the individuals can take arbitrary real values, the averagings are on these real values, and the structure of the social network is not changed, while in our model, the opinions can only take Boolean values, the updates are on the weights of edges, and the structure of the social network can be changed.

The rest of this paper is organized as follows: Section 2, 3, 4 are devoted respectively to Problem I, II, III. In Section 5, we conclude this paper and discuss some future work. The missing proofs can be found in the appendix.

2 Problem I: Fixing a group of individuals

We first introduce some additional notations and definitions.

Let $k \geq 1$, $\bar{x}$ be a list of k distinct vertices in a social network G, and $\bar{a}$ be a list of $0,1$ values of length k, then we use $\bar{x} = \bar{a}$ to denote the partial opinion assignment β such that $\beta(x_i) = a_i$ for each $i : 1 \leq i \leq k$. Moreover, if $S \subseteq V$, then we use $S = 1$ (resp. $S = 0$) to denote the partial opinion assignment β such that $\beta(v) = 1$ (resp. $\beta(v) = 0$) for each $v \in S$.

An edge e in a social network G is a called a *friend* (resp. an *enemy*) edge if $\lambda(e) > 1$ (resp. $0 < \lambda(e) < 1$).

A social network G is *harmonious* if $\lambda(e) > 1$ for each $e \in E$, namely, there are no enemy edges in the network.

A social network G is *balanced* if there are no cycles containing an odd number of enemy edges (cf. [CH56,EK10]). The balanced social network reflects the intuition that "friends of your friends are friends" and "enemies of your enemies are friends". A social network is balanced iff the network can be divided into two antagonistic groups with each group harmonious, more formally, $G = (V, W, E, \lambda)$ such that for each edge $\{v, w\} \in E$, $\lambda(\{v, w\}) > 1$ if $v, w \in V$ or $v, w \in W$, and $0 < \lambda(\{v, w\}) < 1$ if $v \in V, w \in W$ (cf. [CH56,EK10]).

Theorem 1. *If $G = (V, E, \lambda)$ is a harmonious social network, then for any disjoint subsets $S, T \subseteq V$, $\Theta(S = 1, T = 1) \geq \Theta(S = 1, T = 0)$.*

Intuitively, Theorem 1 says that if a social network is harmonious, then for any group of individuals in the network, the individuals in the group are inclined to agree on the public question, in order to maximize the social welfare of the whole group.

Proof. The proof is by an induction on $|E|$, the number of edges in G.

Induction base $|E| = 0$: $\Theta(S = 1, T = 1) = \Theta(S = 1, T = 0) = 1$.

Induction step $|E| > 0$:

Let $U = V \setminus (S \cup T)$, we distinguish between the following two cases:

1. There do not exist $u \in U$ and $v \in T$ such that $\{u, v\} \in E$.
2. There exist $u \in U$ and $v \in T$ such that $\{u, v\} \in E$.

Case 1.

Let G_1 be the graph induced by the vertex set U, and $\bar{u} = u_1 \dots u_k$ be the list of all vertices u in G_1 such that there is $v \in S$ satisfying that $\{u, v\} \in E$.

Let G_2 be the graph induced by the vertex set $S \cup T$.

In the following, we use $\Theta_1(\cdots)$ and $\Theta_2(\cdots)$ to denote the social welfare in G_1 and G_2 respectively.

Then for any $b \in \{0, 1\}$,

$$\Theta(S = 1, T = b) = \sum_{\bar{a} \in \{0,1\}^k} \Theta_1(\bar{u} = \bar{a}) \Lambda(\bar{a}) \Theta_2(S = 1, T = b),$$

where $\Lambda(\bar{a}) = \prod_{\{u_i, v\} : u_i \in \bar{u}, v \in S, \{u_i, v\} \in E} \theta_{\{u_i, v\}}(a_i, 1)$.

Because $S \cup T = V(G_2)$ and G is harmonious, it follows that

$$\Theta_2(S = 1, T = 1) = \left(\prod_{v, w \in S \cup T, \{v, w\} \in E} \lambda(\{v, w\}) \right) \geq \Theta_2(S = 1, T = 0).$$

Therefore, $\Theta(S=1,T=1) \geq \Theta(S=1,T=0)$.

Case 2.

Let $G' = (V, E \setminus \{u,v\}, \lambda')$, where λ' is the restriction of λ to $E \setminus \{u,v\}$.

Then

$$\Theta(S=1,T=1) = \lambda(\{u,v\})\Theta'(S=1,T=1,u=1) + \Theta'(S=1,T=1,u=0),$$

where $\Theta'(S=1,T=1,u=1) = \Theta_{G'}(S=1,T=1,u=1)$, namely, the social welfare of the partial opinion assignment $S=1,T=1,u=1$ in G', similarly for $\Theta'(S=1,T=1,u=0)$. We also have

$$\Theta(S=1,T=0) = \Theta'(S=1,T=0,u=1) + \lambda(\{u,v\})\Theta'(S=1,T=0,u=0),$$

It follows that

$$\begin{aligned}
&\Theta(S=1,T=1) - \Theta(S=1,T=0) \\
=\ &\lambda(\{u,v\})(\Theta'(S=1,T=1,u=1) - \Theta'(S=1,T=0,u=0)) + \\
&(\Theta'(S=1,T=1,u=0) - \Theta'(S=1,T=0,u=1)) \\
=\ &(\lambda(\{u,v\}) - 1)(\Theta'(S=1,T=1,u=1) - \Theta'(S=1,T=0,u=0)) + \\
&(\Theta'(S=1,T=1) - \Theta'(S=1,T=0)).
\end{aligned}$$

The last equation holds because $\Theta'(S=1,T=1,u=1) + \Theta'(S=1,T=1,u=0) = \Theta'(S=1,T=1)$ and $\Theta'(S=1,T=0,u=1) + \Theta'(S=1,T=0,u=0) = \Theta'(S=1,T=0)$.

Because G is harmonious, it follows that $\lambda(\{u,v\}) > 1$, i.e. $\lambda(\{u,v\}) - 1 > 0$.

By the induction hypothesis, $\Theta'(S=1,T=1,u=1) - \Theta'(S=1,T=0,u=0) \geq 0$ and $\Theta'(S=1,T=1) - \Theta'(S=1,T=0) \geq 0$. We conclude that $\Theta(S=1,T=1) \geq \Theta(S=1,T=0)$. □

Based on Theorem 1, a similar result can be obtained for balanced social networks.

Corollary 1. *Suppose $G = (V, W, E, \lambda)$ is a balanced social network, and $\overline{x} = x_1 \dots x_k$ (resp. $\overline{y} = y_1 \dots y_l$) is a list of distinct vertices in V (resp. W). Then for any $\overline{a} \in \{0,1\}^k$ and $\overline{b} \in \{0,1\}^l$, $\Theta(\overline{x}=1, \overline{y}=0) \geq \Theta(\overline{x}=\bar{a}, \overline{y}=\bar{b})$.*

Remark 1. The tuple $\overline{x}$ or $\overline{y}$ in Corollary 1 may be empty.

3 Problem II: Fixing the opinion of a pair of individuals

In this section, we select a pair of individuals, fix the opinion of them, and investigate how the relative positions of the pair of individuals affect the social welfare of the (partial) opinion assignment fixed for them.

We only get results for the special case that the social networks are trees.

Theorem 2. *Let $T = (V, E)$ be a social network that is a tree and x, y be a pair of distinct vertices in T. Then the following facts hold.*

1. *$\Theta(x=1, y=1)$ reaches the minimum and $\Theta(x=1, y=0)$ reaches the maximum if x, y satisfy the following condition:*

- *Either the network is harmonious, and the distance between x, y in $T' = (V, E, \lambda')$ is the maximum, where $\lambda'(e) = \log((\lambda(e)+1)/(\lambda(e)-1))$ for each $e \in E$.*
- *Or the network is not harmonious, $\{x, y\} \in E$, and*
$$\lambda(\{x,y\}) = \min_{\{v,w\}\in E} \lambda(\{v,w\}).$$

2. *$\Theta(x=1, y=1)$ reaches the maximum and $\Theta(x=1, y=0)$ reaches the minimum if x, y satisfy the following condition.*
Let
$$\Omega = \max\left(\begin{array}{c} \left\{ \frac{\lambda(\{v,w\})-1}{\lambda(\{v,w\})+1} \;\middle|\; \{v,w\} \in E, \lambda(\{v,w\}) > 1 \right\} \bigcup \\ \left\{ \frac{(\lambda(\{v,u\})-1)(\lambda(\{u,w\})-1)}{(\lambda(\{v,u\})+1)(\lambda(\{u,w\})+1)} \;\middle|\; \begin{array}{c} v \neq w, \{v,u\}, \{u,w\} \in E, \\ 0 < \lambda(\{v,u\}), \lambda(\{u,w\}) < 1 \end{array} \right\} \end{array}\right).$$
Then
- *Either $\{x, y\} \in E$ and $\frac{\lambda(\{x,y\})-1}{\lambda(\{x,y\})+1} = \Omega$.*
- *Or there exists $z \in V$ such that $\{x, z\}, \{z, y\} \in E$ and*
$$\frac{(\lambda(\{x,z\})-1)(\lambda(\{z,y\})-1)}{(\lambda(\{x,z\})+1)(\lambda(\{z,y\})+1)} = \Omega.$$

Theorem 2 can be deduced from the following lemma.

Lemma 1. *Let $T = (V, E)$ be a social network that is a tree, x, y be a pair of distinct vertices in T, and P be the path between x and y in T. Then for any $a, b \in \{0, 1\}$, we have*
$$\Theta_T(x=a, y=b) = \Theta_P(x=a, y=b) \times \prod_{e \notin P} (\lambda(e)+1).$$

In the following, we show how Theorem 2 follows from Lemma 1. The proof of Lemma 1 is omitted.

Proof (Theorem 2). Let P be the path between x and y in T. Then from Theorem 2, it follows that
$$\begin{aligned}\Theta_T(x=1, y=1) &= \Theta_P(x=1, y=1) \times \prod_{e \notin P} (\lambda(e)+1) \\ &= \frac{\Theta_P(x=1, y=1)}{\prod_{e \in P} (\lambda(e)+1)} \prod_{e \in T} (\lambda(e)+1).\end{aligned}$$

By induction on the length of P, it is easy to show that
$$\Theta_P(x=1, y=1) = \prod_{e \in P} (\lambda(e)+1) + \prod_{e \in P} (\lambda(e)-1).$$

Therefore,
$$\begin{aligned}\Theta_T(x=1, y=1) &= \frac{\prod_{e \in P} (\lambda(e)+1) + \prod_{e \in P} (\lambda(e)-1)}{\prod_{e \in P} (\lambda(e)+1)} \prod_{e \in T} (\lambda(e)+1) \\ &= \left(1 + \prod_{e \in P} \frac{\lambda(e)-1}{\lambda(e)+1}\right) \prod_{e \in T} (\lambda(e)+1).\end{aligned}$$

Similarly,

$$\Theta_T(x=1,y=0) = \left(1 - \prod_{e\in P} \frac{\lambda(e)-1}{\lambda(e)+1}\right) \prod_{e\in T} (\lambda(e)+1).$$

Consequently, $\Theta_T(x=1,y=1)$ reaches the minimum and $\Theta_T(x=1,y=0)$ reaches the maximum iff $\prod_{e\in P} \frac{\lambda(e)-1}{\lambda(e)+1}$ reaches the minimum.

If T is harmonious, let $T' = (V, E, \lambda')$, where

$$\lambda'(e) = \log\left((\lambda(e)+1)/(\lambda(e)-1)\right) \text{ for each } e \in E.$$

Then $\prod_{e\in P} \frac{\lambda(e)-1}{\lambda(e)+1}$ reaches the minimum iff $\prod_{e\in P} \frac{\lambda(e)+1}{\lambda(e)-1}$ reaches the maximum iff the distance between x, y in T' is the maximum.

Otherwise, T is not harmonious, i.e. there are enemy edges in T. If there are an odd number of enemy edges on P, then $\prod_{e\in P} \frac{\lambda(e)-1}{\lambda(e)+1} < 0$. It follows that the minimum value of $\prod_{e\in P} \frac{\lambda(e)-1}{\lambda(e)+1}$ is less than 0. Since $\left|\frac{\lambda(e)-1}{\lambda(e)+1}\right| < 1$ for any edge e, it follows that $\prod_{e\in P} \frac{\lambda(e)-1}{\lambda(e)+1}$ reaches the minimum iff P is a single edge $\{x, y\}$ such that

$$0 < \lambda(\{x,y\}) = \min_{\{v,w\}\in E} \lambda(\{v,w\}) < 1.$$

Symmetrically, $\Theta_T(x=1,y=1)$ reaches the maximum and $\Theta_T(x=1,y=0)$ reaches the minimum iff $\prod_{e\in P} \frac{\lambda(e)-1}{\lambda(e)+1}$ reaches the maximum.

At first, we notice that the maximum value of $\prod_{e\in P} \frac{\lambda(e)-1}{\lambda(e)+1}$ is greater than 0. If T contains at least one friend edge, then this is obvious. Otherwise, all the edges in T are enemy edges, then $\prod_{e\in P} \frac{\lambda(e)-1}{\lambda(e)+1} > 0$ for any path P consisting of an even number of edges.

Because $\left|\frac{\lambda(e)-1}{\lambda(e)+1}\right| < 1$ for any edge e, it follows that $\prod_{e\in P} \frac{\lambda(e)-1}{\lambda(e)+1}$ reaches the maximum iff either $P = \{x, y\}$, $\lambda(\{x, y\}) > 1$, and $\frac{\lambda(\{x,y\})-1}{\lambda(\{x,y\})+1} = \Omega$, or $P = \{x, z\}\{z, y\}$, $0 < \lambda(\{x, z\}), \lambda(\{z, y\}) < 1$, and $\frac{(\lambda(\{x,z\})-1)(\lambda(\{z,y\})-1)}{(\lambda(\{x,z\})+1)(\lambda(\{z,y\})+1)} = \Omega$. □

4 Problem III: Evolution of social networks

In this section, we investigate how social networks evolve by the following updating rule:

> *In each updating step, the weight of every edge* $\{v, w\}$ *changes from* $\lambda(\{v, w\})$ *to* $\Theta(v = 1, w = 1)/\Theta(v = 1, w = 0)$.

The intuition of the rule is that for each edge $\{v, w\}$, the ratio of the social welfare of the agreement of v, w on the public question to the social welfare of the disagreement of v, w reflects the strength of their relationship:

– If $\Theta(v = 1, w = 1)/\Theta(v = 1, w = 0) > 1$, then v, w are inclined to agree on the public question in order to maximize the social welfare, intuitively this means that more likely they are friends,
– If $\Theta(v = 1, w = 1)/\Theta(v = 1, w = 0) = 1$, then the relationship between v, w are neutral,
– If $0 < \Theta(v = 1, w = 1)/\Theta(v = 1, w = 0) < 1$, then more likely v, w are enemies.

For a network G, let $G^{(1)}, G^{(2)}, \dots$ denote the network obtained after $1, 2, \dots$ updating steps. In addition, let $G^{(0)} = G$ by convention.

A social network is said to *balance eventually in the evolution* if

either there is some N such that for every $n \geq N$, $G^{(n)}$ is balanced, or $\lim_{n\to\infty} G^{(n)}$ (the limit graph of $G^{(n)}$) exists and is balanced.

Remark 2. If a social network G is balanced, then according to Corollary 1, for any friend (resp. enemy) edge $\{v, w\}$ in G, $\Theta(v = 1, w = 1) \geq \Theta(v = 1, w = 0)$ (resp. $\Theta(v = 1, w = 1) \leq \Theta(v = 1, w = 0)$), it follows that $G^{(1)}$ is balanced. Similarly, $G^{(2)}$ is balanced and so on. So G balances eventually in the evolution.

A graph $G = (V, E)$ is *biconnected* if $|V| \geq 2$, G is connected and for any $v \in V$, $G \setminus v$, the graph obtained by deleting v from G, is still connected.

Let $G = (V, E)$ be a graph such that $|V| \geq 2$. A *biconnected component* of G is a maximal biconnected subgraph of G. It is known that two distinct biconnected components of a graph share at most one common vertex (cf. [Die05]). The *biconnected-component graph* of a graph G, denoted by $\mathcal{B}(G)$, is the graph such that its vertex set is the set of biconnected components and its edge set is the set of $\{B_1, B_2\}$ such that the biconnected components B_1 and B_2 share a common vertex. The biconnected-component graph $\mathcal{B}(G)$ of a connected graph G containing at least two vertices is in fact a tree (cf. [Die05]).

Theorem 3. *A graph such that each of its biconnected components is a cycle or an edge balances eventually in the evolution.*

Theorem 3 can be easily deduced from the following two lemmas.

Lemma 2. *Let $G = (V, E, \lambda)$ be a social network, $G' = (V, E, \lambda')$ be the network obtained after one updating step. In addition, let $\{v, w\} \in E$ and B be the biconnected component of G containing the edge $\{v, w\}$. Then $\lambda'(\{v, w\}) = \Theta_B(v = 1, w = 1)/\Theta_B(v = 1, w = 0)$.*

Intuitively, Lemma 2 says that the evolution of each biconnected component is independent of all the other biconnected components. In particular, for each edge e which is itself a biconnected component, the weight of e is not changed in the evolution, namely, is always $\lambda(e)$.

With Lemma 2, it is sufficient to show the following lemma in order to prove Theorem 3.

Lemma 3. *A cycle balances eventually in the evolution.*

The rest of this section is devoted to the proof of Lemma 3.

Proof (Lemma 3).

Let $C = v_1 \dots v_n v_1$ be a cycle of length n ($n \geq 3$). For each $i : 1 \leq i \leq n$, let λ_i denote $\lambda(\{v_i, v_{i+1}\})$ (where $v_{n+1} = v_1$ by convention), for the briefness.

Let C' be the cycle obtained after one updating step and λ'_i denote the weight of $\{v_i, v_{i+1}\}$ in C'. Then by induction on n, it is not hard to show that

$$\lambda'_i = \frac{\prod\limits_{k \neq i} (\lambda_k + 1) + \prod\limits_{k \neq i} (\lambda_k - 1)}{\prod\limits_{k \neq i} (\lambda_k + 1) - \prod\limits_{k \neq i} (\lambda_k - 1)} \; \lambda_i.$$

Because of the symmetry in the computation of $\lambda'_1, \dots, \lambda'_n$, we can assume that $\lambda_1 \geq \lambda_2 \dots \geq \lambda_n$ without loss of generality.

If C is already balanced, namely, it contains an even number of enemy edges, then according to Remark 2, C balances eventually in the evolution.

In the following, we assume that C is not balanced, namely there is $i_0 : 1 \leq i_0 \leq n$ such that

$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{i_0} > 1 > \lambda_{i_0+1} \geq \dots \geq \lambda_n$ and $n - i_0$ is odd.

Claim. *The following facts hold.*

1. $\lambda'_j < \lambda_j$ *for* $j : 1 \leq j \leq i_0$ *and* $\lambda'_j > \lambda_j$ *for* $j : i_0 < j \leq n$.
2. $\lambda'_1 \geq \dots \geq \lambda'_{i_0}$ *and* $\lambda'_{i_0+1} \geq \dots \geq \lambda'_n$.
3. $\lambda'_{i_0-1} > 1 > \lambda'_{i_0+2}$.
4. *If* $\lambda'_{i_0} \leq 1$, *then* $\lambda'_{i_0+1} < 1$.
5. *If* $\lambda'_{i_0} = 1$ *(resp.* $\lambda'_{i_0+1} = 1$*), then* $\lambda''_{i_0} < 1$ *(resp.* $\lambda''_{i_0+1} > 1$*) and* $\lambda''_j = \lambda'_j$ *for each* $j \neq i_0$ *(resp.* $j \neq i_0 + 1$*), where* λ''*'s are the weights of edges after two updating steps.*

The proof of the claim is omitted.

From the Claim, if follows that $\lambda'_1 \geq \dots \geq \lambda'_{i_0-1} > 1$ and $1 > \lambda'_{i_0+2} \geq \dots \geq \lambda'_n > 0$.

For λ'_{i_0} and λ'_{i_0+1}, if one of the following situations happens, then we are done.

- If $\lambda'_{i_0} < 1$, then $\lambda'_{i_0+1} < 1$. So there are $(n - i_0 + 1)$ enemy edges in C', and C' is balanced. We conclude that C balances eventually in the evolution.
- If $\lambda'_{i_0} = 1$, then $\lambda'_{i_0+1} < 1$, $\lambda''_{i_0} < 1$ and $\lambda''_j = \lambda'_j$ for each $j \neq i_0$. So $1 > \lambda''_{i_0}, \lambda''_{i_0+1}, \dots, \lambda''_n > 0$, and C'', the cycle obtained after two updating steps, is balanced since it contains $(n - i_0 + 1)$ enemy edges. We conclude that C balances eventually in the evolution.
- If $\lambda'_{i_0} > 1$ and $\lambda'_{i_0+1} > 1$, then there are $(n - i_0 - 1)$ enemy edges in C', so C' is balanced. Therefore, C balances eventually in the evolution.
- If $\lambda'_{i_0} > 1$ and $\lambda'_{i_0+1} = 1$, then $\lambda''_{i_0+1} > 1$ and $\lambda''_j = \lambda'_j$ for each $j \neq i_0 + 1$. So there are $(n - i_0 - 1)$ enemy edges in C'', thus C'' is balanced. We conclude that C balances eventually in the evolution.

If none of the above situations happens for the weights of the edge $\{v_i, v_{i+1}\}$ and $\{v_{i+1}, v_{i+2}\}$ in the evolution, let $C^{(0)}, C^{(1)}, C^{(2)}, \ldots$ denote the cycle obtained after zero, one, two updating steps and so on, then it follows that for any $\ell \geq 0$, we always have $\lambda_{i_0}^{(\ell)} > 1$ and $0 < \lambda_{i_0+1}^{(\ell)} < 1$ in $C^{(\ell)}$.

From the Claim, if follows that

$$\lambda_j^{(0)} > \lambda_j^{(1)} > \lambda_j^{(2)} > \ldots > 1 \text{ for each } j : 1 \leq j \leq i_0,$$

and

$$0 < \lambda_j^{(0)} < \lambda_j^{(1)} < \lambda_j^{(2)} < \ldots < 1 \text{ for each } j : i_0 < j \leq n.$$

Therfore, $\lim_{\ell \to +\infty} \lambda_j^{(\ell)}$ exists for each $j : 1 \leq j \leq n$. It follows that the limit graph $\lim_{\ell \to +\infty} C^{(\ell)}$ exists.

On the other hand, for each $j : 1 \leq j \leq n$, we have

$$\lambda_j' = \frac{\prod_{k \neq j} (\lambda_k + 1) + \prod_{k \neq j} (\lambda_k - 1)}{\prod_{k \neq j} (\lambda_k + 1) - \prod_{k \neq j} (\lambda_k - 1)} \lambda_j.$$

It follows that for each $j : 1 \leq j \leq n$,

$$\lim_{\ell \to +\infty} \lambda_j^{(\ell)} = \frac{\prod_{k \neq j} \left(\lim_{\ell \to +\infty} \lambda_k^{(\ell)} + 1 \right) + \prod_{k \neq j} \left(\lim_{\ell \to +\infty} \lambda_k^{(\ell)} - 1 \right)}{\prod_{k \neq j} \left(\lim_{\ell \to +\infty} \lambda_k^{(\ell)} + 1 \right) - \prod_{k \neq j} \left(\lim_{\ell \to +\infty} \lambda_k^{(\ell)} - 1 \right)} \lim_{\ell \to +\infty} \lambda_j^{(\ell)}.$$

From this, it is deduced that for each $j : 1 \leq j \leq n$, $\prod_{k \neq j} \left(\lim_{\ell \to +\infty} \lambda_k^{(\ell)} - 1 \right) = 0$.

Therefore, there exist $j_1, j_2 : 1 \leq j_1, j_2 \leq n, j_1 \neq j_2$ such that $\lim_{\ell \to +\infty} \lambda_{j_1}^{(\ell)} = 1$ and $\lim_{\ell \to +\infty} \lambda_{j_2}^{(\ell)} = 1$. This means that in the limit graph $\lim_{\ell \to +\infty} C^{(\ell)}$, the weights of at least two edges become into 1, namely, the two edges disappear, so the limit graph $\lim_{\ell \to +\infty} C^{(\ell)}$ is a collection of disjoint paths, thus balanced. We conclude that C balances eventually in the evolution. □

5 Conclusion

In this paper, we proposed a mathematical framework to quantitatively reason about opinions in social networks and considered three problems in this framework.

We first investigated the problem which opinion assignment maximizes the social welfare for a given group of individuals and proved that in harmonious social networks, the complete agreement of opinions in the given group maximizes the social welfare for the group, while in balanced social networks which consist of two antagonistic subnetworks, the opinion assignment which achieves the complete agreement in each of the two subnetworks, with the positive answer in one subnetwork and the negative answer in the other, maximizes the social welfare for the group.

Then we considered the problem that when fixing the opinion of a freely chosen pair of distinct vertices x, y, which pair of x, y will minimize or maximize the social welfare. We established the results for the social networks that are trees. In particular, we proved that in a harmonious tree network $T = (V, E, \lambda)$, the social welfare of the same opinion for x, y is minimized iff the distance between x, y in $T' = (V, E, \lambda')$ is maximized, where $\lambda'(e) = \log((\lambda(e)+1)/(\lambda(e)-1))$.

Finally we considered the evolution of social networks by the updating rule of replacing the weight of each edge $\{v, w\}$ with the ratio of the social welfare of the agreement of v, w to that of the disagreement of v, w. We proved that the networks whose biconnected components are edges or cycles, will eventually evolve into balanced social networks.

There are several obvious open questions left in this paper. For Problem I, it is interesting to get some results for the non-balanced social networks. For Problem II and Problem III, it is interesting to extend the results to the more general social networks. At last, it is also interesting to consider the other problems within the framework besides the three problems considered in this paper.

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A Proofs in Section 2 (Problem I)

Corollary 1. *Suppose $G = (V, W, E, \lambda)$ is a balanced social network, and $\overline{x} = x_1 \dots x_k$ (resp. $\overline{y} = y_1 \dots y_l$) is a list of distinct vertices in V (resp. W). Then for any $\overline{a} \in \{0,1\}^k$ and $\overline{b} \in \{0,1\}^l$, $\Theta(\overline{x} = 1, \overline{y} = 0) \geq \Theta(\overline{x} = \bar{a}, \overline{y} = \bar{b})$.*

Proof. Let $G = (V, W, E, \lambda)$ be a balanced social network.

Define a harmonious network $H = (V \cup W, E, \lambda')$ from G as follows,

$$\lambda'(\{v,w\}) = \begin{cases} \lambda(\{v,w\}), & \text{if } v, w \in V \text{ or } v, w \in W, \\ \frac{1}{\lambda(\{v,w\})}, & \text{if } v \in V, w \in W. \end{cases}$$

Let $\overline{x} = x_1 \dots x_k$ (resp. $\overline{y} = y_1 \dots y_l$) be a list of distinct vertices in V (resp. W).

Claim. *Let F be the set of edges $\{v,w\} \in E$ such that $v \in V$ and $w \in W$. For any $\overline{c} \in \{0,1\}^k$ and $\overline{d} \in \{0,1\}^l$, we have*

$$\Theta_G(\overline{x} = \overline{c}, \overline{y} = \overline{d}) = \left(\prod_{e \in F} \lambda(e) \right) \Theta_H(\overline{x} = \overline{c}, \overline{y} = 1 - \overline{d}),$$

where $1 - \overline{d} = (1 - d_j)_{1 \leq j \leq l}$.

Proof (the Claim).

We first prove the Claim for the special case that $V = \{x_1, \dots, x_k\}$ and $W = \{y_1, \dots, y_l\}$.

Let $G_1 = G[V]$, the subgraph of G induced by V, and $G_2 = G[W]$, the subgraph of G induced by W.

Then it follows that

$$\Theta_G(\overline{x} = \overline{c}, \overline{y} = \overline{d}) = \Theta_{G_1}(\overline{x} = \overline{c}) \; \Theta_{G_2}(\overline{y} = \overline{d}) \prod_{\{x_i, y_j\} \in F} \theta_{\{x_i, y_j\}}(c_i, d_j),$$

and

$$\Theta_H(\overline{x} = \overline{c}, \overline{y} = 1 - \overline{d}) = \Theta_{G_1}(\overline{x} = \overline{c}) \; \Theta_{G_2}(\overline{y} = 1 - \overline{d}) \prod_{\{x_i, y_j\} \in F} \theta'_{\{x_i, y_j\}}(c_i, 1 - d_j),$$

where

$$\theta_{\{x_i, y_j\}}(c_i, d_j) = \begin{cases} \lambda(\{x_i, y_j\}), & \text{if } c_i = d_j, \\ 1, & \text{otherwise.} \end{cases}$$

and

$$\theta'_{\{x_i, y_j\}}(c_i, 1 - d_j) = \begin{cases} 1, & \text{if } c_i = d_j, \\ \frac{1}{\lambda(\{x_i, y_j\})}, & \text{otherwise.} \end{cases}$$

Since $\theta_{\{x_i, y_j\}}(c_i, d_j) = \lambda(\{x_i, y_j\}) \theta'_{\{x_i, y_j\}}(c_i, 1 - d_j)$, we conclude that

$$\Theta_G(\overline{x} = \overline{c}, \overline{y} = \overline{d}) = \left(\prod_{e \in F} \lambda(e) \right) \Theta_H(\overline{x} = \overline{c}, \overline{y} = 1 - \overline{d}).$$

Now we consider the more general case that $V \neq \{x_1, \ldots, x_k\}$ or $W \neq \{y_1, \ldots, y_l\}$. Let $\overline{v} = v_1 \ldots v_r$ be a list of all vertices in $V \setminus \{x_1, \ldots, x_k\}$ and $\overline{w} = w_1 \ldots w_s$ be a list of all vertices in $W \setminus \{y_1, \ldots, y_l\}$.

Then $\Theta_G(\overline{x} = \overline{c}, \overline{y} = \overline{d})$ equals

$$
\begin{aligned}
& \Theta_G(\overline{x} = \overline{c}, \overline{y} = \overline{d}) \\
= & \sum_{\overline{c'} \in \{0,1\}^r, \overline{d'} \in \{0,1\}^s} \Theta_G(\overline{x} = \overline{c}, \overline{v} = \overline{c'}, \overline{y} = \overline{d}, \overline{w} = \overline{d'}) \\
= & \left(\prod_{e \in F} \lambda(e) \right) \sum_{\overline{c'} \in \{0,1\}^r, \overline{d'} \in \{0,1\}^s} \Theta_H(\overline{x} = \overline{c}, \overline{v} = \overline{c'}, \overline{y} = 1 - \overline{d}, \overline{w} = 1 - \overline{d'}) \\
= & \left(\prod_{e \in F} \lambda(e) \right) \Theta_H(\overline{x} = \overline{c}, \overline{y} = 1 - \overline{d}).
\end{aligned}
$$

The claim holds. □

Now we return to the proof of the corollary.

It follows from the Claim that $\Theta_G(\overline{x} = \overline{a}, \overline{y} = \overline{b}) = \left(\prod_{e \in F} \lambda(e) \right) \Theta_H(\overline{x} = \overline{a}, \overline{y} = 1 - \overline{b})$ for any $\overline{a} \in \{0,1\}^k$ and $\overline{b} \in \{0,1\}^l$.

Therefore, $\Theta_G(\overline{x} = 1, \overline{y} = 0) \geq \Theta_G(\overline{x} = \overline{a}, \overline{y} = \overline{b})$ iff $\Theta_H(\overline{x} = 1, \overline{y} = 1) \geq \Theta_H(\overline{x} = \overline{a}, \overline{y} = 1 - \overline{b})$.

Because H is harmonious, from Theorem 1, we conclude that $\Theta_H(\overline{x} = 1, \overline{y} = 1) \geq \Theta_H(\overline{x} = \overline{a}, \overline{y} = 1 - \overline{b})$, so $\Theta_G(\overline{x} = 1, \overline{y} = 0) \geq \Theta_G(\overline{x} = \overline{a}, \overline{y} = \overline{b})$ holds.

The proof of Corollary 1 is complete. □

B Proofs in Section 3 (Problem II)

Lemma 1. *Let $T = (V, E)$ be a social network that is a tree, x, y be a pair of distinct vertices in T, and P be the path between x and y in T. Then for any $a, b \in \{0, 1\}$, we have*

$$\Theta_T(x = a, y = b) = \Theta_P(x = a, y = b) \times \prod_{e \notin P} (\lambda(e) + 1).$$

We first prove the following proposition.

Proposition 1. *Let $T = (V, E)$ be a tree, $v \in T$, and $a \in \{0, 1\}$, let $T \setminus \{v\} = \{v_1, ..., v_m\}$ then*

$$\Theta_T(v = a) = \sum_{\bar{a} \in \{0,1\}^m} \prod_{\{v_i, v_j\} \in E, 0 \leq i,j \leq m} \theta_{\{v_i, v_j\}}(a_i, a_j) = \prod_{e \in E} (1 + \lambda(e)),$$

where $v_0 = v, a_0 = a$ by convention.

Proof (Proposition 1).

Because $\Theta_T(v=1) = \Theta_T(v=0)$, it is sufficient to prove

$$\sum_{a\in\{0,1\},\bar{a}\in\{0,1\}^m} \prod_{\{v_i,v_j\}\in E, 0\le i,j\le m} \theta_{\{v_i,v_j\}}(a_i,a_j) = 2\prod_{e\in E}(1+\lambda(e)).$$

We prove this by induction on m.

Induction base: $m=1$, namely, T is a single edge $\{v, v_1\}$.

Then

$$\sum_{a,a_1\in\{0,1\}} \theta_{\{v,v_1\}}(a,a_1) = 2(1+\lambda(\{v,v_1\}))$$

Induction step: Let $m>1$.

Without loss of generality, suppose that v_m is a leaf in T and $\{v_l, v_m\}\in E$ for some $l : 0\le l<m$ (where $v_0 = v$ by convention). Let $T_1 = T\setminus\{v_m\}$ and $a_0 = a$, then

$$\begin{aligned}
&\sum_{\bar{a}\in\{0,1\}^{m+1}} \prod_{\{v_i,v_j\}\in E} \theta_{\{v_i,v_j\}}(a_i,a_j)\\
&= \sum_{a_0\dots a_{m-1}\in\{0,1\}^m} \sum_{a_m\in\{0,1\}} \left(\theta_{\{v_l,v_m\}}(a_l,a_m) \prod_{\{v_i,v_j\}\in T_1} \theta_{\{v_i,v_j\}}(a_i,a_j)\right)\\
&= \sum_{a_0\dots a_{m-1}\in\{0,1\}^m} \left((1+\lambda(\{v_l,v_m\})) \prod_{\{v_i,v_j\}\in T_1} \theta_{\{v_i,v_j\}}(a_i,a_j)\right)\\
&= (1+\lambda(\{v_l,v_m\})) \sum_{a_0\dots a_{m-1}\in\{0,1\}^m} \prod_{\{v_i,v_j\}\in T_1} \theta_{\{v_i,v_j\}}(a_i,a_j)\\
&= (1+\lambda(\{v_l,v_m\})) \left(2\prod_{\{v_i,v_j\}\in T_1}(1+\lambda(\{v_i,v_j\}))\right) \text{ (By induction hypothesis)}\\
&= 2\prod_{e\in E}(1+\lambda(e)).
\end{aligned}$$

□

Proof (Lemma 1). Suppose $P = xz_1\dots z_t y$.

Let $T\setminus P$ denote the graph obtained from T by deleting all the edges on P (Note that the vertices on P are not deleted).

The graph $T\setminus P$ consists of $t+2$ connected components, $T_0, T_1, \dots, T_t, T_{t+1}$, such that $x\in T_0$, $y\in T_{t+1}$, and $z_i\in T_i$ for each $i : 1\le i\le t$. It is evident that each T_i is a tree.

For each $i : 0\le i\le t+1$, let $\overline{v_i} = (v_i^1, ..., v_i^{k_i})$ be the list of all vertices of T_i different from z_i (where $z_0 = x$ and $z_{t+1} = y$ by convention). Then

$$\begin{aligned}
&\Theta_T(x=a, y=b)\\
&= \sum_{\left(\overline{a_i}\in\{0,1\}^{k_i}\right)_{0\le i\le t+1}, \bar{c}\in\{0,1\}^t} \Theta_T\left(x=a, y=b, \bar{z}=\bar{c}, (\overline{v_i}=\overline{a_i})_{0\le i\le t+1}\right)\\
&= \sum_{\bar{c}\in\{0,1\}^t}\left(\sum_{\left(\overline{a_i}\in\{0,1\}^{k_i}\right)_{0\le i\le t+1}} \Theta_T\left(x=a, y=b, \bar{z}=\bar{c}, (\overline{v_i}=\overline{a_i})_{0\le i\le t+1}\right)\right)
\end{aligned}$$

$$= \sum_{\bar{c}\in\{0,1\}^t} \left(\sum_{\left(\overline{a_i}\in\{0,1\}^{k_i}\right)_{0\le i\le t+1}} \Theta_P(x=a, y=b, \bar{z}=\bar{c}) \prod_{i=0}^{t+1} \Theta_{T_i}(z_i=c_i, \overline{v_i}=\overline{a_i}) \right)$$

$$= \sum_{\bar{c}\in\{0,1\}^t} \left(\Theta_P(x=a, y=b, \bar{z}=\bar{c}) \left(\sum_{\left(\overline{a_i}\in\{0,1\}^{k_i}\right)_{0\le i\le t+1}} \prod_{i=0}^{t+1} \Theta_{T_i}(z_i=c_i, \overline{v_i}=\overline{a_i}) \right) \right),$$

where $c_0 = a, c_{t+1} = b$ by convention.

On the other hand, we have

$$\sum_{\left(\overline{a_i}\in\{0,1\}^{k_i}\right)_{0\le i\le t+1}} \left(\prod_{i=0}^{t+1} \Theta_{T_i}(z_i=c_i, \overline{v_i}=\overline{a_i}) \right)$$

$$= \sum_{\left(\overline{a_i}\in\{0,1\}^{k_i}\right)_{0\le i\le t}} \left(\sum_{\overline{a_{t+1}}\in\{0,1\}^{k_{t+1}}} \left(\prod_{i=0}^{t+1} \Theta_{T_i}(z_i=c_i, \overline{v_i}=\overline{a_i}) \right) \right)$$

$$= \sum_{\left(\overline{a_i}\in\{0,1\}^{k_i}\right)_{0\le i\le t}} \left(\sum_{\overline{a_{t+1}}\in\{0,1\}^{k_{t+1}}} \left(\prod_{i=0}^{t} \Theta_{T_i}(z_i=c_i, \overline{v_i}=\overline{a_i}) \right) \Theta_{T_{t+1}}(z_{t+1}=c_{t+1}, \overline{v_{t+1}}=\overline{a_{t+1}}) \right)$$

$$= \sum_{\left(\overline{a_i}\in\{0,1\}^{k_i}\right)_{0\le i\le t}} \left(\left(\prod_{i=0}^{t} \Theta_{T_i}(z_i=c_i, \overline{v_i}=\overline{a_i}) \right) \sum_{\overline{a_{t+1}}\in\{0,1\}^{k_{t+1}}} \Theta_{T_{t+1}}(z_{t+1}=c_{t+1}, \overline{v_{t+1}}=\overline{a_{t+1}}) \right)$$

$$= \sum_{\left(\overline{a_i}\in\{0,1\}^{k_i}\right)_{0\le i\le t}} \left(\left(\prod_{i=0}^{t} \Theta_{T_i}(z_i=c_i, \overline{v_i}=\overline{a_i}) \right) \Theta_{T_{t+1}}(z_{t+1}=c_{t+1}) \right)$$

$$= \Theta_{T_{t+1}}(z_{t+1}=c_{t+1}) \sum_{\left(\overline{a_i}\in\{0,1\}^{k_i}\right)_{0\le i\le t}} \left(\prod_{i=0}^{t} \Theta_{T_i}(z_i=c_i, \overline{v_i}=\overline{a_i}) \right)$$

$$= \cdots$$

$$= \prod_{i=0}^{t+1} \Theta_{T_i}(z_i=c_i).$$

Therefore,

$$\Theta_T(x=a, y=b)$$

$$= \sum_{\bar{c}\in\{0,1\}^t} \left(\Theta_P(x=a, y=b, \bar{z}=\bar{c}) \left(\sum_{\left(\overline{a_i}\in\{0,1\}^{k_i}\right)_{0\le i\le t+1}} \prod_{i=0}^{t+1} \Theta_{T_i}(z_i=c_i, \overline{v_i}=\overline{a_i}) \right) \right)$$

$$= \sum_{\bar{c}\in\{0,1\}^t} \left(\Theta_P(x=a, y=b, \bar{z}=\bar{c}) \prod_{i=0}^{t+1} \Theta_{T_i}(z_i=c_i) \right)$$

From Proposition 1, it follows that

$$\prod_{i=0}^{t+1} \Theta_{T_i}(z_i = c_i) = \prod_{i=0}^{t+1} \prod_{e \in T_i} (1+\lambda(e)) = \prod_{e \notin P} (1+\lambda(e)).$$

Consequently

$$\begin{aligned}
&\Theta_T(x=a, y=b) \\
&= \sum_{\bar{c} \in \{0,1\}^t} \left(\Theta_P(x=a, y=b, \bar{z}=\bar{c}) \prod_{i=0}^{t+1} \Theta_{T_i}(z_i = c_i) \right) \\
&= \sum_{\bar{c} \in \{0,1\}^t} \left(\Theta_P(x=a, y=b, \bar{z}=\bar{c}) \prod_{e \notin P} (1+\lambda(e)) \right) \\
&= \left(\prod_{e \notin P} (1+\lambda(e)) \right) \sum_{\bar{c} \in \{0,1\}^t} \Theta_P(x=a, y=b, \bar{z}=\bar{c}) \\
&= \Theta_P(x=a, y=b) \times \prod_{e \notin P} (1+\lambda(e)).
\end{aligned}$$

The proof of Lemma 1 is complete. □

C Proofs in Section 4 (Problem III)

Lemma 2. *Let $G = (V, E, \lambda)$ be a social network, $G' = (V, E, \lambda')$ be the network obtained after one updating step. In addition, let $\{v, w\} \in E$ and B be the biconnected component of G containing the edge $\{v, w\}$. Then $\lambda'(\{v, w\}) = \Theta_B(v = 1, w = 1)/\Theta_B(v = 1, w = 0)$.*

Proof. Suppose $G = (V, E, \lambda)$ is a social network and $G' = (V, E, \lambda')$ is the network obtained after one updating step.

Let $\{v, w\} \in E$, let B be the (unique) biconnected component of G containing the edge $\{v, w\}$.

Without loss of generality, assume that G is connected since different connected components evolve independently.

Let $F_1, \ldots, F_k$ be the connected components of $\mathcal{B}(G) \setminus \{B\}$, namely, the graph obtained from the biconnected-component graph $\mathcal{B}(G)$ by deleting the vertex B. Note that each vertex in F_i ($1 \le i \le k$) corresponds to one biconnected component of G. For each $i : 1 \le i \le k$, let H_i be the subgraph of G which is the union of the biconnected components of G corresponding to the vertices in F_i. It is not hard to see that each H_i ($1 \le i \le k$) has exactly one vertex, say u_i, shared with the biconnected component B.

Then for any $a, b \in \{0, 1\}$,

$$\Theta_G(v=a, w=b) = \sum_{\bar{c} \in \{0,1\}^k} \left(\Theta_B(v=a, w=b, \bar{u}=\bar{c}) \prod_{i=1}^{k} \Theta_{H_i}(u_i = c_i) \right).$$

Because for each $i : 1 \le i \le k$, $\Theta_{H_i}(u_i = 1) = \Theta_{H_i}(u_i = 0)$. It follows that

$$\Theta_G(v = a, w = b) = \left(\prod_{i=1}^{k} \Theta_{H_i}(u_i = 1)\right) \sum_{\bar{c} \in \{0,1\}^k} \Theta_B(v = a, w = b, \bar{u} = \bar{c})$$
$$= \left(\prod_{i=1}^{k} \Theta_{H_i}(u_i = 1)\right) \Theta_B(v = a, w = b).$$

Therefore,

$$\lambda'(\{v, w\}) = \frac{\Theta_G(v=1,w=1)}{\Theta_G(v=1,w=0)} = \frac{\left(\prod_{i=1}^{k} \Theta_{H_i}(u_i=1)\right)\Theta_B(v=1,w=1)}{\left(\prod_{i=1}^{k} \Theta_{H_i}(u_i=1)\right)\Theta_B(v=1,w=0)}.$$

It follows that $\lambda'(\{v, w\}) = \Theta_B(v = 1, w = 1)/\Theta_B(v = 1, w = 0)$. □

Claim in Lemma 3. *The following facts hold.*

1. *$\lambda'_j < \lambda_j$ for each $j : 1 \le j \le i_0$ and $\lambda'_j > \lambda_j$ for each $j : i_0 < j \le n$.*
2. *$\lambda'_1 \ge \ldots \ge \lambda'_{i_0}$ and $\lambda'_{i_0+1} \ge \cdots \ge \lambda'_n$.*
3. *$\lambda'_{i_0-1} > 1 > \lambda'_{i_0+2}$.*
4. *If $\lambda'_{i_0} \le 1$, then $\lambda'_{i_0+1} < 1$.*
5. *If $\lambda'_{i_0} = 1$ (resp. $\lambda'_{i_0+1} = 1$), then $\lambda''_{i_0} < 1$ (resp. $\lambda''_{i_0+1} > 1$) and $\lambda''_j = \lambda'_j$ for each $j \ne i_0$ (resp. $j \ne i_0 + 1$), where λ'''s are the weights of edges after two updating steps.*

Proof (the Claim).

1. Because C is not balanced and $n - i_0$ is odd, it follows that if $j \le i_0$, then $\prod_{k \ne j} (\lambda_k - 1) < 0$, thus $\lambda'_j < \lambda_j$; otherwise, $j > i_0$, we have $\prod_{k \ne j} (\lambda_k - 1) > 0$, thus $\lambda'_j > \lambda_j$.

2. For each $1 \le j < n$, consider $\lambda'_j - \lambda'_{j+1}$.

$$\lambda'_j - \lambda'_{j+1} = \frac{\prod\limits_{k \ne j} (\lambda_k + 1) + \prod\limits_{k \ne j} (\lambda_k - 1)}{\prod\limits_{k \ne j} (\lambda_k + 1) - \prod\limits_{k \ne j} (\lambda_k - 1)} \lambda_j - \frac{\prod\limits_{k \ne j+1} (\lambda_k + 1) + \prod\limits_{k \ne j+1} (\lambda_k - 1)}{\prod\limits_{k \ne j+1} (\lambda_k + 1) - \prod\limits_{k \ne j+1} (\lambda_k - 1)} \lambda_{j+1}$$

The sign of $\lambda'_j - \lambda'_{j+1}$ is equal to that of

$$\left(\prod_{k \ne j} (\lambda_k + 1) + \prod_{k \ne j} (\lambda_k - 1)\right)\left(\prod_{k \ne j+1} (\lambda_k + 1) - \prod_{k \ne j+1} (\lambda_k - 1)\right)\lambda_j -$$
$$\left(\prod_{k \ne j} (\lambda_k + 1) - \prod_{k \ne j} (\lambda_k - 1)\right)\left(\prod_{k \ne j+1} (\lambda_k + 1) + \prod_{k \ne j+1} (\lambda_k - 1)\right)\lambda_{j+1}$$
$$= (\lambda_j - \lambda_{j+1}) \left[(\lambda_j + 1)(\lambda_{j+1} + 1)x^2 - 2(\lambda_j + \lambda_{j+1})xy - (\lambda_j - 1)(\lambda_{j+1} - 1)y^2\right]$$

where $x = \prod_{k\neq j,j+1}(\lambda_k+1)$ and $y = \prod_{k\neq j,j+1}(\lambda_k-1)$.

Case 1. $j < i_0$: the sign of y is equal to that of $(-1)^{n-i_0} = -1 < 0$.

Because $(\lambda_j - 1)(\lambda_{j+1}-1) > 0$, $y^2 < x^2$ and $(\lambda_j - 1)(\lambda_{j+1}-1) < (\lambda_j + 1)(\lambda_{j+1}+1)$, it follows that $(\lambda_j+1)(\lambda_{j+1}+1)x^2 - (\lambda_j-1)(\lambda_{j+1}-1)y^2 > 0$.

Moreover, $-2(\lambda_j + \lambda_{j+1})xy > 0$, thus $\lambda'_j - \lambda'_{j+1} \geq 0$ for each $j < i_0$.

Case 2. $j > i_0$: the sign of y is equal to that of $(-1)^{n-i_0-2} = -1 < 0$.

The discussion is similar to Case 1.

3. $\lambda'_{i_0-1} - 1$ has the same sign as

$$\begin{aligned}
&\left(\prod_{k\neq i_0-1}(\lambda_k+1) + \prod_{k\neq i_0-1}(\lambda_k-1)\right)\lambda_{i_0-1} - \left(\prod_{k\neq i_0-1}(\lambda_k+1) - \prod_{k\neq i_0-1}(\lambda_k-1)\right)\\
&= \left((\lambda_{i_0}+1)x + (\lambda_{i_0}-1)y\right)\lambda_{i_0-1} - \left((\lambda_{i_0}+1)x - (\lambda_{i_0}-1)y\right)\\
&= (\lambda_{i_0-1}-1)(\lambda_{i_0}+1)x + (\lambda_{i_0-1}-1)(\lambda_{i_0}-1)y\\
&= (\lambda_{i_0-1}-1)\left((\lambda_{i_0}+1)x + (\lambda_{i_0}-1)y\right)
\end{aligned}$$

where $x = \prod_{k\neq i_0-1,i_0}(\lambda_k+1)$ and $y = \prod_{k\neq i_0-1,i_0}(\lambda_k-1)$.

It is not hard to see that $y < 0$ since $n - i_0$ is odd.

Because $|x| > |y|$ and $\lambda_{i_0}+1 > \lambda_{i_0}-1 > 0$, it follows that $(\lambda_{i_0}+1)x + (\lambda_{i_0} - 1)y > 0$.

Therefore, $\lambda'_{i_0-1} - 1 > 0$, i.e. $\lambda'_{i_0-1} > 1$.

Similarly, $1 - \lambda'_{i_0+2}$ has the same sign as

$$\begin{aligned}
&\left(\prod_{k\neq i_0+2}(\lambda_k+1) - \prod_{k\neq i_0+2}(\lambda_k-1)\right) - \left(\prod_{k\neq i_0+2}(\lambda_k+1) + \prod_{k\neq i_0+2}(\lambda_k-1)\right)\lambda_{i_0+2}\\
&= \left((\lambda_{i_0+1}+1)x - (\lambda_{i_0+1}-1)y\right) - \left((\lambda_{i_0+1}+1)x + (\lambda_{i_0+1}-1)y\right)\lambda_{i_0+2}\\
&= (1+\lambda_{i_0+1})(1-\lambda_{i_0+2})x + (1-\lambda_{i_0+1})(1+\lambda_{i_0+2})y\\
&= (1-\lambda_{i_0+1}\lambda_{i_0+2})(x+y) + (\lambda_{i_0+1}-\lambda_{i_0+2})(x-y)
\end{aligned}$$

where $x = \prod_{k\neq i_0+1,i_0+2}(\lambda_k+1)$ and $y = \prod_{k\neq i_0+1,i_0+2}(\lambda_k-1)$

Because $x+y, x-y > 0$, $1 - \lambda_{i_0+1}\lambda_{i_0+2} > 0$ and $\lambda_{i_0+1} \geq \lambda_{i_0+2}$, it follows that $1 - \lambda'_{i_0+2} > 0$, i.e. $1 > \lambda'_{i_0+2}$.

4. To the contrary, suppose that $\lambda'_{i_0} \leq 1$ and $\lambda'_{i_0+1} \geq 1$.

Then

$$\left(\prod_{k\neq i_0}(\lambda_k+1) + \prod_{k\neq i_0}(\lambda_k-1)\right)\lambda_{i_0} \leq \left(\prod_{k\neq i_0}(\lambda_k+1) - \prod_{k\neq i_0}(\lambda_k-1)\right)$$

and

$$\left(\prod_{k\neq i_0+1}(\lambda_k+1) + \prod_{k\neq i_0+1}(\lambda_k-1)\right)\lambda_{i_0+1} \geq \left(\prod_{k\neq i_0+1}(\lambda_k+1) - \prod_{k\neq i_0+1}(\lambda_k-1)\right).$$

Let $x = \prod_{k\neq i_0, i_0+1} (\lambda_k + 1)$ and $y = \prod_{k\neq i_0, i_0+1} (\lambda_k - 1)$, then

$$((\lambda_{i_0+1} + 1)x + (\lambda_{i_0+1} - 1)y)\lambda_{i_0} \leq (\lambda_{i_0+1} + 1)x - (\lambda_{i_0+1} - 1)y$$

and

$$((\lambda_{i_0} + 1)x + (\lambda_{i_0} - 1)y)\lambda_{i_0+1} \geq (\lambda_{i_0} + 1)x - (\lambda_{i_0} - 1)y.$$

It follows that $(\lambda_{i_0} - 1)(1 + \lambda_{i_0+1})x \leq (\lambda_{i_0} + 1)(1 - \lambda_{i_0+1})y$ and $(\lambda_{i_0} + 1)(1 - \lambda_{i_0+1})x \leq (\lambda_{i_0} - 1)(1 + \lambda_{i_0+1})y$.

It is not hard to see that $y > 0$ since $n - i_0$ is odd, it follows that $\frac{x}{y} \leq \frac{(\lambda_{i_0}+1)(1-\lambda_{i_0+1})}{(\lambda_{i_0}-1)(1+\lambda_{i_0+1})}$ and $\frac{x}{y} \leq \frac{(\lambda_{i_0}-1)(1+\lambda_{i_0+1})}{(\lambda_{i_0}+1)(1-\lambda_{i_0+1})}$, therefore $\frac{x^2}{y^2} \leq 1$, contradicting to the fact that $x > y > 0$.

5. If $\lambda'_{i_0} = 1$, then $\lambda'_{i_0+1} < 1$ according to 4. From 2,3, it follows that $\lambda'_1 \geq \ldots \geq \lambda'_{i_0-1} > 1 > \lambda'_{i_0+1} \geq \ldots \lambda'_n$.

Because

$$\lambda''_{i_0} = \frac{\prod_{k\neq i_0} (\lambda'_k + 1) + \prod_{k\neq i_0} (\lambda'_k - 1)}{\prod_{k\neq i_0} (\lambda'_k + 1) - \prod_{k\neq i_0} (\lambda'_k - 1)} \lambda'_{i_0},$$

$\prod_{k\neq i_0} (\lambda'_k - 1) < 0$, and $\lambda'_{i_0} = 1$, it follows that $\lambda''_{i_0} < 1$.

Since $\lambda'_{i_0} = 1$, from the definition, it is easy to see that $\lambda''_j = \lambda'_j$ for $j \neq i_0$.

The argument for the situation that $\lambda'_{i_0+1} = 1$ is similar. $\square$