A Note on the Characterization of $\text{TL}[\text{EF}]^\star$

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Abstract

In this note, we give a new proof for Bojańczyk & Walukiewicz’s effective characterization of $\text{TL}[\text{EF}]$ (the fragments of Computation Tree Logic (CTL), with $\text{EF}$ modality only) following the Ehrenfeucht-Fraisse game approach. Then, we extend the proof to the effective characterization of $\text{TL}[\text{EFns}]$ ($\text{Fns}$ is the non-strict “future” temporal operator, while $\text{F}$ is the strict one).

Key words: formal languages, branching-time temporal logic, tree languages, Ehrenfeucht-Fraisse game

1. Introduction

The definability problem for logics on trees is to decide whether a given regular tree language is definable in a logic. This kind of problem has proven to be rather difficult. For instance, the definability problem for first order logic on trees ($\text{FO}(<)$) has been a longstanding open problem since the 80’s of the last century despite several partial results [6,4,1,2].

In [3], Bojańczyk & Walukiewicz made a breakthrough in the definability problem for logics on trees by giving effective characterizations for several sublogics of CTL, namely $\text{TL}[\text{EX}]$, $\text{TL}[\text{EF}]$ and $\text{TL}[\text{EX,EF}]$. While the proofs of the characterization for $\text{TL}[\text{EX}]$ and $\text{TL}[\text{EX,EF}]$ in [3] were elegant and short, the proof for $\text{TL}[\text{EF}]$ was very intricate.

One of the main reasons for the intricacy of the proof of $\text{TL}[\text{EF}]$ in [3] is that the proof was constructive. To avoid the intricacy, in this note, we give an existential proof for the characterization of $\text{TL}[\text{EF}]$ following the Ehrenfeucht-Fraisse games approach, similar to the proof of the characterization of the fragment of LTL that only uses the operator “F”, “sometimes in the future” [8]. Moreover, we extend this proof to the characterization of $\text{TL}[\text{EFns}]$ ($\text{Fns}$ is the non-strict “future” operator, while $\text{F}$ is strict), which was mentioned to be open in [3].

The remaining sections are organized as follows: in Section 2, the syntax and semantics of $\text{TL}[\text{EF}]$ and $\text{TL}[\text{EFns}]$ are defined; and in Section 3, some definitions and notations are introduced; in Section 4, a new proof of characterization of $\text{TL}[\text{EF}]$ is given; then in Section 5, the effective characterization of $\text{TL}[\text{EFns}]$ is established; finally in Section 6, we give some conclusions and remarks.

2. Syntax and Semantics of $\text{TL}[\text{EF}]$ and $\text{TL}[\text{EFns}]$

Let $\Sigma$ be a finite alphabet, then the syntax of $\text{TL}[\text{EF}]$ is defined by the following rules.

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1 One of the referees pointed out that the characterization of $\text{TL}[\text{EFns}]$ has been independently announced by Zoltan Ésik and Szabolcs Iván at a workshop of CSL’06 on formal languages: http://www.inf.u-szeged.hu/~csl06/ws.php.
A binary tree domain is a prefix closed nonempty subset of \{0,1\} such that for all \(v \in \{0,1\}\), \(v0\) is in the domain iff \(v1\) is in the domain, in other words, each inner node has two sons. Evidently \(\varepsilon\) is in all tree domains, which is called the root of the tree. The prefix relation on the tree domain is denoted by \(<\).

Let \(\Sigma\) be a finite alphabet. A \(\Sigma\)-labelled finite binary tree is a function from a binary tree domain to \(\Sigma\). If \(t\) is a \(\Sigma\)-labelled finite binary tree, then the tree domain of \(t\) is denoted by \(dom(t)\). For any \(v \in dom(t)\), the label of \(v\) in \(t\) is denoted by \(t(v)\). In particular, \(t(\varepsilon)\) is the label of the root of \(t\).

If \(v \in dom(t)\), then \(t|_v\) denotes the subtree of \(t\) below \(v\) (including \(v\)).

Let \(T_\Sigma\) denote the set of all \(\Sigma\)-labelled finite binary trees.

The semantics of \(TL[EF]\) are defined as follows.

\(t \models p_\alpha\) if \(t(\varepsilon) = \alpha\), where \(\alpha \in \Sigma\).

\(t \models \neg \varphi_1\) if not \(t \models \varphi_1\).

\(t \models \varphi_1 \lor \varphi_2\) if \(t \models \varphi_1\) or \(t \models \varphi_2\).

\(t \models EF\varphi_1\) if there is some \(v \in dom(t)\), \(v > \varepsilon\) such that \(t|_v \models \varphi_1\).

The syntax of \(TL[EF_{ns}]\) is defined by the same rules in (1) with \(EF\varphi_1\) replaced by \(EF_{ns}\varphi_1\).

The semantics of \(EF_{ns}\varphi_1\) is defined as follows:

\(t \models EF_{ns}\varphi_1\) if there is \(v \in dom(t)\) such that \(t|_v \models \varphi_1\) (note that here \(v\) may be \(\varepsilon\)).

Let \(\varphi\) be a \(TL[EF]\) or \(TL[EF_{ns}]\) formula, then the closure of \(\varphi\), denoted by \(cl(\varphi)\), is defined to be the smallest set of formulas containing \(\varphi\) and closed under negations and subformulas.

A tree language \(L\) is said to be \(TL[EF]\) (\(TL[EF_{ns}]\) respectively)-definable if there is a formula \(\varphi\) in \(TL[EF]\) (\(TL[EF_{ns}]\) respectively) such that \(L = \{t \in T_\Sigma | t \models \varphi\}\).

Since \(EF_{ns}\varphi \equiv \varphi \lor EF\varphi\), \(TL[EF_{ns}]\) can be seen as a sublogic of \(TL[EF]\). Moreover, \(TL[EF]\) is more expressive than \(TL[EF_{ns}]\). For instance, the property “the tree has at least depth two and all its nodes are labelled by \(a\)” can be expressed by \(TL[EF]\) formula: \(p_a \land EFp_a \land \neg EF\neg p_a\), which, nevertheless, is not expressible in \(TL[EF_{ns}]\).

3. Notations and definitions

Basically, we follow the notations in [3]. But for the reader’s convenience, we recall the relevant notations and definitions.
Let $s$ be a tree and $a \in \Sigma$. Then $s[a]$ denotes the tree the same as $s$ except that the root of $s$ is labelled by $a$ now.

Let $s$ be a tree. Then the delayed type of $s$, denoted by $\text{dtype}(s)$, is a function from $\Sigma$ to $\text{Types}(L)$ such that $\text{dtype}(s)(a) = \text{type}(s[a])$.

It is evident that $\text{dtype}(a(s, t))$ has nothing to do with $a$, consequently we can write $\text{dtype}(s, t)$ simply. It is also easy to see that if $s \sim L$ $s'$ and $t \sim L$ $t'$, then $\text{dtype}(s, t) = \text{dtype}(s', t')$. Thus we can write $\text{dtype}(\alpha, \beta)$ to denote the delayed type $\text{dtype}(s, t)$ with $\text{type}(s) = \alpha$ and $\text{type}(t) = \beta$.

4. New proof of characterization of $\text{TL[EF]}$

Before the proof, we give some definitions, propositions and lemmas.

**Definition 1 (TL[EF] EF-Game)** Let $s, t$ be two trees. Then the $k$-round Ehrenfeucht-Fra"issé game on $s, t$ is played by two players spoiler and duplicator in turn:

- $0$-round game: if $s(\varepsilon) \neq t(\varepsilon)$, then spoiler wins, otherwise duplicator wins.

- $k$-round game ($k > 0$): if $s(\varepsilon) \neq t(\varepsilon)$, then spoiler wins.

Otherwise spoiler should select some non-root node in one of the two trees, say $v$ in $s$. If he fails to do so (namely both trees have only one node), then duplicator wins.

Otherwise duplicator should select some non-root node in the other tree, say $w$ in $t$. If she fails to do so (namely exactly one of the two trees has only one node), then spoiler wins.

Otherwise spoiler and duplicator play the $(k+1)$-round game on $s|v$ and $t|w$.

We say that spoiler or duplicator has a winning strategy in the $k$-round $\text{TL[EF]}$ EF-game on $s, t$ if he or she can win regardless of the moves by the opponent.

It is not hard to see that if spoiler has a winning strategy in the $k$-round $\text{TL[EF]}$ EF-game on $s, t$, then he has a winning strategy in the $(k+1)$-round game as well. Similarly if duplicator has a winning strategy in the $(k+1)$-round $\text{TL[EF]}$ EF-game on $s, t$, then she has a winning strategy in the $k$-round game as well.

Similar to Corollary 2.2 in [5], we have the following proposition.

**Proposition 1** Let $L$ be a tree language. If there is some $k \geq 0$ such that for all $s \in L$ and $t \notin L$, spoiler has a winning strategy in the $k$-round $\text{TL[EF]}$ EF-game on $s, t$; then $L$ is $\text{TL[EF]}$ definable.

**Definition 2 (EF-admissible)** $L$ is said to be EF-admissible if the following three properties are satisfied:

- $[P1]$ $\text{dtype}(\alpha, \beta) = \text{dtype}(\beta, \alpha)$
- $[P2]$ if $\alpha_0 \approx \beta_0$ and $\alpha_1 \approx \beta_1$, then $\text{dtype}(\alpha_0, \alpha_1) = \text{dtype}(\beta_0, \beta_1)$
- $[P3]$ if $\alpha \preceq \beta$, then $\text{dtype}(\alpha, \beta) = \text{dtype}(\beta, \beta)$

where $\alpha, \beta(i = 0, 1), \alpha, \beta \in \text{Types}(L)$.

**Definition 3 (DSCCS-dependent)** $L$ is said to be delayed-strongly-connected-component-set dependent (DSCCS-dependent) if for any trees $s$ and $t$ such that $\text{DSCCS}(s) = \text{DSCCS}(t)$, we have that $\text{dtype}(s) = \text{dtype}(t)$.

**Theorem 1 ([3,1])** Let $L$ be a regular tree language. Then the following three conditions are equivalent:

- (i) $L$ is $\text{TL[EF]}$-definable.
- (ii) $L$ is EF-admissible.
- (iii) $L$ is DSCCS-dependent.

Proof.

(i) $\Rightarrow$ (ii):

Suppose that $L$ is $\text{TL[EF]}$-definable. $[P1]$ is evident since $\text{TL[EF]}$ can't distinguish between the left and right sons.

The proof of $[P2]$ is exactly Lemma 3.3.7 in [1]. The proof of $[P3]$ is exactly Lemma 3.3.6 in [1].

(ii) $\Rightarrow$ (iii):

Essentially the proof has been given in section 3.3.1 of [1].

Suppose that $L$ is EF-admissible.

Let $s, t$ be two trees such that $\text{DSCCS}(s) = \text{DSCCS}(t)$. If $\text{DSCCS}(s) = \text{DSCCS}(t) = \emptyset$, then evidently $\text{dtype}(s) = \text{dtype}(t)$.

Now we assume $\text{DSCCS}(s) = \text{DSCCS}(t) \neq \emptyset$.

Let $\alpha_i = \text{type}(s|\varepsilon)$ and $\beta_i = \text{type}(t|\varepsilon)$, where $i = 0, 1$. Then $[\alpha_i] \in \text{DSCCS}(t)$ and $[\beta_i] \in \text{DSCCS}(s)$, where $i = 0, 1$. There are three cases.

Case 1: there is $i$ such that $[\alpha_0], [\alpha_1] \in \text{SCCS}(s|\varepsilon)$.

Then $\alpha_0, \alpha_1 \preceq \beta_i$ and $\beta_i \preceq \alpha_j, \beta_{i-1} \preceq \alpha_{j'}$ for some $j, j'$. Thus $\alpha_j \approx \beta_i$ and $\alpha_{j'} \approx \beta_i, \beta_{i-1} \approx \beta_i$.

So

$$\text{dtype}(\alpha_0, \alpha_1) = \text{dtype}(\alpha_{j'}, \alpha_{j-1}) = \text{dtype}(\beta_i, \alpha_{j-1})$$

$$\text{dtype}(\beta_i, \beta_i) = \text{dtype}(\beta_i, \beta_{i-1}) = \text{dtype}(\beta_0, \beta_1)$$

The first and last equations above are according to $[P1]$ in the definition 2; the second equation is according to $[P2]$; the third and fourth equations are according to $[P3]$.

Case II: there is $i$ such that $[\beta_0], [\beta_1] \in \text{SCCS}(s|\varepsilon)$. 

Similar to Case I.

Case III: neither I nor II holds.

Then there is \( i \) such that \([\alpha_0] \in SC(S(t_{i0}))\) and \([\alpha_1] \in SC(S(t_{i1}))\) and there is \( j \) such that \([\beta_0] \in SC(S(s_{j0}))\) and \([\beta_1] \in SC(S(s_{j1}))\). There are four subcases.

Subcase III.I: \( i = j = 0 \)
Then \( \alpha_0 \geq \beta_0, \alpha_1 \leq \beta_1, \beta_0 \leq \alpha_0 \) and \( \beta_1 \leq \alpha_1 \). So we have \( \alpha_0 \approx \beta_0 \) and \( \alpha_1 \approx \beta_1 \).
Then \( dtype(\alpha_0, \alpha_1) = dtype(\beta_0, \beta_1) \) according to \([P2]\).

Subcase III.II: \( i = j = 1 \)
Then \( \alpha_0 \leq \beta_1, \alpha_1 \geq \beta_0, \beta_0 \leq \alpha_1 \) and \( \beta_1 \leq \alpha_0 \). So we have \( \alpha_0 \approx \beta_1 \) and \( \alpha_1 \approx \beta_0 \).
Then according to \([P2]\) and \([P1]\), \( dtype(\alpha_0, \alpha_1) = dtype(\beta_0, \beta_1) \) as well.

Subcase III.III: \( i = j \neq 0 \)
Then \( \alpha_0 \leq \beta_0, \alpha_1 \leq \beta_1, \beta_0 \leq \alpha_1 \) and \( \beta_1 \leq \alpha_0 \). So we have \( \alpha_0 \approx \beta_0, \alpha_1 \approx \beta_1 \).
Then \( dtype(\alpha_0, \alpha_1) = dtype(\beta_0, \beta_1) \) according to \([P2]\).

Subcase III.IV: \( i = 1 = j \)
Then \( \alpha_0 \leq \beta_1, \alpha_1 \geq \beta_0, \beta_0 \leq \alpha_1 \) and \( \beta_1 \leq \alpha_0 \). So we have \( \alpha_0 \approx \beta_1 \approx \alpha_1 \approx \beta_0 \).
Then \( dtype(\alpha_0, \alpha_1) = dtype(\beta_0, \beta_1) \) according to \([P2]\).

(iii) \( \Rightarrow \) (i):
Suppose \( L \) is DSCCS-dependent.

According to Proposition 1, it suffices to prove that for all \( s, t \) such that \( type(s) \neq type(t) \), spoiler has a winning strategy in the \((drank(s) + drank(t))\)-round game on \( s, t \) (because if this is true, then spoiler has a winning strategy in the \((2 \cdot |SCCS(L)|)\)-round game on \( s, t \) for all \( s \in L \) and \( t \notin L \)).

Induction on \( drank(s) + drank(t) \).
Let \( s(\varepsilon) = a \) and \( t(\varepsilon) = b \).
Induction base: \( drank(s) + drank(t) = 0 \).
Then both \( s \) and \( t \) have only one node. Since \( type(s) \neq type(t) \), then \( a \neq b \), consequently spoiler wins in the 0-round game.
Induction step: \( drank(s) + drank(t) > 0 \).
If \( a \neq b \), then spoiler wins.
Otherwise \( DSCCS(s) \neq DSCCS(t) \) since \( type(s) \neq type(t) \) and \( L \) is DSCCS-dependent. Consequently there is \( [\gamma] \in DSCCS(s) \setminus DSCCS(t) \) or \( [\gamma] \in DSCCS(t) \setminus DSCCS(s) \). Here we consider the former case, the latter case can be considered similarly.

Then spoiler selects \( v > \varepsilon \) in \( s \) such that \( type(s_{[\gamma]}) = [\gamma] \) and \( v \) is maximal in this sense.
If \( t \) has only one node, then spoiler wins.
Otherwise duplicator selects \( w > \varepsilon \) in \( t \).

Since \( [type(s_{[\gamma]})] = [\gamma] \) and \( v \) is maximal, \( [\gamma] \notin DSCCS(s_{[\gamma]}) \), so \( drank(s_{[\gamma]}) < drank(s) \). Then \( drank(s_{[\gamma]}) + drank(t_{[\gamma]}) < drank(s) + drank(t) \).

We also have that \( type(s_{[\gamma]}) \neq type(t_{[\gamma]}) \) for otherwise \( [\gamma] = [type(s_{[\gamma]})] = [type(t_{[\gamma]})] \in DSCCS(t) \), a contradiction.

Then according to the induction hypothesis, spoiler has a winning strategy in the \((drank(s_{[\gamma]}) + drank(t_{[\gamma]}))\)-round game on \( s_{[\gamma]} \) and \( t_{[\gamma]} \). Consequently spoiler has a winning strategy in the \((drank(s) + drank(t) - 1)\)-round game on \( s_{[\gamma]} \) and \( t_{[\gamma]} \).

Thus we conclude that spoiler has a winning strategy in the \((drank(s) + drank(t))\)-round game on \( s, t \).

5. Characterization of TL[EFns]

Before giving the characterization of TL[EFns], we give some definitions, propositions and lemmas.

**Definition 4 (TL[EFns] EF-Game)** Let \( s, t \) be two trees. Then the \( k \)-round Ehrenfeucht-Fraïssé game on \( s, t \) is played by two players spoiler and duplicator in turn:

0-round game: if \( s(\varepsilon) \neq t(\varepsilon) \), then spoiler wins, otherwise duplicator wins.

\( k \)-round game \((k > 0)\): if \( s(\varepsilon) \neq t(\varepsilon) \), then spoiler wins.

Otherwise spoiler selects some node in one of the two trees, say \( v \) in \( s \). And duplicator selects some node in the other tree, say \( w \) in \( t \). Then spoiler and duplicator play the \((k-1)\)-round game on \( s_{[v]} \) and \( t_{[w]} \).

Similar to the TL[EF] EF-game, we have that if spoiler has a winning strategy in the \( k \)-round TL[EFns] EF-game on \( s, t \), then he has a winning strategy in the \((k + 1)\)-round game as well. Similarly if duplicator has a winning strategy in the \((k + 1)\)-round TL[EFns] EF-game on \( s, t \), then she has a winning strategy in the \( k \)-round game as well.

Similar to Proposition 1, we have the following proposition for TL[EFns].

**Proposition 2** Let \( L \) be a tree language. If there is some \( k \geq 0 \) such that for all \( s \in L \) and \( t \notin L \), spoiler has a winning strategy in the \( k \)-round TL[EFns] EF-game on \( s, t \); then \( L \) is TL[EFns] definable.

Let \( L \) be a tree language and \( \alpha \in Types(L) \). The root letters of \( \alpha \), denoted by \( letters(\alpha) \), is defined to be \( \{ \alpha \in \Sigma \mid \text{there is } t \text{ such that } t(\varepsilon) = a, type(t) = \alpha \} \).

**Definition 5 (EFns-admissible)** A tree language \( L \) is said to be EFns-admissible if it is EF-admissible...
Lemma 1 Let $L$ be $\text{EF}_{\text{NS}}$-admissible and $s,t$ be trees. If $\text{SCCS}(s) = \text{SCCS}(t)$ and $s(\varepsilon) = t(\varepsilon)$, then $\text{type}(s) = \text{type}(t)$.

Proof. Let $a = s(\varepsilon) = t(\varepsilon)$, $\alpha = \text{type}(s)$ and $\beta = \text{type}(t)$.

Since $L$ is $\text{EF}_{\text{NS}}$-admissible, then it is $\text{EF}$-admissible, so $\text{DSCS}$-dependent according to Theorem 1. Consequently $d\text{type}(a(s,s)) = d\text{type}(a(t,t))$ since $\text{DSCS}(a(s,s)) = \text{SCCS}(s) = \text{SCCS}(t) = \text{DSCS}(a(t,t))$. So $\text{type}(a(s,s)) = \text{type}(a(t,t))$.

Because $a \in \text{rletters}(\alpha)$ and $a \in \text{rletters}(\beta)$, according to [P4] in Definition 5, $\alpha = a(\alpha,\alpha) = \text{type}(a(s,s)) = \text{type}(a(t,t))$.

The following lemma is obvious.

Lemma 2 Let $L$ be defined by $\text{TL}[\text{EF}_{\text{NS}}]$ formula $\varphi$ and $s,t$ be two trees. If $s$ and $t$ satisfy the same formulas in $\text{cl}(\varphi)$, then $\text{type}(s) = \text{type}(t)$.

Theorem 2 Let $L$ be a regular tree language. Then the following two conditions are equivalent:

(i) $L$ is $\text{TL}[\text{EF}_{\text{NS}}]$-definable.

(ii) $L$ is $\text{EF}_{\text{NS}}$-admissible.

Proof.

(i) $\Rightarrow$ (ii):

Suppose that $L$ is $\text{TL}[\text{EF}_{\text{NS}}]$ definable.

Since $\text{TL}[\text{EF}_{\text{NS}}]$ can be seen as a sublogic of $\text{TL}[\text{EF}]$, we know that $L$ is $\text{EF}$-admissible from Theorem 1.

Now we consider [P4].

Let $s$ be a tree such that $\text{type}(s) = \alpha$, $s(\varepsilon) = a$.

Let $t = a(s,s)$. We can prove that for all $\text{TL}[\text{EF}_{\text{NS}}]$ formula $\varphi$, $s \models \varphi$ iff $t \models \varphi$ by induction on the structure of $\varphi$.

Because $L$ is $\text{TL}[\text{EF}_{\text{NS}}]$ definable, then according to Lemma 2, we have $\text{type}(s) = \text{type}(t), a(\alpha,\alpha) = \alpha$.

(ii) $\Rightarrow$ (i):

Suppose that $L$ is $\text{EF}_{\text{NS}}$-admissible.

According to Proposition 2, it suffices to prove that for all $s,t$ with $\text{type}(s) \neq \text{type}(t)$, spoiler has a winning strategy in the $(3 \cdot (\text{rank}(s) + \text{rank}(t)))$-round game on $s$ and $t$.

Induction on $\text{rank}(s) + \text{rank}(t)$.

Let $\text{type}(s) = \alpha$, $\text{type}(t) = \beta$, $s(\varepsilon) = a$, $t(\varepsilon) = b$.

Base case: $\text{rank}(s) + \text{rank}(t) = 2$ (because $\text{rank}(s), \text{rank}(t) \geq 1$).

Then $\text{rank}(s) = \text{rank}(t) = 1$, $\text{SCCS}(s) = \{\alpha\}$ and $\text{SCCS}(t) = \{\beta\}$.

Spoiler selects some leaf $v$ in $s$. And duplicator selects $w$ in $t$.

Spoiler selects some leaf $w'$ in $t$ such that $w' \geq w$. And duplicator has no choice but to select $v$ in $s$.

Because $\text{type}(s,v) = \alpha$ and $\text{type}(t,w') = \beta$, we have $s(v) \neq t(w')$, spoiler wins. Consequently spoiler has a winning strategy in the $2$-round game on $s,t$. Thus spoiler has a winning strategy in the $(3 \cdot (\text{rank}(s) + \text{rank}(t)))$-round game on $s,t$.

Induction step: $\text{rank}(s) + \text{rank}(t) > 2$.

If $a \neq b$, then spoiler wins.

Otherwise we have that $\text{SCCS}(s) \neq \text{SCCS}(t)$ according to Lemma 1. There are three cases.

Case I: there is $[\gamma] \in \text{SCCS}(s) \setminus \{[\alpha]\}$ and $[\gamma] \notin \text{SCCS}(t)$.

Evidently $\gamma < \alpha$.

Spoiler selects some $v > \varepsilon$ in $s$ such that $[\text{type}(s,v)] = [\gamma]$ and $v$ is maximal in this sense.

Duplicator selects some $w$ in $t$.

Since $[\alpha] \notin \text{SCCS}(s,w)$, we have $\text{rank}(s,v) < \text{rank}(s)$, thus $\text{rank}(s,v) + \text{rank}(t,w) < \text{rank}(s) + \text{rank}(t)$.

We also have $\text{type}(s,w) \neq \text{type}(t,w)$ because otherwise $[\gamma] = [\text{type}(s,w)] = [\text{type}(t,w)] \in \text{SCCS}(t)$, a contradiction.

Then according to the induction hypothesis, spoiler has a winning strategy in the $(3 \cdot (\text{rank}(s) + \text{rank}(t)))$-round game on $s,w$ and $t,w$. Thus spoiler has a winning strategy in the $(3 \cdot (\text{rank}(s) + \text{rank}(t)) - 1)$-round game on $s,w$ and $t,w$.

We conclude that spoiler has a winning strategy in the $(3 \cdot (\text{rank}(s) + \text{rank}(t)))$-round game on $s,t$.

Case II: there is $[\gamma] \in \text{SCCS}(t) \setminus \{[\beta]\}$ and $[\gamma] \notin \text{SCCS}(s)$.

Similar to Case I.

Case III: neither I nor II holds.

Then $\text{SCCS}(s) \setminus \{[\alpha]\} \subseteq \text{SCCS}(t)$ and $\text{SCCS}(t) \setminus \{[\beta]\} \subseteq \text{SCCS}(s)$.

We must have that $[\alpha] \notin \text{SCCS}(t)$ or $[\beta] \notin \text{SCCS}(s)$. For otherwise $\text{SCCS}(s) \subseteq \text{SCCS}(t)$ and $\text{SCCS}(t) \subseteq \text{SCCS}(s)$, $\text{SCCS}(s) = \text{SCCS}(t)$, then according to Lemma 1, $\alpha = \beta$, a contradiction.

Without loss of generality, suppose that $[\alpha] \notin \text{SCCS}(t)$.

Then spoiler selects some $v$ in $s$ such that $[\text{type}(s,v)] = [\alpha]$ and $v$ is maximal in this sense.

Duplicator selects some $w$ in $t$.

If $s(v) \neq t(w)$, then spoiler wins.

Otherwise if $\text{type}(t,w) \prec \text{type}(t,w)$, then we have that $\text{rank}(t,w) < \text{rank}(t)$, and $\text{rank}(s,v) + \text{rank}(t,w) < \text{rank}(s) + \text{rank}(t)$.

Moreover, $\text{type}(s,v) \neq \text{type}(t,w)$ because otherwise $[\alpha] = [\text{type}(s,v)] = [\text{type}(t,w)] \in \text{SCCS}(t)$, a contradiction.
Then according to the induction hypothesis, spoiler has a winning strategy in the $(3 \cdot (\text{rank}(s_{|v}) + \text{rank}(t_{|w})))$-round game on $s_{|v}$ and $t_{|w}$. Thus spoiler has a winning strategy in the $(3 \cdot (\text{rank}(s) + \text{rank}(t)) - 1)$-round game on $s_{|v}$ and $t_{|w}$.

So spoiler has a winning strategy in the $(3 \cdot (\text{rank}(s) + \text{rank}(t)))$-round game on $s$, $t$ in this case.

**Otherwise** $[\text{type}(t_{|w})] = [\text{type}(t)] = [\beta]$.

If $v$ is a leaf in $s$, then spoiler can select some leaf $w' \geq w$ in $t$, duplicator has to select $v'$ in $s$. Since $[\text{type}(s_{|v})] = [\alpha]$, we have $s(v) \neq t(w')$ for otherwise $[\alpha] = [\text{type}(s_{|v})] = [\text{type}(t_{|w})] \in \text{SCCS}(t)$, a contradiction. So spoiler wins in this case.

**Otherwise** $v$ is an inner node of $s$. Let $a' = s(v) = t(w)$, $\text{type}(s_{|v}) = \alpha'$, $\text{type}(t_{|w}) = \beta'$ and $\alpha_i = \text{type}(s_{|\alpha_i})$ where $i = 0, 1$ (see Fig. 1).

Because $v$ is maximal, $\alpha_0, \alpha_1 < \alpha$. Since $\text{SCCS}(s_{\alpha}) \subseteq \text{SCCS}(t)$, it must be that $[\alpha_0], [\alpha_1] \in \text{SCCS}(t)$, so $\alpha_0, \alpha_1 \leq \beta$. In fact, we must have that $\alpha_0, \alpha_1 \leq \beta$.

To the contrary, suppose that $[\alpha_0] = [\beta]$ (or $[\alpha_1] = [\beta]$).

Because $[\alpha_0] = [\beta]$ and $\alpha_1 \leq \beta$, we have that $\text{dtype}(\alpha_0, \alpha_1) = \text{dtype}(\beta, \alpha_1) = \text{dtype}(\beta', \beta') = (\text{The first and third equations are by [P2], second by [P3]). Consequently}

$$a' = \alpha' \circ \alpha_0, \alpha_1 \circ dtype(\alpha_0, \alpha_1)(a') = dtype(\beta', \beta')(a') = a' (\beta', \beta') = \beta'$$

(The last equation above holds because $a' \in \text{letters}(\beta')$ and [P4]).

Then $[\alpha] = [\alpha'] = [\beta'] = [\beta] \in \text{SCCS}(t)$, a contradiction.

So it must be that $\alpha_0, \alpha_1 < \beta$. Then $[\beta] \notin \text{SCCS}(s_{|v})$.

Now spoiler can select $w' \geq w$ in $t$ such that $[\text{type}(t_{|w'})] = [\beta]$ and $w'$ is maximal in this sense.

Duplicator selects $v' \geq v$ in $s$.

If $s(v') \neq t(w')$, then spoiler wins.

**Otherwise** if $v' > v$, then $\text{type}(s_{|v'}) \prec v$ since $v$ is maximal. In this case, we can get that $\text{rank}(s_{|v'}) + \text{rank}(t_{|w'}) < \text{rank}(s) + \text{rank}(t)$ and $\text{type}(s_{|v'}) \neq \text{type}(t_{|w'})$ and spoiler has a winning strategy in the $(3 \cdot (\text{rank}(s) + \text{rank}(t)))$-round game on $s$, $t$ by the induction hypothesis.

**Otherwise** $v' = v$. In this case $\text{type}(s_{|w'}) \neq \text{type}(t_{|w'})$ for otherwise $[\alpha] = [\text{type}(s_{|w'})] = [\text{type}(t_{|w'})] \in \text{SCCS}(t)$, a contradiction.

Since $L$ is DSCCS-dependent according to Theorem 1, we have that $\text{DSCCS}(s_{|w'}) \neq \text{DSCCS}(t_{|w'})$.

Without loss of generality, suppose that there is $[\gamma] \in \text{DSCCS}(s_{|w'}) \cup \text{DSCCS}(t_{|w'})$, the other case is similar.

We know that $[\alpha] \notin \text{SCCS}(t)$ and $[\beta] \notin \text{SCCS}(s_{|w'})$. Thus we have that $[\text{type}(s_{|w'})] = [\alpha] \notin \text{SCCS}(t_{|w'})$ and $[\text{type}(t_{|w'})] = [\beta] \notin \text{SCCS}(s_{|w'})$.

So $[\gamma] \notin \text{DSCCS}(s_{|w'}) \cup \{[\text{type}(t_{|w'})]\} = \text{SCCS}(t_{|w'})$.

Then spoiler can select $v'' > v'$ in $s$ such that $[\text{type}(s_{|w''})] = [\gamma]$.

Duplicator selects $w'' \geq w'$ in $t$ (see Fig. 2).

**6. Conclusions and Remarks**

In this note, we give a new proof of characterization of TL[EF] following the Ehrenfeucht-Fraïssé
games approach and extend this proof to the characterization of TL[EFns].

The property $P_4$ in the characterization of TL[EFns] (Definition 5) can be seen as one kind of binary-tree extension of stutter-invariance concept of words [7].

We define the binary-stutter-invariance as follows: Let $L$ be a tree language, $t$ be a tree and $t'$ be a tree obtained from $t$ by applying the following operation: replacing subtree $t|_v$ ($v \in \text{dom}(t)$) by $a(t|_v, t|_v)$ ($a = t(v)$) in $t$ (see Fig. 3). If ($t \in L$ iff $t' \in L$ for any $t$ and $t'$ stated above), then we say that $L$ is binary-stutter-invariant.

From Theorem 1 and Theorem 2, it is easy to see that a TL[EF]-definable tree language $L$ is TL[EFns]-definable iff $L$ is binary-stutter-invariant.

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