Differential Games Based on Invariant Sets Generation

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***Abstract*— Differential games play an important role in collision avoidance, motion planning and control of aircrafts, and related applications. The central problem is the computation of the set of initial states from which the ego player can enforce safety specifications over a specified time horizon. In this paper we study differential games based on invariant sets generation, where the ego player aims to perpetually force the system to satisfy certain safety specification while the mutual other player attempts to enforce a violation of the safety specification. This game is studied within the Hamilton-Jacobi reachability framework via computing two new robust controlled invariant sets, i.e., the lower robust controlled invariant set and the upper robust controlled invariant set (Definition 2). These two robust controlled invariant sets are respectively characterized as the zero level set of the unique bounded continuous viscosity solution to a Hamilton-Jacobi equation with sup-inf Hamiltonian and inf-sup Hamiltonian. This is the main contribution of this work. The uniqueness and continuity property of viscosity solutions facilitates the use of contemporary numerical methods to solve this game. Two examples, including a Moore-Greitzer jet engine model, are used to illustrate our approach.**

I. Introduction

Differential games, i.e., dynamic games featuring an evolution governed by differential equations, have many important applications in engineering domains, e.g., in the analysis of collision avoidance [29], [37], energy management [15] and safe reinforcement learning [32]. They model a form of strategic interactions among rational players, where each player makes decisions in light of its own preference while expecting adversarial actions from the mutual other player. As the resulting winning strategies are robust against any possible action of the adversary, differential games have received growing interest as a model facilitating synthesis of reliable control strategies for safety-critical systems.

Differential games were initiated by Rufus Isaacs in the early 1950s when he studied military pursuit-evasion problems while working in the Rand Corporation. The pursuit-evasion game he studied is a two-player zero-sum game, where the players have completely opposite interests [23]. A challenging class of differential games is to determine the set of states from which the ego player is able to stay away from an avoid set, regardless of opposing actions of the mutual other player. This set goes by many names in the literature such as discriminating kernels [2], [13], backward reachable sets [28] and stable bridges [35]. Differential games can be solved within the Hamilton-Jacobi reachability framework. Hamilton-Jacobi reachability analysis addresses reachability problems by exploiting the link to optimal control through viscosity solutions of Hamilton-Jacobi equations [4]. It extends the use of Hamilton-Jacobi equations, which are widely used in optimal control theory [5], [6], to perform reachability analysis over both finite time horizons [25], [28], [26], [1], [18] and the infinite time horizon [12], [21], [22]. While computationally intensive, Hamilton-Jacobi reachability approaches are still appealing nowadays due to the availability of contemporary numerical methods [16] and modern numerical tools such as [27], [10], which allow solving associated game problems conveniently. Within the Hamilton-Jacobi reachability framework, continuity and uniqueness of solutions are desirable from a numeric computation point of view, since rigorous convergence results for many popular numerical approximations, e.g., finite difference schemes, ENO schemes and WENO schemes, to the derived Hamilton-Jacobi equation require continuity and uniqueness of the solution. Unfortunately, reachability analysis under state constraints may induce discontinuities of viscosity solutions, see for instance [24], [5], [17], [7], [14], [30], [8], [3], unless the dynamics satisfy special assumptions at the boundary of state-constraint sets, e.g., inward pointing qualification assumption [33], [34], [9], outward pointing condition [19] and vanishing on the boundary [7]. These conditions are, however, restrictive and viscosity solutions can therefore be discontinuous in general. Recently, without requiring such assumptions, the work in [11] infers a modified Hamilton-Jacobi equation and considers reachability problems over finite time horizons for state-constrained systems with control inputs. The modified Hamilton-Jacobi equation exhibits a unique continuous viscosity solution. Based on the Hamilton-Jacobi formulation in [11], the work in [26] studies the finite-time games for state-constrained systems. The work in [18] further investigates differential games over finite time horizons where the target set, the state constraint set and the dynamic are allowed to be time-varying. Recently, the work in [39] studies the problem of computing robust invariant sets over the infinite time horizon for state-constrained perturbed systems without control inputs, where a robust invariant set is a set of states such that every possible trajectory starting from it never violates a given state constraint, irrespective of the actual perturbation. In [39] the maximal robust invariant set is described as the zero level set of the unique continuous viscosity solution to a Hamilton-Jacobi equation.

In this paper we extend the Hamilton-Jacobi formulation in [39] to address differential games based on invariant sets generation. In the game, we consider the problem of computing two new robust controlled invariant sets, i.e., the lower and upper robust controlled invariant sets. For a given

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state-constraint set, the lower robust controlled invariant set is a set of initial states such that for any neighborhood of the state-constraint set and any finite time horizon, there exists a nonanticipative strategy for the ego player which makes the system stay inside the neighborhood over the finite time horizon, irrespective of actions of the mutual other player; the upper one is a set of initial states such that for any neighborhood of the state-constraint set, any finite time horizon and any nonanticipative strategy of the mutual other player, there exists an action for the ego player which makes the system stay inside the neighborhood over the finite time horizon. We characterize the lower robust controlled invariant set as the zero level set of the unique bounded continuous viscosity solution to a Hamilton-Jacobi equation with sup-inf Hamiltonian and the upper robust controlled invariant set as the zero level set of the unique bounded continuous viscosity solution to a Hamilton-Jacobi equation with inf-sup Hamiltonian, respectively. Under the classical Isaacs condition, these two sets coincide. To the best of our knowledge this is the first work on the use of Hamilton-Jacobi equations to address differential games with the lower and upper robust controlled invariant sets. Two examples, including a Moore-Greitzer jet engine model, are employed to demonstrate our approach.

This paper is structured as follows: Section II gives an introduction of differential games of interest in this paper, including the notion of lower and upper robust controlled invariant sets. Section III formulates the computation of both lower and upper robust controlled invariant sets within the framework of Hamilton-Jacobi type partial differential equation. After demonstrating our approach on two examples in Section IV, we conclude this paper in Section V.

II. Differential Game Formulation

In this section we introduce the notations and definitions used in the rest of this paper. The following basic notations will be used throughout this paper: $\mathbb{R}^n$ denotes the set of n-dimensional real vectors. $\|\boldsymbol{x}\|$ denotes the 2-norm, i.e., $\|\boldsymbol{x}\| := \sqrt{\sum_{i=1}^n x_i^2}$, where $\boldsymbol{x} = (x_1, \dots, x_n)^\top$. $C^\infty(\mathbb{R}^n)$ denotes the set of smooth functions over $\mathbb{R}^n$. Vectors are denoted by boldface letters.

We consider systems with dynamics given by

$$\begin{cases} \dot{\boldsymbol{x}}(s) = \boldsymbol{f}(\boldsymbol{x}(s), \boldsymbol{u}(s), \boldsymbol{d}(s)) \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathcal{X}. \end{cases} \tag{1}$$

Here we assume that $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d}) : \mathbb{R}^n \times U \times D \mapsto \mathbb{R}^n$ is continuous over $\boldsymbol{x}$, $\boldsymbol{u}$ and $\boldsymbol{d}$, and locally Lipschitz over $\boldsymbol{x}$ uniformly for $\boldsymbol{u} \in U$ and $\boldsymbol{d} \in D$. The sets $\mathcal{X}$, U and D are compact subsets of finite dimensional spaces $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^l$ respectively, and the controls $\boldsymbol{u}(\cdot) : [0, \infty) \mapsto U$ and $\boldsymbol{d}(\cdot) : [0, \infty) \mapsto D$ are measurable functions. We define

$$\mathcal{U} = \{\boldsymbol{u}(\cdot) : [0, \infty) \mapsto U, \text{measurable}\} \text{ and}$$
$$\mathcal{D} = \{\boldsymbol{d}(\cdot) : [0, \infty) \mapsto D, \text{measurable}\}$$

as the respective sets of control functions.

Throughout this paper we will investigate differential games in which the ego player wants to control the system to enforce safety specifications while the mutual other player attempts to prevent this. For this reason, we will usually interpret $\boldsymbol{u}(\cdot)$ as a control action while considering $\boldsymbol{d}(\cdot)$ as an adversarial perturbation. The trajectory of system (1) under the control of $\boldsymbol{u}(\cdot) \in \mathcal{U}$ and $\boldsymbol{d}(\cdot) \in \mathcal{D}$ is denoted by $\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\boldsymbol{u},\boldsymbol{d}}(\cdot) : \mathbb{R} \mapsto \mathbb{R}^n$ with $\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\boldsymbol{u},\boldsymbol{d}}(0) = \boldsymbol{x}_0$. The game is investigated in the framework of non-anticipative strategies, whose concepts are formally presented in Definition 1.

Definition 1: We say that a map $\boldsymbol{\alpha}(\cdot) : \mathcal{D} \mapsto \mathcal{U}$ is a non-anticipative strategy (for the ego player) if it satisfies the following condition:

For $\boldsymbol{d}_1(\cdot), \boldsymbol{d}_2(\cdot) \in \mathcal{D}$ and $s \geq 0$ for which $\boldsymbol{d}_1(t) = \boldsymbol{d}_2(t)$ for almost every $t \in [0, s]$, $\boldsymbol{\alpha}(\boldsymbol{d}_1)(t)$ and $\boldsymbol{\alpha}(\boldsymbol{d}_2)(t)$ coincide for almost every $t \in [0, s]$. The set of non-anticipative strategies $\boldsymbol{\alpha}(\cdot)$ for the ego player is denoted by Γ.

Non-anticipative strategies for the mutual other player $\boldsymbol{\beta}(\cdot) : \mathcal{U} \mapsto \mathcal{D}$ are defined similarly. Its set is denoted by Δ.

Based on the non-anticipative strategies in Definition 1, we define two new robust controlled invariant sets, i.e., the lower robust controlled invariant set and the upper robust controlled invariant set.

Definition 2: Let $\mathcal{X} = \{\boldsymbol{x} \in \mathbb{R}^n \mid h(\boldsymbol{x}) \leq 0\}$ and $\mathcal{X}_\epsilon = \{\boldsymbol{x} \in \mathbb{R}^n \mid h(\boldsymbol{x}) \leq \epsilon\}$ be sets in $\mathbb{R}^n$, where $\mathcal{X}$ is compact and $h(\boldsymbol{x})$ is bounded and locally Lipschitz continuous in $\mathbb{R}^n$,

1) The lower robust controlled invariant set $\mathcal{R}^-$ of system (1) is the set of initial states $\boldsymbol{x}$ such that for any $\epsilon > 0$ and any $T \geq 0$, there exists a non-anticipative strategy $\boldsymbol{\alpha}(\cdot) \in \Gamma$ such that for any perturbation $\boldsymbol{d}(\cdot) \in \mathcal{D}$ the corresponding trajectory $\boldsymbol{\phi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t)$ stays inside $\mathcal{X}_\epsilon$ for $t \in [0, T]$, i.e.,

$$\mathcal{R}^- = \left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \begin{array}{l} \forall \epsilon > 0, \forall T \geq 0, \exists \boldsymbol{\alpha}(\cdot) \in \Gamma, \forall \boldsymbol{d}(\cdot) \in \mathcal{D}, \\ \forall t \in [0, T], \boldsymbol{\phi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t) \in \mathcal{X}_\epsilon \end{array} \right\}.$$

2). The upper robust controlled invariant set $\mathcal{R}^+$ of system (1) is the set of initial states $\boldsymbol{x}$ such that for any $T \geq 0$ and any $\epsilon > 0$ and any non-anticipative strategy $\boldsymbol{\beta}(\cdot) \in \Delta$, there exists a control $\boldsymbol{u}(\cdot) \in \mathcal{U}$ such that the trajectory $\boldsymbol{\phi}_{\boldsymbol{x}}^{\boldsymbol{u},\boldsymbol{\beta}(\boldsymbol{u})}(t)$ stays inside $\mathcal{X}_\epsilon$ for $t \in [0, T]$, i.e.,

$$\mathcal{R}^+ = \left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \begin{array}{l} \forall \epsilon > 0, \forall T \geq 0, \forall \boldsymbol{\beta}(\cdot) \in \Delta, \exists \boldsymbol{u}(\cdot) \in \mathcal{U}, \\ \forall t \in [0, T], \boldsymbol{\phi}_{\boldsymbol{x}}^{\boldsymbol{u},\boldsymbol{\beta}(\boldsymbol{u})}(t) \in \mathcal{X}_\epsilon \end{array} \right\}.$$

Note that the assumption on the boundedness of $h(\boldsymbol{x})$ over $\boldsymbol{x} \in \mathbb{R}^n$ is not restrictive since if $h(\boldsymbol{x})$ is unbounded, then $h(\boldsymbol{x}) := \frac{h(\boldsymbol{x})}{1+h^2(\boldsymbol{x})}$ is bounded and the set $\mathcal{X}$ is still equal to $\{\boldsymbol{x} \in \mathbb{R}^n \mid h(\boldsymbol{x}) \leq 0\}$.

The differential game of interest in this paper is on the computation of the sets $\mathcal{R}^-$ and $\mathcal{R}^+$ based on the following assumption.

Assumption 1: Both the sets $\mathcal{R}^-$ and $\mathcal{R}^+$ are not empty and have nonempty interior.

From Definition 2, we have the following inference.

Corollary 1: $\mathcal{R}^- \subseteq \mathcal{X}$ and $\mathcal{R}^+ \subseteq \mathcal{X}$.

Proof: Let $\boldsymbol{x} \in \mathcal{R}^-$ but $\boldsymbol{x} \notin \mathcal{X}$.

Obviously, there exists $\epsilon_1 > 0$ such that $h(\boldsymbol{x}) = \epsilon_1$.

Therefore,

$$\exists \epsilon < \epsilon_1, \exists T = 0, \forall \boldsymbol{\alpha}(\cdot) \in \Gamma, \exists \boldsymbol{d}(\cdot) \in \mathcal{D}, \\ \exists t \in [0, T], \boldsymbol{\phi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t) \notin \mathcal{X}_\epsilon,$$

contradicting $\boldsymbol{x} \in \mathcal{R}^-$. Therefore, $\boldsymbol{x} \in \mathcal{X}$ and thus $\mathcal{R}^- \subseteq \mathcal{X}$.

Analogously, we have $\mathcal{R}^+ \subseteq \mathcal{X}$. ■

III. Characterization of Robust Controlled Invariant Sets

In this section we characterize the lower and upper robust controlled invariant sets using Hamilton-Jacobi equations with sup-inf and inf-sup Hamiltonians respectively.

A. System Reformulation

As $\boldsymbol{f}$ is assumed to be locally Lipschitz continuous in system (1), the existence of a global solution $\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\boldsymbol{u},\boldsymbol{d}}(t)$ over $t \in [0, \infty)$ is not guaranteed for any initial state $\boldsymbol{x}_0 \in \mathbb{R}^n$. However, the existence of global solutions is a prerequisite for constructing Hamilton-Jacobi partial differential equations in the Hamilton-Jacobi reachability framework. As in [39], [38], in this subsection we construct a system, to which the global solution over $t \in [0, \infty)$ starting from any initial state $\boldsymbol{x}_0 \in \mathbb{R}^n$ exists. Also, its solution coincides with the solution to system (1) over a compact set B. The compact set B is chosen to satisfy

$$\mathcal{X} \subseteq B \text{ and } \partial B \cap \partial \mathcal{X} = \emptyset, \tag{2}$$

resulting in that there exists $\epsilon' > 0$ such that

$$\mathcal{X}_\epsilon \subseteq B \text{ and } \partial \mathcal{X}_\epsilon \cap \partial B = \emptyset, \forall \epsilon \in [0, \epsilon']. \tag{3}$$

Such a set B exists since the set $\mathcal{X}$ is a compact set in $\mathbb{R}^n$. The auxiliary system is of the following form:

$$\dot{\boldsymbol{x}}(s) = \boldsymbol{F}(\boldsymbol{x}(s), \boldsymbol{u}(s), \boldsymbol{d}(s)), \tag{4}$$

where $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d}) : \mathbb{R}^n \times U \times D \mapsto \mathbb{R}^n$, which is globally Lipschitz continuous over $\boldsymbol{x} \in \mathbb{R}^n$ uniformly for $\boldsymbol{u} \in U$ and $\boldsymbol{d} \in D$ with Lipschitz constant L_f, i.e.,

$$\|\boldsymbol{F}(\boldsymbol{x}_1, \boldsymbol{u}, \boldsymbol{d}) - \boldsymbol{F}(\boldsymbol{x}_2, \boldsymbol{u}, \boldsymbol{d})\| \leq L_f \|\boldsymbol{x}_1 - \boldsymbol{x}_2\| \tag{5}$$

for $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$, $\boldsymbol{u} \in U$ and $\boldsymbol{d} \in D$, where L_f is the Lipschitz constant of $\boldsymbol{f}$ over B. Moreover, $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d}) = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d})$ over $B \times U \times D$.

The existence of system (4) is guaranteed by Kirszbraun's extension theorem for Lipschitz maps [36]. For instance,

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d}) := \inf_{\boldsymbol{y} \in B} (\boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{d}) + A L_f \|\boldsymbol{x} - \boldsymbol{y}\|) \tag{6}$$

satisfies such requirement, where A is an n-dimensional vector with each component being equal to one. Thus, for any $(\boldsymbol{x}_0, \boldsymbol{u}(\cdot), \boldsymbol{d}(\cdot)) \in B \times \mathcal{U} \times \mathcal{D}$, there exists a unique absolutely continuous trajectory $\boldsymbol{x}(t) = \boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{u},\boldsymbol{d}}(t)$ satisfying (4) a.e. with $\boldsymbol{x}(0) = \boldsymbol{x}_0$ for $t \in [0, \infty)$ [Theorem 5.5, Section III, [5]]. Since $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d}) = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d})$ over $B \times U \times D$, we have the conclusion that given $\boldsymbol{x}_0 \in B$, $\boldsymbol{u}(\cdot) \in \mathcal{U}$, $\boldsymbol{d}(\cdot) \in \mathcal{D}$ and $T \geq 0$, if $\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{u},\boldsymbol{d}}(t) \in B$ for $t \in [0, T]$, $\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\boldsymbol{u},\boldsymbol{d}}(t) \in B$ for $t \in [0, T]$ holds and further $\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\boldsymbol{u},\boldsymbol{d}}(t) = \boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{u},\boldsymbol{d}}(t)$ for $t \in [0, T]$.

Moreover, the sets $\mathcal{R}^-$ and $\mathcal{R}^+$ for system (1) coincide with the corresponding sets for system (4). Since $\mathcal{X}_{\epsilon_1} \subseteq \mathcal{X}_{\epsilon_2}$ for $0 < \epsilon_1 \leq \epsilon_2$,

$$\mathcal{R}^- = \left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \begin{array}{l} \forall \epsilon \in [0, \epsilon'], \forall T \geq 0, \exists \boldsymbol{\alpha}(\cdot) \in \Gamma, \\ \forall \boldsymbol{d}(\cdot) \in \mathcal{D}, \forall t \in [0, T], \boldsymbol{\phi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t) \in \mathcal{X}_\epsilon \end{array} \right\}$$

and

$$\mathcal{R}^+ = \left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \begin{array}{l} \forall \epsilon \in [0, \epsilon'], \forall T \geq 0, \forall \boldsymbol{\beta}(\cdot) \in \Delta, \\ \exists \boldsymbol{u}(\cdot) \in \mathcal{U}, \forall t \in [0, T], \boldsymbol{\phi}_{\boldsymbol{x}}^{\boldsymbol{u},\boldsymbol{\beta}(\boldsymbol{u})}(t) \in \mathcal{X}_\epsilon \end{array} \right\}$$

holds, where ϵ' is defined in (3). Therefore, we have that

$$\mathcal{R}^- = \left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \begin{array}{l} \forall \epsilon > 0, \forall T \geq 0, \exists \boldsymbol{\alpha}(\cdot) \in \Gamma, \forall \boldsymbol{d}(\cdot) \in \mathcal{D}, \\ \forall t \in [0, T], \boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t) \in \mathcal{X}_\epsilon \end{array} \right\}$$

and

$$\mathcal{R}^+ = \left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \begin{array}{l} \forall \epsilon > 0, \forall T \geq 0, \forall \boldsymbol{\beta}(\cdot) \in \Delta, \exists \boldsymbol{u}(\cdot) \in \mathcal{U}, \\ \forall t \in [0, T], \boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{u},\boldsymbol{\beta}(\boldsymbol{u})}(t) \in \mathcal{X}_\epsilon \end{array} \right\}.$$

This indicates that both the lower and upper robust controlled invariant sets can be obtained based on system (4). In the rest we consider system (4) instead of system (1).

B. Hamilton-Jacobi Equations

In order to obtain Hamilton-Jacobi equations for characterizing the sets $\mathcal{R}^-$ and $\mathcal{R}^+$, for any solution $\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{u},\boldsymbol{d}}(\cdot)$ of system (4) with an initial value $\boldsymbol{x}$ we associate a payoff which depends on $(\boldsymbol{u}(\cdot), \boldsymbol{d}(\cdot)) \in \mathcal{U} \times \mathcal{D}$ and is denoted by

$$J(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d}) := \sup_{t \in [0, \infty)} e^{-\gamma t} h(\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{u},\boldsymbol{d}}(t)), \tag{7}$$

where γ is a scalar constant valued in $(0, \infty)$.

For the payoff $J(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d})$, we respectively define the lower value function $V^-(\boldsymbol{x})$ and upper value function $V^+(\boldsymbol{x})$ as follows:

$$V^-(\boldsymbol{x}) := \inf_{\boldsymbol{\alpha}(\cdot) \in \Gamma} \sup_{\boldsymbol{d}(\cdot) \in \mathcal{D}} J(\boldsymbol{x}, \boldsymbol{\alpha}(\boldsymbol{d}), \boldsymbol{d}) \tag{8}$$

and

$$V^+(\boldsymbol{x}) := \sup_{\boldsymbol{\beta}(\cdot) \in \Delta} \inf_{\boldsymbol{u}(\cdot) \in \mathcal{U}} J(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\beta}(\boldsymbol{u})). \tag{9}$$

We will show that the zero level sets of the lower value function $V^-(\boldsymbol{x})$ and the upper value function $V^+(\boldsymbol{x})$ are respectively the lower robust controlled invariant set $\mathcal{R}^-$ and the upper robust controlled invariant set $\mathcal{R}^+$, i.e., $\mathcal{R}^- = \{\boldsymbol{x} \in \mathbb{R}^n \mid V^-(\boldsymbol{x}) = 0\}$ and $\mathcal{R}^+ = \{\boldsymbol{x} \in \mathbb{R}^n \mid V^+(\boldsymbol{x}) = 0\}$. Before justifying this statement, we need an intermediate proposition stating that both the lower value function V^- and the upper value function V^+ are non-negative and bounded over $\mathbb{R}^n$.

Proposition 1: The lower value function $V^-(\boldsymbol{x})$ is non-negative and bounded over $\boldsymbol{x} \in \mathbb{R}^n$. Analogously, the upper value function $V^+(\boldsymbol{x})$ is non-negative and bounded over $\boldsymbol{x} \in \mathbb{R}^n$ as well.

Proof: We just prove the statement pertinent to the lower value function $V^-(\boldsymbol{x})$. The similar proof procedure applies to the upper value function $V^+(\boldsymbol{x})$ as well.

Since $h(\boldsymbol{x})$ is bounded over $\mathbb{R}^n$, we have that

$$\lim_{t\to\infty} e^{-\gamma t} h(\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t)) = 0$$

for $\boldsymbol{\alpha}(\cdot)\in\Gamma$, $\boldsymbol{d}(\cdot)\in\mathcal{D}$ and $\boldsymbol{x}\in\mathbb{R}^n$. Since $J(\boldsymbol{x},\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}) \geq \lim_{t\to\infty} e^{-\gamma t} h(\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t))$ for $\boldsymbol{\alpha}(\cdot)\in\Gamma$, $\boldsymbol{d}(\cdot)\in\mathcal{D}$ and $\boldsymbol{x}\in\mathbb{R}^n$, this implies that

$$J(\boldsymbol{x},\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}) \geq 0, \forall \boldsymbol{\alpha}(\cdot)\in\Gamma, \forall \boldsymbol{d}(\cdot)\in\mathcal{D}, \forall \boldsymbol{x}\in\mathbb{R}^n.$$

Thus,

$$\sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} J(\boldsymbol{x},\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}) \geq 0, \forall \boldsymbol{\alpha}(\cdot)\in\Gamma, \forall \boldsymbol{x}\in\mathbb{R}^n.$$

Consequently, $V^-(\boldsymbol{x})\geq 0$ for $\boldsymbol{x}\in\mathbb{R}^n$.

The boundedness of V^- is guaranteed by the fact that

$$J(\boldsymbol{x},\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}) \leq M, \forall \boldsymbol{\alpha}(\cdot)\in\Gamma, \forall \boldsymbol{d}(\cdot)\in\mathcal{D}, \forall \boldsymbol{x}\in\mathbb{R}^n,$$

where M is a positive value such that $|h(\boldsymbol{x})|\leq M$ over $\boldsymbol{x}\in\mathbb{R}^n$. Thus, $V^-(\boldsymbol{x})\leq M$ over $\boldsymbol{x}\in\mathbb{R}^n$. ■

Lemma 1: $\mathcal{R}^- = \{\boldsymbol{x} \mid V^-(\boldsymbol{x})=0\}$ and $\mathcal{R}^+ = \{\boldsymbol{x} \mid V^+(\boldsymbol{x})=0\}$.

Proof: 1. For the statement $\mathcal{R}^- = \{\boldsymbol{x} \mid V^-(\boldsymbol{x})=0\}$, we first prove $\mathcal{R}^- \subseteq \{\boldsymbol{x} \mid V^-(\boldsymbol{x})=0\}$.

Assume that $\boldsymbol{x}\in\mathcal{R}^-$ and ϵ is an arbitrary positive number. Since $h(\boldsymbol{x})$ is bounded over $\boldsymbol{x}\in\mathbb{R}^n$, there exists $M>0$ such that $h(\boldsymbol{x})\leq M$ over $\boldsymbol{x}\in\mathbb{R}^n$. Also, since $\lim_{t\to\infty} Me^{-\gamma t}=0$, there exists $T>0$ such that $Me^{-\gamma t}\leq\epsilon$ for $t\geq T$.

Since $\boldsymbol{x}\in\mathcal{R}^-$, there exists $\boldsymbol{\alpha}_{\epsilon,T}(\cdot)\in\Gamma$ such that

$$\sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} \sup_{t\in[0,T]} h(\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}_{\epsilon,T}(\boldsymbol{d}),\boldsymbol{d}}(t)) \leq \epsilon$$

and thus

$$\begin{aligned}
&\sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} \sup_{t\in[0,\infty)} e^{-\gamma t} h(\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}_{\epsilon,T}(\boldsymbol{d}),\boldsymbol{d}}(t))\\
&= \sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} \max\{\sup_{t\in[0,T]} e^{-\gamma t} h(\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}_{\epsilon,T}(\boldsymbol{d}),\boldsymbol{d}}(t)),\\
&\qquad\qquad \sup_{t\in[T,\infty)} e^{-\gamma t} h(\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}_{\epsilon,T}(\boldsymbol{d}),\boldsymbol{d}}(t))\}\\
&\leq \max\{\sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} \sup_{t\in[0,T]} e^{-\gamma t} h(\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}_{\epsilon,T}(\boldsymbol{d}),\boldsymbol{d}}(t)),\\
&\qquad \sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} \sup_{t\in[T,\infty)} e^{-\gamma t} h(\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}_{\epsilon,T}(\boldsymbol{d}),\boldsymbol{d}}(t))\}\\
&\leq \epsilon.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
V^-(\boldsymbol{x}) &= \inf_{\boldsymbol{\alpha}(\cdot)\in\Gamma} \sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} J(\boldsymbol{x},\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d})\\
&\leq \sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} J(\boldsymbol{x},\boldsymbol{\alpha}_{\epsilon,T}(\boldsymbol{d}),\boldsymbol{d}) \leq \epsilon.
\end{aligned}$$

Since ϵ is an arbitrary positive number, $V^-(\boldsymbol{x})\leq 0$. In addition, according to Proposition 1 which states that $V^-(\boldsymbol{x})\geq 0$ over $\mathbb{R}^n$, we conclude that $\mathcal{R}^-\subseteq\{\boldsymbol{x}\in\mathbb{R}^n \mid V^-(\boldsymbol{x})=0\}$.

Next, we prove that $\{\boldsymbol{x}\in\mathbb{R}^n \mid V^-(\boldsymbol{x})=0\}\subseteq\mathcal{R}^-$.

Assume that $\boldsymbol{x}_0\in\{\boldsymbol{x}\in\mathbb{R}^n \mid V^-(\boldsymbol{x})=0\}$ but $\boldsymbol{x}_0\notin\mathcal{R}^-$. Therefore, we have

$$\begin{aligned}
&\exists\epsilon>0, \exists T\geq 0, \forall\boldsymbol{\alpha}(\cdot)\in\Gamma, \exists\boldsymbol{d}(\cdot)\in\mathcal{D},\\
&\qquad\qquad \exists t\in[0,T], \boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t)\notin\mathcal{X}_\epsilon.
\end{aligned}$$

Therefore, $\sup_{t\in[0,\infty)} e^{-\gamma t} h(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t)) \geq e^{-\gamma T}\epsilon$ for $\boldsymbol{\alpha}(\cdot)\in\Gamma$ and consequently

$$\inf_{\boldsymbol{\alpha}(\cdot)\in\Gamma} \sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} \sup_{t\in[0,\infty)} e^{-\gamma t} h(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t)) \geq e^{-\gamma T}\epsilon,$$

contradicting $V^-(\boldsymbol{x}_0)=0$. Thus, $\boldsymbol{x}_0\in\mathcal{R}^-$ and further $\{\boldsymbol{x} \mid V^-(\boldsymbol{x}_0)=0\}\subseteq\mathcal{R}^-$.

In summary, we have $\mathcal{R}^-=\{\boldsymbol{x}\in\mathbb{R}^n \mid V^-(\boldsymbol{x})=0\}$.

2. We prove that $\mathcal{R}^+=\{\boldsymbol{x} \mid V^+(\boldsymbol{x})=0\}$, and first prove that $\mathcal{R}^+\subseteq\{\boldsymbol{x} \mid V^+(\boldsymbol{x})=0\}$.

Let $\boldsymbol{x}\in\mathcal{R}^+$ and $V^+(\boldsymbol{x})=\delta>0$. We will derive a contradiction. Due to $V^+(\boldsymbol{x})=\delta>0$, there exists $\boldsymbol{\beta}_1(\cdot)\in\Delta$ such that $\inf_{\boldsymbol{u}(\cdot)\in\mathcal{U}} J(\boldsymbol{x},\boldsymbol{u},\boldsymbol{\beta}_1(\boldsymbol{u}))>\frac{\delta}{2}$, implying that $J(\boldsymbol{x},\boldsymbol{u},\boldsymbol{\beta}_1(\boldsymbol{u}))>\frac{\delta}{2}$ for all $\boldsymbol{u}(\cdot)\in\mathcal{U}$. Due to the fact that there exists $T'>0$ such that

$$e^{-\gamma t} h(\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{u},\boldsymbol{\beta}(\boldsymbol{u})}(t)) \leq e^{-\gamma T'} M \leq \frac{\delta}{2},$$

for $t\geq T', \boldsymbol{u}(\cdot)\in\mathcal{U}$ and $\boldsymbol{\beta}(\cdot)\in\Delta$, where M is a positive value such that $|h(\boldsymbol{x})|\leq M$ over $\boldsymbol{x}\in\mathbb{R}^n$, there exists $T_{\boldsymbol{u}}\in[0,T']$ for $\boldsymbol{u}(\cdot)\in\mathcal{U}$ such that

$$e^{-\gamma T_{\boldsymbol{u}}} h(\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{u},\boldsymbol{\beta}_1(\boldsymbol{u})}(T_{\boldsymbol{u}})) > \frac{\delta}{2}$$

and therefore, $\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{u},\boldsymbol{\beta}_1(\boldsymbol{u})}(T_{\boldsymbol{u}})\notin\mathcal{X}_{\frac{\delta}{2}}, \forall\boldsymbol{u}(\cdot)\in\mathcal{U}$, contradicting $\boldsymbol{x}\in\mathcal{R}^+$. Thus, $\mathcal{R}^+\subseteq\{\boldsymbol{x} \mid V^+(\boldsymbol{x})\leq 0\}$ holds. In addition, according to Proposition 1 which states that $V^+(\boldsymbol{x})\geq 0$ for $\boldsymbol{x}\in\mathbb{R}^n$, we have $\mathcal{R}^+\subseteq\{\boldsymbol{x}\in\mathbb{R}^n \mid V^+(\boldsymbol{x})=0\}$.

Next, we show that $\{\boldsymbol{x}\in\mathbb{R}^n \mid V^+(\boldsymbol{x})=0\}\subseteq\mathcal{R}^+$. Let $V^+(\boldsymbol{x})=0$ but $\boldsymbol{x}\notin\mathcal{R}^+$. According to the concept of $\mathcal{R}^+$ in Definition 2, we have that

$$\begin{aligned}
&\exists\epsilon>0, \exists T\geq 0, \exists\boldsymbol{\beta}(\cdot)\in\Delta, \forall\boldsymbol{u}(\cdot)\in\mathcal{U},\\
&\qquad\qquad \exists t\in[0,T], h(\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{u},\boldsymbol{\beta}(\boldsymbol{u})}(t))>\epsilon.
\end{aligned}$$

Therefore, $\sup_{\boldsymbol{\beta}(\cdot)\in\Delta}\inf_{\boldsymbol{u}(\cdot)\in\mathcal{U}} J(\boldsymbol{x},\boldsymbol{u},\boldsymbol{\beta}(\boldsymbol{u})) \geq e^{-\gamma T}\epsilon$, which contradicts $V^+(\boldsymbol{x})=0$. Therefore, we conclude that $\{\boldsymbol{x}\in\mathbb{R}^n \mid V^+(\boldsymbol{x})=0\}\subseteq\mathcal{R}^+$.

In summary, $\{\boldsymbol{x}\in\mathbb{R}^n \mid V^+(\boldsymbol{x})=0\}=\mathcal{R}^+$. ■

From Lemma 1, if the lower value function $V^-(\boldsymbol{x})$ and the upper value function $V^+(\boldsymbol{x})$ are computed, the sets $\mathcal{R}^-$ and $\mathcal{R}^+$ can be obtained. We in the sequel show that the lower value function $V^-(\boldsymbol{x}):\mathbb{R}^n\to\mathbb{R}$ and the upper value function $V^+(\boldsymbol{x}):\mathbb{R}^n\to\mathbb{R}$ are respectively the unique continuous and bounded viscosity solution to Eq. (10) and Eq. (11):

$$\min\Big\{\gamma V^-(\boldsymbol{x}) - H^-(\boldsymbol{x},\frac{\partial V^-(\boldsymbol{x})}{\partial\boldsymbol{x}}), V^-(\boldsymbol{x})-h(\boldsymbol{x})\Big\}=0 \tag{10}$$

and

$$\min\Big\{\gamma V^+(\boldsymbol{x}) - H^+(\boldsymbol{x},\frac{\partial V^+(\boldsymbol{x})}{\partial\boldsymbol{x}}), V^+(\boldsymbol{x})-h(\boldsymbol{x})\Big\}=0, \tag{11}$$

for $\boldsymbol{x}\in\mathbb{R}^n$, where $H^-(\boldsymbol{x},\boldsymbol{p})=\sup_{\boldsymbol{d}\in D}\inf_{\boldsymbol{u}\in U}\boldsymbol{p}\cdot\boldsymbol{F}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{d})$ and $H^+(\boldsymbol{x},\boldsymbol{p})=\inf_{\boldsymbol{u}\in U}\sup_{\boldsymbol{d}\in D}\boldsymbol{p}\cdot\boldsymbol{F}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{d})$ are the sup-inf and inf-sup Hamiltonians respectively. The concept of viscosity solutions to Eq. (10) and Eq. (11) is presented in Definition 3.

Definition 3: [5] A locally bounded and continuous function $V(\boldsymbol{x})$ on $\mathbb{R}^n$ is a viscosity solution of Eq. (10) (Eq. (11)), if for any test function $v(\boldsymbol{x}) \in C^\infty(\mathbb{R}^n)$ such that $V(\boldsymbol{x}) - v(\boldsymbol{x})$ attains a local minimum at $\boldsymbol{x}_0 \in \mathbb{R}^n$,

$$\begin{aligned} &\min\{\gamma V(\boldsymbol{x}_0) - H^-(\boldsymbol{x}_0, \boldsymbol{p}), V(\boldsymbol{x}_0) - h(\boldsymbol{x}_0)\} \geq 0 \\ &(\min\{\gamma V(\boldsymbol{x}_0) - H^+(\boldsymbol{x}_0, \boldsymbol{p}), V(\boldsymbol{x}_0) - h(\boldsymbol{x}_0)\} \geq 0) \end{aligned} \tag{12}$$

holds, i.e., $V(\boldsymbol{x})$ is a viscosity supersolution; 2) for any test function $v(\boldsymbol{x}) \in C^\infty(\mathbb{R}^n)$ such that $V(\boldsymbol{x}) - v(\boldsymbol{x})$ attains a local maximum at $\boldsymbol{x}_0 \in \mathbb{R}^n$,

$$\begin{aligned} &\min\{\gamma V(\boldsymbol{x}_0) - H^-(\boldsymbol{x}_0, \boldsymbol{p}), V(\boldsymbol{x}_0) - h(\boldsymbol{x}_0)\} \leq 0 \\ &(\min\{\gamma V(\boldsymbol{x}_0) - H^+(\boldsymbol{x}_0, \boldsymbol{p}), V(\boldsymbol{x}_0) - h(\boldsymbol{x}_0)\} \leq 0) \end{aligned} \tag{13}$$

holds, i.e., $V(\boldsymbol{x})$ is a viscosity subsolution, where $\boldsymbol{p} = \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}}|_{\boldsymbol{x}=\boldsymbol{x}_0}$.

Firstly, we show that both $V^-(\boldsymbol{x})$ and $V^+(\boldsymbol{x})$ are uniformly continuous.

Lemma 2: Both the lower value function $V^-(\boldsymbol{x})$ and the upper value function $V^+(\boldsymbol{x})$ are uniformly continuous over $\mathbb{R}^n$.

Proof: We just prove the statement related to the lower value function $V^-(\boldsymbol{x})$. The one for the upper value function $V^+(\boldsymbol{x})$ can be justified following the same procedure.

$$\begin{aligned} &|V(\boldsymbol{x}_1) - V(\boldsymbol{x}_2)| \\ &\leq |\inf_{\boldsymbol{\alpha}(\cdot)\in\Gamma} \sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} J(\boldsymbol{x}_1, \boldsymbol{\alpha}(\boldsymbol{d}), \boldsymbol{d}) \\ &\qquad - \inf_{\boldsymbol{\alpha}(\cdot)\in\Gamma} \sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} J(\boldsymbol{x}_2, \boldsymbol{\alpha}(\boldsymbol{d}), \boldsymbol{d})| \\ &\leq \sup_{\boldsymbol{\alpha}(\cdot)\in\Gamma} \sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} |J(\boldsymbol{x}_1, \boldsymbol{\alpha}(\boldsymbol{d}), \boldsymbol{d}) - J(\boldsymbol{x}_2, \boldsymbol{\alpha}(\boldsymbol{d}), \boldsymbol{d})| \\ &\leq \sup_{\boldsymbol{\alpha}(\cdot)\in\Gamma} \sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} \sup_{t\in[0,\infty)} e^{-\gamma t}|h(\boldsymbol{\psi}_{\boldsymbol{x}_2}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t)) - h(\boldsymbol{\psi}_{\boldsymbol{x}_1}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t))|. \end{aligned}$$

Since $h(\boldsymbol{x})$ is bounded over $\boldsymbol{x} \in \mathbb{R}^n$, for arbitrary $\epsilon > 0$, there exists $T_\epsilon \geq 0$ such that

$$\begin{aligned} &\sup_{\boldsymbol{\alpha}(\cdot)\in\Gamma} \sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} \sup_{t\in[T_\epsilon,\infty)} e^{-\gamma t}|h(\boldsymbol{\psi}_{\boldsymbol{x}_2}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t)) - h(\boldsymbol{\psi}_{\boldsymbol{x}_1}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t))| \\ &\leq \epsilon. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sup_{\boldsymbol{\alpha}(\cdot)\in\Gamma} \sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} \sup_{t\in[0,\infty)} e^{-\gamma t}|h(\boldsymbol{\psi}_{\boldsymbol{x}_2}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t)) - h(\boldsymbol{\psi}_{\boldsymbol{x}_1}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t))| \\ &\leq \max\{ \\ &\quad \sup_{\boldsymbol{\alpha}(\cdot)\in\Gamma} \sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} \sup_{t\in[0,T_\epsilon)} e^{-\gamma t}|h(\boldsymbol{\psi}_{\boldsymbol{x}_2}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t)) - h(\boldsymbol{\psi}_{\boldsymbol{x}_1}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t))|, \\ &\quad \sup_{\boldsymbol{\alpha}(\cdot)\in\Gamma} \sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} \sup_{t\in[T_\epsilon,\infty)} e^{-\gamma t}|h(\boldsymbol{\psi}_{\boldsymbol{x}_2}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t)) - h(\boldsymbol{\psi}_{\boldsymbol{x}_1}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t))| \\ &\} \\ &\leq \max\{ \\ &\quad \sup_{\boldsymbol{\alpha}(\cdot)\in\Gamma} \sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} \sup_{t\in[0,T_\epsilon)} e^{-\gamma t}|h(\boldsymbol{\psi}_{\boldsymbol{x}_2}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t)) - h(\boldsymbol{\psi}_{\boldsymbol{x}_1}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t))|, \\ &\quad \epsilon \\ &\} \\ &\leq \max\{L_h e^{L_f T_\epsilon}\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|, \epsilon\}, \end{aligned}$$

where L_h ad L_f are the Lipschitz constants of h and $\boldsymbol{F}$ over the bounded set $\Omega(B_1) = \{\boldsymbol{x} \mid \boldsymbol{x} = \boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t), t \in [0, T_\epsilon], \boldsymbol{x}_0 \in B_1, \boldsymbol{\alpha}(\cdot) \in \Gamma, \boldsymbol{d}(\cdot) \in \mathcal{D}\}$ with B_1 being a compact set in $\mathbb{R}^n$ covering $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ respectively. The boundedness of the set $\Omega(B_1)$ can be obtained based on the fact that $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d}) : \mathbb{R}^n \times U \times D \mapsto \mathbb{R}^n$, which is globally Lipschitz continuous over $\boldsymbol{x} \in \mathbb{R}^n$ uniformly for $\boldsymbol{u} \in U$ and $\boldsymbol{d} \in D$ with Lipschitz constant L_f, and

$$\begin{aligned} &\|\boldsymbol{\psi}_{\boldsymbol{x}_1}^{\boldsymbol{\alpha_1}(\boldsymbol{d}_1),\boldsymbol{d}_1}(t) - \boldsymbol{\psi}_{\boldsymbol{x}_2}^{\boldsymbol{\alpha_2}(\boldsymbol{d}_2),\boldsymbol{d}_2}(t)\| \\ &\leq \|\boldsymbol{x}_1 - \boldsymbol{x}_2\| + L_f \int_0^t \|\boldsymbol{\psi}_{\boldsymbol{x}_1}^{\boldsymbol{\alpha_1}(\boldsymbol{d}_1),\boldsymbol{d}_1}(s) - \boldsymbol{\psi}_{\boldsymbol{x}_2}^{\boldsymbol{\alpha_2}(\boldsymbol{d}_2),\boldsymbol{d}_2}(s)\| ds \\ &\leq e^{L_f T_\epsilon}\|\boldsymbol{x}_1 - \boldsymbol{x}_2\| \end{aligned}$$

for $t \in [0, T_\epsilon]$, $\boldsymbol{\alpha_1}(\cdot) \in \Gamma, \boldsymbol{\alpha_2}(\cdot) \in \Gamma, \boldsymbol{d}_1(\cdot) \in \mathcal{D}$ and $\boldsymbol{d}_2(\cdot) \in \mathcal{D}$.

Since there exists $\delta > 0$ satisfying $\|\boldsymbol{x}_1 - \boldsymbol{x}_2\| \leq \delta$ such that $\sup_{\boldsymbol{\alpha}(\cdot)\in\Gamma} \sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} \sup_{t\in[0,T_\epsilon)} e^{-\gamma t}|h(\boldsymbol{\psi}_{\boldsymbol{x}_2}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t)) - h(\boldsymbol{\psi}_{\boldsymbol{x}_1}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t))| \leq \epsilon$, there exists $\delta > 0$ satisfying $\|\boldsymbol{x}_1 - \boldsymbol{x}_2\| \leq \delta$ such that $|V(\boldsymbol{x}_1) - V(\boldsymbol{x}_2)| \leq 2\epsilon$.

Due to the arbitrariness of ϵ, we shows the uniform continuity of $V^-(\boldsymbol{x})$. ■

The following theorem states that the lower value function $V^-(\boldsymbol{x})$ and the upper value function $V^+(\boldsymbol{x})$ are the viscosity solution to Eq. (10) and Eq. (11) respectively.

Theorem 1: The lower value function $V^-(\boldsymbol{x})$ and the upper value function $V^+(\boldsymbol{x})$ are respectively the viscosity solution to Eq. (10) and Eq. (11).

Proof: The proof follows the one of Theorem 1.10 in Section VIII in [5] with some modifications. The details can be found in Appendix. ■

Further, we show the uniqueness of viscosity solutions to Eq. (10) and Eq. (11).

Theorem 2: The lower value function $V^-(\boldsymbol{x})$ and the upper value function $V^+(\boldsymbol{x})$ are respectively the unique viscosity solution to Eq. (10) and Eq. (11).

Proof: The proof follows the one of Theorem 2.12 in Section III in [5] with some modifications. It is based on proving a comparison principle: If $V_1(\boldsymbol{x})$ and $V_2(\boldsymbol{x})$ are bounded continuous functions over $\boldsymbol{x} \in \mathbb{R}^n$, and they are respectively a viscosity sub and supersolution to Eq. (10) (Eq. (11)), then $V_1(\boldsymbol{x}) \leq V_2(\boldsymbol{x})$ in $\mathbb{R}^n$. Obviously, if such comparison principle holds, the uniqueness of bounded continuous solutions to Eq. (10) (Eq. (11)) is guaranteed.

The details can be found in Appendix. ■

We will show that $V^-(\boldsymbol{x}) = V^+(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathbb{R}^n$ based on Eq. (10) and Eq. (11) under the Isaacs condition.

Theorem 3: $V^-(\boldsymbol{x}) \leq V^+(\boldsymbol{x})$ holds for $\boldsymbol{x} \in \mathbb{R}^n$ and consequently $\mathcal{R}^+ \subseteq \mathcal{R}^-$. Moreover, if $H^-(\boldsymbol{x}, \boldsymbol{p}) = H^+(\boldsymbol{x}, \boldsymbol{p})$ for $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{p} \in \mathbb{R}^n$, then $V^-(\boldsymbol{x}) = V^+(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathbb{R}^n$ and thus $\mathcal{R}^- = \mathcal{R}^+$.

Proof: Since the upper value function $V^+(\boldsymbol{x})$ is a supersolution to Eq. (11), we have that for $v(\boldsymbol{x}) \in C^\infty(\mathbb{R}^n)$ such that $V^-(\boldsymbol{x}) - v(\boldsymbol{x})$ attains a local minimum at $\boldsymbol{x}_0 \in \mathbb{R}^n$,

$$\min\{\gamma V^+(\boldsymbol{x}_0) - H^+(\boldsymbol{x}_0, \boldsymbol{p}), V^+(\boldsymbol{x}_0) - h(\boldsymbol{x}_0)\} \geq 0.$$

Since $H^-(\boldsymbol{x},\boldsymbol{p}) \leq H^+(\boldsymbol{x},\boldsymbol{p})$ for $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{p} \in \mathbb{R}^n$, we have that

$$\min\{\gamma V^+(\boldsymbol{x}_0) - H^-(\boldsymbol{x}_0,\boldsymbol{p}), V^+(\boldsymbol{x}_0) - h(\boldsymbol{x}_0)\} \geq 0,$$

where $\boldsymbol{p} = \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}}\mid_{\boldsymbol{x}=\boldsymbol{x}_0}$. Therefore, the upper value function $V^+(\boldsymbol{x})$ is also a supersolution to Eq. (10). According to the comparison statement in the proof of Theorem 2, $V^-(\boldsymbol{x}) \leq V^+(\boldsymbol{x})$ holds for $\boldsymbol{x} \in \mathbb{R}^n$.

Also, since $V^-(\boldsymbol{x}) \geq 0$ over $\mathbb{R}^n$ according to Proposition 1, $\{\boldsymbol{x} \in \mathbb{R}^n \mid V^+(\boldsymbol{x}) = 0\} \subseteq \{\boldsymbol{x} \in \mathbb{R}^n \mid V^-(\boldsymbol{x}) = 0\}$ holds. Furthermore, from Lemma 1, we have $\mathcal{R}^+ \subseteq \mathcal{R}^-$.

Obviously, the fact that $H^-(\boldsymbol{x},\boldsymbol{p}) = H^+(\boldsymbol{x},\boldsymbol{p})$ for $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{p} \in \mathbb{R}^n$ implies that $V^-(\boldsymbol{x}) = V^+(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathbb{R}^n$ and thus $\{\boldsymbol{x} \mid V^-(\boldsymbol{x}) = 0\} = \{\boldsymbol{x} \mid V^+(\boldsymbol{x}) = 0\}$. According to Lemma 1, we have $\mathcal{R}^- = \mathcal{R}^+$. ∎

Remark 1: Since $\boldsymbol{F}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{d}) : \mathbb{R}^n \times U \times D \mapsto \mathbb{R}^n$ is continuous over $\boldsymbol{x}$, $\boldsymbol{u}$ and $\boldsymbol{d}$, according to Theorem 2.3 in Chapter VIII in [5], if U and D are convex spaces, the sets $\{\boldsymbol{u} \in U \mid H(\boldsymbol{u},\overline{\boldsymbol{d}}) \geq t\}$ and $\{\boldsymbol{d} \in D \mid H(\overline{\boldsymbol{u}},\boldsymbol{d}) \geq t\}$ are convex for all $t \in \mathbb{R}$, $\overline{\boldsymbol{u}} \in U$, $\overline{\boldsymbol{d}} \in D$, where $H(\boldsymbol{u},\boldsymbol{d}) = \boldsymbol{p}\cdot\boldsymbol{F}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{d})$. Then $H^-(\boldsymbol{x},\boldsymbol{p}) = H^+(\boldsymbol{x},\boldsymbol{p})$.

The simplest system for Theorem 2.3 in Chapter VIII in [5] to hold is the one being affine in the control variables $\boldsymbol{u} \in U$ and $\boldsymbol{d} \in D$, that is, $\boldsymbol{F}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{d}) = \boldsymbol{F}_1(\boldsymbol{x}) + \boldsymbol{F}_2(\boldsymbol{x})\boldsymbol{u} + \boldsymbol{F}_3(\boldsymbol{x})\boldsymbol{d}$, where U and D are convex compact sets in $\mathbb{R}^m$ and $\mathbb{R}^l$ respectively. This implies that $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{d}) = \boldsymbol{F}_1(\boldsymbol{x}) + \boldsymbol{F}_2(\boldsymbol{x})\boldsymbol{u} + \boldsymbol{F}_3(\boldsymbol{x})\boldsymbol{d}$ for $\boldsymbol{x} \in B$.

IV. Examples

In this section we illustrate our approach on two examples. For numerical implementation of solving Eq. (10) and Eq. (11), we use the ROC-HJ solver [10][1].

Example 1: We consider a two-dimensional system of the following form adopted from [39]:

$$\begin{aligned} \dot{x} &= -0.5x, \\ \dot{y} &= 10x^2 - (u-d)^2 y, \end{aligned} \tag{14}$$

where $U = [-0.2, 0.2]$, $D = [-0.2, 0.2]$ and $\mathcal{X} = \{(x,y)^\top \mid h(x,y) \leq 0\}$ with $h(x,y) = \frac{x^2+y^2-1}{1+(x^2+y^2-1)^2}$.

Let $B = [-1.1, 1.1] \times [-1.1, 1.1]$. Obviously, the set B satisfies (2). We can construct an auxiliary system $\dot{\boldsymbol{x}}(s) = \boldsymbol{F}(\boldsymbol{x}(s),\boldsymbol{u}(s),\boldsymbol{d}(s))$ of the form (4) such that $\boldsymbol{F}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{d}) = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{d})$ for $(\boldsymbol{x},\boldsymbol{u},\boldsymbol{d}) \in B \times U \times D$. According to Lemma 1 and Theorem 1, the lower and upper robust controlled invariant sets, i.e., $\mathcal{R}^-$ and $\mathcal{R}^+$, can be computed by solving Eq. (10) and Eq. (11) respectively. We perform numerical computations on the set B and use uniform grids of 4×10^4 to solve Eq. (10) and Eq. (11).

For this example the Isaacs condition, i.e., $H^-(\boldsymbol{x},\boldsymbol{p}) = H^+(\boldsymbol{x},\boldsymbol{p})$ for $\boldsymbol{p} \in \mathbb{R}^n$ and $\boldsymbol{x} \in \mathbb{R}^n$, does not hold. Indeed, e.g., for $\boldsymbol{p} = (1,1)$ and $\boldsymbol{x} \in B$, we get $\boldsymbol{p}\cdot\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{d}) = -0.5x + 10x^2 + (u-d)^2 y$, implying

$$\sup_{\boldsymbol{d}\in D}\inf_{\boldsymbol{u}\in U} \boldsymbol{p}\cdot\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{d}) = \begin{cases} -0.5x + 10x^2, & \text{if } y \leq 0 \wedge \boldsymbol{x} \in B \\ -0.5x + 10x^2 - 0.04y, & \text{if } y > 0 \wedge \boldsymbol{x} \in B \end{cases},$$

but

$$\inf_{\boldsymbol{u}\in U}\sup_{\boldsymbol{d}\in D} \boldsymbol{p}\cdot\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{d}) = \begin{cases} -0.5x + 10x^2 - 0.04y, & \text{if } y \leq 0 \wedge \boldsymbol{x} \in B \\ -0.5x + 10x^2, & \text{if } y > 0 \wedge \boldsymbol{x} \in B \end{cases}.$$

Thus, we cannot guarantee that $\mathcal{R}^- = \mathcal{R}^+$.

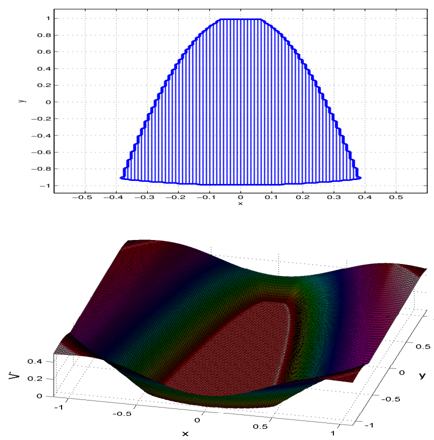

Fig. 1. Above: The blue region denotes the computed lower robust controlled invariant set $\mathcal{R}^-$ for Example 1. Below: An illustration of level sets of the computed lower value function $V^-(\boldsymbol{x})$.

The level sets of computed functions $V^-(\boldsymbol{x})$ and $V^+(\boldsymbol{x})$ are illustrated in Fig. 1 and 2 respectively. The level sets displayed in Fig. 1 and 2 further confirm that the lower value function $V^-(\boldsymbol{x})$ and the upper value function $V^+(\boldsymbol{x})$ are non-negative, as stated in Proposition 1.

The computed lower robust controlled invariant set $\mathcal{R}^- = \{\boldsymbol{x} \mid V^-(\boldsymbol{x}) = 0\}$ and the upper robust controlled invariant set $\mathcal{R}^+ = \{\boldsymbol{x} \mid V^+(\boldsymbol{x}) = 0\}$ are respectively illustrated in Fig. 1 and 2. The comparison of them is illustrated in Fig. 3. The visualized result in Fig. 3 indicates that $\mathcal{R}^+ \subseteq \mathcal{R}^-$, as claimed in Theorem 3.

Example 2: **Moore-Greitzer jet engine model.** We test our approach on the following system coming from [31], corresponding to a Moore-Greitzer jet engine model:

$$\begin{aligned} \dot{x} &= -y - \frac{3}{2}x^2 - \frac{1}{2}x^3 + d, \\ \dot{y} &= (0.8076 + u)x - 0.9424y, \end{aligned}$$

where $\mathcal{X} = \{\boldsymbol{x} \mid h(\boldsymbol{x}) \leq 0\}$ with $h(\boldsymbol{x}) = \frac{x^2+y^2-0.25}{1+(x^2+y^2-0.25)^2}$, $d \in [-0.02, 0.02]$ and $u \in [-0.01, 0.01]$.

From [31], we know that $u(\boldsymbol{x}) = 0.8076x - 0.9424y$ is a controller that guarantees the existence of a robust invariant

[1]https://uma.ensta-paristech.fr/soft/ROC-HJ/

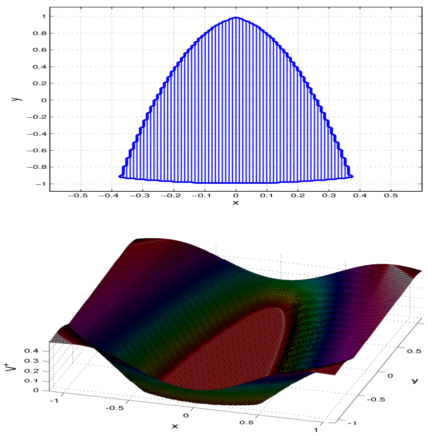

Fig. 2. Above: The blue region denotes the computed upper robust controlled invariant set $\mathcal{R}^+$ for Example 1. Below: An illustration of level sets of the computed upper value function $V^+(\boldsymbol{x})$.

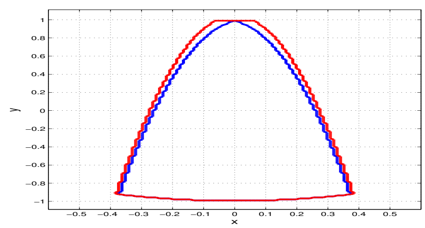

Fig. 3. An illustration of comparison of the computed lower robust controlled invariant set $\mathcal{R}^-$ and upper robust controlled invariant set $\mathcal{R}^+$ for Example 1. Red and blue curves denote the boundaries of computed sets $\mathcal{R}^-$ and $\mathcal{R}^+$, respectively.

set of the following system

$$\dot{x} = -y - \frac{3}{2}x^2 - \frac{1}{2}x^3 + d,$$
$$\dot{y} = u$$

where $d \in [-0.02, 0.02]$. In our example we change the coefficient 0.8076 of the variable x in $u(\boldsymbol{x})$ to $0.8076 + u$ with $u \in [-0.01, 0.01]$.

Let $B = [-0.51, 0.51] \times [-0.51, 0.51]$. Obviously, the set B satisfies (2). We can construct an auxiliary system (4) with $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d})$ of the form (6).

Since $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d})$ can be written as $\boldsymbol{f}_1(\boldsymbol{x}) + \boldsymbol{f}_2(\boldsymbol{x})\boldsymbol{u} + \boldsymbol{f}_3(\boldsymbol{x})\boldsymbol{d}$, and U and D are convex, we obtain that $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d})$ satisfies Theorem 2.3 in Chapter VIII of [5], as illustrated in Remark 1. Furthermore, according to Remark 1 and Theorem 3, we have $V^-(\boldsymbol{x}) = V^+(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathbb{R}^n$ and thus $\mathcal{R}^- = \mathcal{R}^+$. Thus, the lower and upper robust controlled invariant sets, i.e., $\mathcal{R}^-$ and $\mathcal{R}^+$, can be computed by solving either Eq. (10) or Eq. (11).

We perform numerical computations on the set B and use uniform grids of 4×10^4 to solve Eq. (10). The computed sets $\mathcal{R}^- = \{\boldsymbol{x} \mid V^-(\boldsymbol{x}) = 0\}$ and $\mathcal{R}^+ = \{\boldsymbol{x} \mid V^+(\boldsymbol{x}) = 0\}$ are illustrated in Fig. 4, which also illustrates the level sets of functions $V^-(\boldsymbol{x})$ and $V^+(\boldsymbol{x})$. The level sets displayed in Fig. 4 further confirm that both functions $V^-(\boldsymbol{x})$ and $V^+(\boldsymbol{x})$

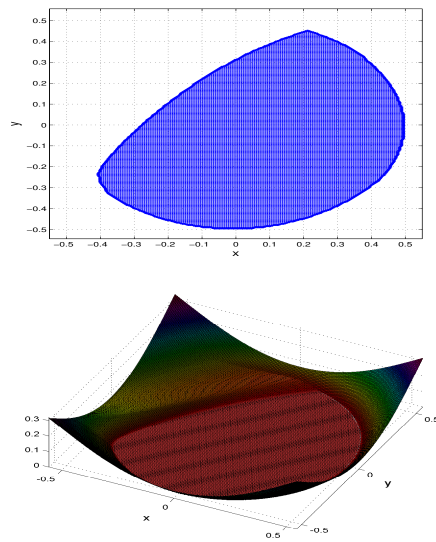

Fig. 4. Above: The blue region denotes the computed lower robust controlled invariant set $\mathcal{R}^-$ and upper robust controlled invariant set $\mathcal{R}^+$. Below: An illustration of level sets of the computed upper value function $V^-(\boldsymbol{x})$ and upper value function $V^+(\boldsymbol{x})$ for Example 2.

are non-negative, as stated in Proposition 1.

V. Conclusion and Future Work

In this paper we considered differential games based on the computation of two new robust controlled invariant sets, i.e., the lower and upper robust controlled invariant sets. This game was studied within the Hamilton-Jacobi reachability framework, in which the lower robust controlled invariant set is characterized as the zero level set of the unique bounded continuous viscosity solution to a Hamilton-Jacobi equation with sup-inf Hamiltonian while the upper robust controlled invariant set is characterized as the zero level set of the unique bounded continuous viscosity solution to a Hamilton-Jacobi equation with inf-sup Hamiltonian. Two examples, including one adopted from a Moore-Greitzer jet engine model, were employed to illustrate our approach.

In our future work, we would investigate the condition under which Assumption 1 holds, and the relationship between the lower (upper) robust controlled invariant set $\mathcal{R}^-$ ($\mathcal{R}^+$) and the lower (upper) controlled invariant set $\mathcal{R}^{*-}$ ($\mathcal{R}^{*+}$), where the concepts of both the lower controlled invariant set $\mathcal{R}^{*-}$ and the upper controlled invariant set $\mathcal{R}^{*+}$ are given in Definition 4.

Definition 4: Let $\mathcal{X} = \{\boldsymbol{x} \in \mathbb{R}^n \mid h(\boldsymbol{x}) \leq 0\}$ be a compact set in $\mathbb{R}^n$, where $h(\boldsymbol{x})$ is a bounded and locally Lipschitz continuous function in $\mathbb{R}^n$,

1) The lower controlled invariant set $\mathcal{R}^{*-}$ of system (1) is the set of initial states $\boldsymbol{x}$ such that for any $T \geq 0$, there exists a non-anticipative strategy $\boldsymbol{\alpha}(\cdot) \in \Gamma$ such that for any perturbation $\boldsymbol{d}(\cdot) \in \mathcal{D}$ the corresponding trajectory $\boldsymbol{\phi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t)$ stays inside $\mathcal{X}$ for $t \in [0, T]$, i.e.,

$$\mathcal{R}^{*-} = \left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \begin{array}{l} \forall T \geq 0, \exists \boldsymbol{\alpha}(\cdot) \in \Gamma, \forall \boldsymbol{d}(\cdot) \in \mathcal{D}, \\ \forall t \in [0, T], \boldsymbol{\phi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t) \in \mathcal{X} \end{array} \right\}.$$

2). The upper controlled invariant set $\mathcal{R}^{*+}$ of system (1) is the set of initial states $\boldsymbol{x}$ such that for any $T \geq 0$ and any non-anticipative strategy $\boldsymbol{\beta}(\cdot) \in \Delta$, there exists a control $\boldsymbol{u}(\cdot) \in \mathcal{U}$ such that the trajectory $\boldsymbol{\phi}_{\boldsymbol{x}}^{\boldsymbol{u},\boldsymbol{\beta}(\boldsymbol{u})}(t)$ stays inside $\mathcal{X}$ for $t \in [0, T]$, i.e.,

$$\mathcal{R}^{*+} = \left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \begin{array}{l} \forall T \geq 0, \forall \boldsymbol{\beta}(\cdot) \in \Delta, \exists \boldsymbol{u}(\cdot) \in \mathcal{U}, \\ \forall t \in [0, T], \boldsymbol{\phi}_{\boldsymbol{x}}^{\boldsymbol{u},\boldsymbol{\beta}(\boldsymbol{u})}(t) \in \mathcal{X} \end{array} \right\}.$$

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APPENDIX

Before justifying that $V^-(\boldsymbol{x})$ and $V^+(\boldsymbol{x})$ are respectively the viscosity solution to Eq. (10) and (11), we first introduce two auxiliary lemmas, i.e., Lemma 3 and Lemma 4.

Lemma 3: For $\boldsymbol{x} \in \mathbb{R}^n$ and $t \geq 0$, we have

$$V^-(\boldsymbol{x}) = \inf_{\boldsymbol{\alpha}(\cdot)\in\Gamma} \sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} \max\{e^{-\gamma t}V^-(\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t)), \sup_{\tau\in[0,t]} e^{-\gamma\tau}h(\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(\tau))\} \quad (15)$$

and

$$V^+(\boldsymbol{x}) = \sup_{\boldsymbol{\beta}(\cdot)\in\Delta} \inf_{\boldsymbol{u}(\cdot)\in\mathcal{U}} \max\{e^{-\gamma t}V^+(\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{u},\boldsymbol{\beta}(\boldsymbol{u})}(t)), \sup_{\tau\in[0,t]} e^{-\gamma\tau}h(\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{u},\boldsymbol{\beta}(\boldsymbol{u})}(\tau))\}. \quad (16)$$

Proof: Let

$$W(\boldsymbol{x},t) = \inf_{\boldsymbol{\alpha}(\cdot)\in\Gamma} \sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} \max\{ e^{-\gamma t}V^-(\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t)), \sup_{\tau\in[0,t]} e^{-\gamma\tau}h(\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(\tau)) \}.$$

We will show that for every $\epsilon > 0$, $V^-(\boldsymbol{x}) \leq W(\boldsymbol{x},t)+2\epsilon$ and $V^-(\boldsymbol{x}) \geq W(\boldsymbol{x},t) - 3\epsilon$. Then since $\epsilon > 0$ is arbitrary, $V^-(\boldsymbol{x}) = W(\boldsymbol{x},t)$.

1. $V^-(\boldsymbol{x}) \leq W(\boldsymbol{x},t)+2\epsilon$. Fix $\epsilon > 0$ and choose $\boldsymbol{\alpha}_1(\cdot) \in \Gamma$ such that

$$W(\boldsymbol{x},t) \geq -\epsilon + \sup_{\boldsymbol{d}_1(\cdot)\in\mathcal{D}} \max\{ e^{-\gamma t}V^-(\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}_1(\boldsymbol{d}_1),\boldsymbol{d}_1}(t)), \sup_{\tau\in[0,t]} e^{-\gamma\tau}h(\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}_1(\boldsymbol{d}_1),\boldsymbol{d}_1}(\tau)) \}.$$

Similarly, choose $\boldsymbol{\alpha}_2(\cdot) \in \Gamma$ such that

$$V^-(\boldsymbol{y}) \geq \sup_{\boldsymbol{d}_2(\cdot)\in\mathcal{D}} \sup_{\tau\in[t,\infty)} e^{-\gamma(\tau-t)}h(\boldsymbol{\psi}_{\boldsymbol{y}}^{\boldsymbol{\alpha}_2(\boldsymbol{d}_2),\boldsymbol{d}_2}(\tau-t)) - \epsilon,$$

where $\boldsymbol{y} = \boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}_1(\boldsymbol{d}_1),\boldsymbol{d}_1}(t)$.

Let

$$\boldsymbol{d}(\tau) = \begin{cases} \boldsymbol{d}_1(\tau) & \text{if } \tau \in [0,t) \\ \boldsymbol{d}_2(\tau-t) & \text{if } \tau \in [t,\infty) \end{cases}$$

and

$$\boldsymbol{\alpha}(\boldsymbol{d})(\tau) = \begin{cases} \boldsymbol{\alpha}_1(\boldsymbol{d})(\tau) & \text{if } \tau \in [0,t) \\ \boldsymbol{\alpha}_2(\boldsymbol{d})(\tau-t) & \text{if } \tau \in [t,\infty) \end{cases}.$$

It is easy to see that $\boldsymbol{\alpha}(\cdot) : \mathcal{D} \mapsto \mathcal{U}$ is non-anticipative. By uniqueness, $\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(\tau) = \boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}_1(\boldsymbol{d}_1),\boldsymbol{d}_1}(\tau)$ if $\tau \in [0,t)$, and $\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(\tau) = \boldsymbol{\psi}_{\boldsymbol{y}}^{\boldsymbol{\alpha}_2(\boldsymbol{d}_2),\boldsymbol{d}_2}(\tau-t)$ if $\tau \in [t,\infty)$.

Hence,

$$\begin{aligned} &W(\boldsymbol{x},t) \\ &\geq \sup_{\boldsymbol{d}_1(\cdot)\in\mathcal{D}} \sup_{\boldsymbol{d}_2(\cdot)\in\mathcal{D}} \max\{ \\ &\sup_{\tau\in[t,\infty)} e^{-\gamma\tau}h(\boldsymbol{\psi}_{\boldsymbol{y}}^{\boldsymbol{\alpha}_2(\boldsymbol{d}_2),\boldsymbol{d}_2}(\tau-t)), \\ &\sup_{\tau\in[0,t]} e^{-\gamma\tau}h(\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}_1(\boldsymbol{d}_1),\boldsymbol{d}_1}(\tau)) \\ &\} - 2\epsilon \\ &\geq \sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} \sup_{\tau\in[0,\infty)} e^{-\gamma\tau}h(\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(\tau)) - 2\epsilon \\ &\geq V^-(\boldsymbol{x}) - 2\epsilon. \end{aligned}$$

Therefore, $V^-(\boldsymbol{x}) \leq W(\boldsymbol{x},t) + 2\epsilon$.

2. $V^-(\boldsymbol{x}) \geq W(\boldsymbol{x},t)-3\epsilon$. Fix $\epsilon > 0$ and choose $\boldsymbol{\alpha}(\cdot) \in \Gamma$ such that

$$V^-(\boldsymbol{x}) \geq \sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} \sup_{t\in[0,\infty)} e^{-\gamma t}h(\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t)) - \epsilon.$$

By the definition of $W(\boldsymbol{x},t)$, we have

$$W(\boldsymbol{x},t) \leq \sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} \max\{ \max_{\tau\in[0,t]} e^{-\gamma\tau}h(\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(\tau)), e^{-\gamma t}V^-(\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(t)) \}.$$

Hence there exists $\boldsymbol{d}_1(\cdot) \in \mathcal{D}$ such that

$$W(\boldsymbol{x},t) \leq \max\{ \max_{\tau\in[0,t]} e^{-\gamma\tau}h(\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}(\boldsymbol{d}_1),\boldsymbol{d}_1}(\tau)), e^{-\gamma t}V^-(\boldsymbol{y}) \} + \epsilon. \quad (17)$$

where $\boldsymbol{y} = \boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}(\boldsymbol{d}_1),\boldsymbol{d}_1}(t)$. Moreover, we have that for $\tau \in [t,\infty)$,

$$V^-(\boldsymbol{y}) \leq \sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} \sup_{\tau\in[t,\infty)} e^{-\gamma(\tau-t)}h(\boldsymbol{\psi}_{\boldsymbol{y}}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(\tau-t)),$$

so there exists $\boldsymbol{d}_2(\cdot) \in \mathcal{D}$ such that

$$V^-(\boldsymbol{y}) \leq \sup_{\tau\in[t,\infty)} e^{-\gamma(\tau-t)}h(\boldsymbol{\psi}_{\boldsymbol{y}}^{\boldsymbol{\alpha}(\boldsymbol{d}_2),\boldsymbol{d}_2}(\tau-t)) + \epsilon. \quad (18)$$

We define

$$\boldsymbol{d}(\tau) = \begin{cases} \boldsymbol{d}_1(\tau) \text{ if } \tau \in [0,t) \\ \boldsymbol{d}_2(\tau-t) \text{ if } \tau \in [t,\infty) \end{cases}.$$

Therefore, combining (17) and (18), we have

$$W(\boldsymbol{x},t) \leq \sup_{\tau\in[0,\infty)} e^{-\gamma\tau}h(\boldsymbol{\psi}_{\boldsymbol{x}}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(\tau)) + 2\epsilon,$$

which together with (V) implies $V^-(\boldsymbol{x}) \geq W(\boldsymbol{x},t) - 3\epsilon$.

The above procedure can be employed to prove that V^+ satisfies the dynamic programming principle (16) as well. ■

Lemma 4: Let $v \in C^\infty(\mathbb{R}^n)$.

(1). If $\gamma v(\boldsymbol{x}_0) - H^-(\boldsymbol{x}_0, \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\boldsymbol{x}_0}) \leq -\theta < 0$, then, for sufficiently small $\delta > 0$, there exists $\boldsymbol{d}(\cdot) \in \mathcal{D}$ such that for all $\boldsymbol{\alpha}(\cdot) \in \Gamma$ and all $s \in [0,\delta]$,

$$\gamma v(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(s)) - \tilde{v}(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(s), \boldsymbol{\alpha}(\boldsymbol{d})(s), \boldsymbol{d}(s)) \leq -\frac{\theta}{2},$$

where $\tilde{v}(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(s), \boldsymbol{\alpha}(\boldsymbol{d})(s), \boldsymbol{d}(s)) = \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(s)} \cdot \boldsymbol{F}(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(s), \boldsymbol{\alpha}(\boldsymbol{d})(s), \boldsymbol{d}(s))$.

(2). If $\gamma v(\boldsymbol{x}_0) - H^-(\boldsymbol{x}_0, \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\boldsymbol{x}_0}) \geq \theta > 0$, then, for sufficiently small $\delta > 0$, there exists $\boldsymbol{\alpha}(\cdot) \in \Gamma$ such that for all $\boldsymbol{d}(\cdot) \in \mathcal{D}$ and all $s \in [0, \delta]$,

$$\gamma v(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(s)) - \tilde{v}(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(s), \boldsymbol{\alpha}(\boldsymbol{d})(s), \boldsymbol{d}(s)) \geq \frac{\theta}{2},$$

where $\tilde{v}(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(s), \boldsymbol{\alpha}(\boldsymbol{d})(s), \boldsymbol{d}(s)) = \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(s)} \cdot \boldsymbol{F}(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(s), \boldsymbol{\alpha}(\boldsymbol{d})(s), \boldsymbol{d}(s))$.

(3). If $\gamma v(\boldsymbol{x}_0) - H^+(\boldsymbol{x}_0, \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\boldsymbol{x}_0}) \geq \theta > 0$, then, for sufficiently small $\delta > 0$, there exists $\boldsymbol{u}(\cdot) \in \mathcal{U}$ such that for all $\boldsymbol{\beta}(\cdot) \in \Delta$ and all $s \in [0, \delta]$,

$$\gamma v(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{u},\boldsymbol{\beta}(\boldsymbol{u})}(s)) - \widehat{v}(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{u},\boldsymbol{\beta}(\boldsymbol{u})}(s), \boldsymbol{u}(s), \boldsymbol{\beta}(\boldsymbol{u})(s)) \geq \frac{\theta}{2},$$

where $\widehat{v}(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{u},\boldsymbol{\beta}(\boldsymbol{u})}(s), \boldsymbol{u}(s), \boldsymbol{\beta}(\boldsymbol{u})(s)) = \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{u},\boldsymbol{\beta}(\boldsymbol{u})}} \cdot \boldsymbol{F}(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{u},\boldsymbol{\beta}(\boldsymbol{u})}(s), \boldsymbol{u}(s), \boldsymbol{\beta}(\boldsymbol{u})(s))$.

(4). If $\gamma v(\boldsymbol{x}_0) - H^+(\boldsymbol{x}_0, \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\boldsymbol{x}_0}) \leq -\theta < 0$, then, for sufficiently small $\delta > 0$, there exists $\boldsymbol{\beta}(\cdot) \in \Delta$ such that for all $\boldsymbol{u}(\cdot) \in \mathcal{U}$ and all $s \in [0, \delta]$,

$$\gamma v(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{u},\boldsymbol{\beta}(\boldsymbol{u})}(s)) - \widehat{v}(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{u},\boldsymbol{\beta}(\boldsymbol{u})}(s), \boldsymbol{u}(s), \boldsymbol{\beta}(\boldsymbol{u})(s)) \leq -\frac{\theta}{2},$$

where $\widehat{v}(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{u},\boldsymbol{\beta}(\boldsymbol{u})}(s), \boldsymbol{u}(s), \boldsymbol{\beta}(\boldsymbol{u})(s)) = \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{u},\boldsymbol{\beta}(\boldsymbol{u})}} \cdot \boldsymbol{F}(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{u},\boldsymbol{\beta}(\boldsymbol{u})}(s), \boldsymbol{u}(s), \boldsymbol{\beta}(\boldsymbol{u})(s))$.

Proof: The proofs of statements 1 and 2 are given. The statements 3 and 4 can be justified similarly.

1. Since $\gamma v(\boldsymbol{x}_0) - H^-(\boldsymbol{x}_0, \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\boldsymbol{x}_0}) \leq -\theta < 0$, there exists $\boldsymbol{d}_0 \in D$ such that

$$\gamma v(\boldsymbol{x}_0) - \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\boldsymbol{x}_0} \cdot \boldsymbol{F}(\boldsymbol{x}_0, \boldsymbol{u}, \boldsymbol{d}_0) \leq -\frac{3}{4}\theta < 0, \forall \boldsymbol{u} \in U.$$

Also, since $v \in C^\infty$, $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d})$ is continuous over $(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d})$, there exists $\delta_{\boldsymbol{u}}$ for $\boldsymbol{u} \in U$ such that for $\boldsymbol{x}$ satisfying $\|\boldsymbol{x} - \boldsymbol{x}_0\| \leq \delta_{\boldsymbol{u}}$,

$$\gamma v(\boldsymbol{x}) - \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}} \cdot \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d}_0) \leq -\frac{3}{5}\theta < 0.$$

Since U is a compact set in $\mathbb{R}^m$, there exist finitely many distinct points $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_l \in U$ with positive values $\delta_1, \ldots, \delta_l$ such that

$$U \subset \cup_{i=1}^l \{\boldsymbol{u} \mid \|\boldsymbol{u} - \boldsymbol{u}_i\| \leq \delta_i\}$$

and

$$\gamma v(\boldsymbol{x}) - \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}} \cdot \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d}_0) \leq -\frac{1}{2}\theta < 0$$

for $\boldsymbol{x}$ satisfying $\|\boldsymbol{x} - \boldsymbol{x}_0\| \leq \delta_i$ and $\boldsymbol{u}$ satisfying $\|\boldsymbol{u} - \boldsymbol{u}_i\| \leq \delta_i$, where $i = 1, \ldots, l$. Therefore,

$$\gamma v(\boldsymbol{x}) - \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}} \cdot \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d}_0) \leq -\frac{1}{2}\theta < 0, \forall \boldsymbol{u} \in U$$

for $\boldsymbol{x}$ satisfying $\|\boldsymbol{x} - \boldsymbol{x}_0\| \leq \delta' = \min_{i=1,\ldots,l} \delta_i$.

Let Ω be a compact set in $\mathbb{R}^n$ which covers all states traversed by trajectories starting from $\boldsymbol{x}_0$ within a finite time interval $[0, \delta'']$, and M be the upper bound of $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d})$ over $\Omega \times U \times D$. We have

$$\|\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{u},\boldsymbol{d}}(t) - \boldsymbol{x}_0\| = \int_{\tau=0}^t \|\boldsymbol{F}(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau), \boldsymbol{d}(\tau))\| d\tau \leq Mt$$

for $t \in [0, \delta'']$, $\boldsymbol{u}(\cdot) \in \mathcal{U}$ and $\boldsymbol{d}(\cdot) \in \mathcal{D}$. Therefore, there exists $\delta > 0$ with $\delta \leq \delta''$ such that

$$\|\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{u},\boldsymbol{d}}(t) - \boldsymbol{x}_0\| \leq \delta', \forall t \in [0, \delta], \forall \boldsymbol{u}(\cdot) \in \mathcal{U}, \forall \boldsymbol{d}(\cdot) \in \mathcal{D}. \tag{19}$$

We choose a measurable function $\boldsymbol{d}' : [0, \infty) \mapsto D$ with $\boldsymbol{d}'(s) = \boldsymbol{d}_0$ for $s \in [0, \infty)$. Obviously, $\boldsymbol{d}'(\cdot) \in \mathcal{D}$. Therefore, we haves that for $\boldsymbol{u}(\cdot) \in \mathcal{U}$ and $s \in [0, \delta]$,

$$\gamma v(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{u},\boldsymbol{d}'}(s)) - \tilde{v}(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{u},\boldsymbol{d}'}(s), \boldsymbol{u}(s), \boldsymbol{d}'(s)) \leq -\frac{\theta}{2},$$

implying that for all $\boldsymbol{\alpha}(\cdot) \in \Gamma$ and all $s \in [0, \delta]$,

$$\gamma v(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}'),\boldsymbol{d}'}(s)) - \tilde{v}(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}'),\boldsymbol{d}'}(s), \boldsymbol{\alpha}(\boldsymbol{d}')(s), \boldsymbol{d}'(s)) \leq -\frac{\theta}{2}.$$

where $\tilde{v}(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{u},\boldsymbol{d}'}(s), \boldsymbol{u}(s), \boldsymbol{d}'(s)) = \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{u},\boldsymbol{d}'}(s)} \cdot \boldsymbol{F}(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{u},\boldsymbol{d}'}(s), \boldsymbol{u}(s), \boldsymbol{d}'(s))$ and $\tilde{v}(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}'),\boldsymbol{d}'}(s), \boldsymbol{\alpha}(\boldsymbol{d}')(s), \boldsymbol{d}'(s)) = \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}'),\boldsymbol{d}'}(s)} \cdot \boldsymbol{F}(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}'),\boldsymbol{d}'}(s), \boldsymbol{\alpha}(\boldsymbol{d}')(s), \boldsymbol{d}'(s))$

2. Since $\gamma v(\boldsymbol{x}_0) - H^-(\boldsymbol{x}_0, \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\boldsymbol{x}_0}) \geq \theta > 0$, there exists a corresponding $\boldsymbol{u}_{\boldsymbol{d}_0} \in U$ for every $\boldsymbol{d}_0 \in D$ such that

$$\gamma v(\boldsymbol{x}_0) - \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\boldsymbol{x}_0} \cdot \boldsymbol{F}(\boldsymbol{x}_0, \boldsymbol{u}_{\boldsymbol{d}_0}, \boldsymbol{d}_0) \geq \frac{3}{4}\theta > 0.$$

Since $v \in C^\infty$ and $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{d})$ is continuous over $\boldsymbol{x}$, $\boldsymbol{u}$ and $\boldsymbol{d}$, there exists $\delta' > 0$ such that for $\boldsymbol{d} \in D$ satisfying $\|\boldsymbol{d} - \boldsymbol{d}_0\| \leq \delta'$ and $\boldsymbol{x}$ satisfying $\|\boldsymbol{x} - \boldsymbol{x}_0\| \leq \delta'$,

$$\gamma v(\boldsymbol{x}) - \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}} \cdot \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}_{\boldsymbol{d}_0}, \boldsymbol{d}) \geq \frac{3}{5}\theta > 0.$$

Since D is a compact set in $\mathbb{R}^l$, there exist finitely many distinct points $\boldsymbol{d}_1, \ldots, \boldsymbol{d}_l \in D$ with positive values $\delta_1, \ldots, \delta_l$ such that $D \subset \cup_{i=1}^l \{\boldsymbol{d} \mid \|\boldsymbol{d} - \boldsymbol{d}_i\| \leq \delta_i\}$. Moreover, there exists $\boldsymbol{u}_{\boldsymbol{d}_i} \in U$ such that for $\boldsymbol{d}$ satisfying $\|\boldsymbol{d} - \boldsymbol{d}_i\| \leq \delta_i$ and $\boldsymbol{x}$ satisfying $\|\boldsymbol{x} - \boldsymbol{x}_0\| \leq \delta_i$,

$$\gamma v(\boldsymbol{x}) - \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}} \cdot \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}_{\boldsymbol{d}_i}, \boldsymbol{d}) \geq \frac{1}{2}\theta > 0$$

holds, where $i = 1, \ldots, l$.

Setting $\boldsymbol{\nu} : D \mapsto U$ such that $\boldsymbol{\nu}(\boldsymbol{d}) = \boldsymbol{u}_{\boldsymbol{d}_i}$ if $\boldsymbol{d} \in \{\boldsymbol{d} \mid \|\boldsymbol{d} - \boldsymbol{d}_i\| \leq \delta_i\} \setminus \cup_{j=1}^{i-1} \{\boldsymbol{d} \mid \|\boldsymbol{d} - \boldsymbol{d}_j\| \leq \delta_j\}$ for $i = 1, \ldots, l$, we have that for $\boldsymbol{x}$ satisfying $\|\boldsymbol{x} - \boldsymbol{x}_0\| \leq \delta' = \min_{i=1,\ldots,l} \delta_i$,

$$\gamma v(\boldsymbol{x}) - \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}} \cdot \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{\nu}(\boldsymbol{d}), \boldsymbol{d}) \geq \frac{1}{2}\theta > 0, \forall \boldsymbol{d} \in D.$$

Furthermore, like (19), we obtain that there exists $\delta > 0$ such that

$$\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\nu}(\boldsymbol{d}),\boldsymbol{d}}(s) \in \{\boldsymbol{x} \mid \|\boldsymbol{x} - \boldsymbol{x}_0\| \leq \delta'\}, \forall s \in [0, \delta], \forall \boldsymbol{d}(\cdot) \in \mathcal{D}.$$

Let $\boldsymbol{\alpha}(\cdot) : \mathcal{D} \mapsto \mathcal{U}$ be $\boldsymbol{\alpha}(\boldsymbol{d})(s) = \boldsymbol{\nu}(\boldsymbol{d}(s))$ for $s \geq 0$. It is obvious that $\boldsymbol{\alpha}(\cdot) \in \Gamma$. Consequently, there exist $\delta > 0$

and a strategy $\boldsymbol{\alpha}(\cdot) \in \Gamma$ such that for all $\boldsymbol{d}(\cdot) \in \mathcal{D}$ and all $s \in [0, \delta]$,

$$\gamma v(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(s)) - \tilde{v}(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(s), \boldsymbol{\alpha}(\boldsymbol{d})(s), \boldsymbol{d}(s)) \geq \frac{\theta}{2}.$$

where $\tilde{v}(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(s), \boldsymbol{\alpha}(\boldsymbol{d})(s), \boldsymbol{d}(s)) = \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(s)} \cdot \boldsymbol{F}(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(s), \boldsymbol{\alpha}(\boldsymbol{d})(s), \boldsymbol{d}(s))$. ■

The proof of Theorem 1: *Proof:* We just show the proof of the statement pertinent to V^-. We will prove that V^- is both viscosity sub and super-solution to Eq. (10) according to Definition 3.

Firstly, we prove that V^- is a sub-solution to Eq. (10). Let $v \in C^\infty(\mathbb{R}^n)$ such that $V^- - v$ attains a local maximum at $\boldsymbol{x}_0$. Without loss of generality, assume that this maximum is zero, i.e., $V^-(\boldsymbol{x}_0) = v(\boldsymbol{x}_0)$. According to the continuity of $V^-(\boldsymbol{x})$ and $v(\boldsymbol{x})$, there exists a positive value $\overline{\delta}$ such that

$$V^-(\boldsymbol{x}) - v(\boldsymbol{x}) \leq 0$$

for $\boldsymbol{x}$ satisfying $\|\boldsymbol{x} - \boldsymbol{x}_0\| \leq \overline{\delta}$. Suppose (13) is false. Then there definitely exists a positive value ϵ_1 such that

$$h(\boldsymbol{x}_0) \leq v(\boldsymbol{x}_0) - \epsilon_1 \text{ and} \tag{20}$$

$$\gamma v(\boldsymbol{x}_0) - H^-(\boldsymbol{x}_0, \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\boldsymbol{x}_0}) \geq \epsilon_1 \tag{21}$$

hold. Therefore, for the former inequality, i.e., $h(\boldsymbol{x}_0) \leq v(\boldsymbol{x}_0) - \epsilon_1$, there exists a sufficiently small $\delta_1 > 0$ with $\delta_1 \leq \overline{\delta}$ such that for $\boldsymbol{x}$ satisfying $\|\boldsymbol{x} - \boldsymbol{x}_0\| \leq \delta_1$ and t satisfying $0 \leq t \leq \delta_1$,

$$e^{-\gamma t} h(\boldsymbol{x}) \leq v(\boldsymbol{x}_0) - \frac{\epsilon_1}{2}.$$

According to Lemma 4, (21) implies that for sufficiently small $\delta > 0$, there exists a strategy $\boldsymbol{\alpha}_1(\cdot) \in \Gamma$ such that for all $\boldsymbol{d}(\cdot) \in \mathcal{D}$ and all $s \in [0, \delta]$,

$$\gamma v(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}_1(\boldsymbol{d}),\boldsymbol{d}}(s)) - \tilde{v}(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}_1(\boldsymbol{d}),\boldsymbol{d}}(s), \boldsymbol{\alpha}_1(\boldsymbol{d})(s), \boldsymbol{d}(s)) \geq \frac{\epsilon_1}{2}, \tag{22}$$

where $\tilde{v}(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}_1(\boldsymbol{d}),\boldsymbol{d}}(s), \boldsymbol{\alpha}_1(\boldsymbol{d})(s), \boldsymbol{d}(s)) = \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}_1(\boldsymbol{d}),\boldsymbol{d}}(s)} \cdot \boldsymbol{F}(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}_1(\boldsymbol{d}),\boldsymbol{d}}(s), \boldsymbol{\alpha}_1(\boldsymbol{d})(s), \boldsymbol{d}(s))$. δ can be chosen such that $\|\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}_1(\boldsymbol{d}),\boldsymbol{d}}(s) - \boldsymbol{x}_0\| \leq \delta_1, \forall s \in [0, \delta], \forall \boldsymbol{d}(\cdot) \in \mathcal{D}$. Since $v \in C^\infty(\mathbb{R}^n)$, by applying Grönwall's inequality [20] to (22) with the time interval [0, δ], we have

$$v(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}_1(\boldsymbol{d}),\boldsymbol{d}}(\delta)) \leq e^{\delta\gamma} v(\boldsymbol{x}_0) + \frac{\epsilon_1}{2\gamma}(1 - e^{\delta\gamma}).$$

Therefore,

$$e^{-\delta\gamma} v(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}_1(\boldsymbol{d}),\boldsymbol{d}}(\delta)) \leq v(\boldsymbol{x}_0) - \frac{\epsilon_1}{2\gamma}(1 - e^{-\delta\gamma}).$$

Furthermore, since $V^-(\boldsymbol{x}) \leq v(\boldsymbol{x})$ for $\boldsymbol{x}$ satisfying $\|\boldsymbol{x} - \boldsymbol{x}_0\| \leq \delta_1$ with $V^-(\boldsymbol{x}_0) = v(\boldsymbol{x}_0)$ as well as $V^- \geq 0$, we have

$$e^{-\delta\gamma} V^-(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}_1(\boldsymbol{d}),\boldsymbol{d}}(\delta)) \leq V^-(\boldsymbol{x}_0) - \frac{\epsilon_1}{2\gamma}(1 - e^{-\delta\gamma}).$$

Therefore, according to (15), we finally have

$$\begin{aligned}
&V^-(\boldsymbol{x}_0) \\
&= \inf_{\boldsymbol{\alpha}(\cdot)\in\Gamma} \sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} \max\big\{ \\
&e^{-\gamma\delta} V^-(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(\delta)), \sup_{\tau\in[0,\delta]} e^{-\gamma\tau} h(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(\tau)) \\
&\big\} \\
&\leq \sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} \max\big\{ \\
&e^{-\gamma\delta} V^-(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}_1(\boldsymbol{d}),\boldsymbol{d}}(\delta)), \sup_{\tau\in[0,\delta]} e^{-\gamma\tau} h(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}_1(\boldsymbol{d}),\boldsymbol{d}}(\tau)) \\
&\big\} \\
&\leq \max\big\{ \\
&e^{-\gamma\delta} V^-(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}_1(\boldsymbol{d}_1),\boldsymbol{d}_1}(\delta)), \sup_{\tau\in[0,\delta]} e^{-\gamma\tau} h(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}_1(\boldsymbol{d}_1),\boldsymbol{d}_1}(\tau)) \\
&\big\} + \epsilon_3 \\
&\leq V^-(\boldsymbol{x}_0) - \min\{\frac{\epsilon_1}{2}, \frac{\epsilon_1}{2\gamma}(1 - e^{-\delta\gamma})\} + \epsilon_3 \\
&< V^-(\boldsymbol{x}_0),
\end{aligned} \tag{23}$$

which is a contradiction. In (23), $\boldsymbol{d}_1(\cdot) \in \mathcal{D}$ satisfies

$$\begin{aligned}
&\sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} \max\big\{ \\
&e^{-\gamma\delta} V^-(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}_1(\boldsymbol{d}),\boldsymbol{d}}(\delta)), \sup_{\tau\in[0,\delta]} e^{-\gamma\tau} h(\psi_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}_1(\boldsymbol{d}),\boldsymbol{d}}(\tau)) \\
&\big\} \\
&\leq \max\big\{ \\
&e^{-\gamma\delta} V^-(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}_1(\boldsymbol{d}_1),\boldsymbol{d}_1}(\delta)), \sup_{\tau\in[0,\delta]} e^{-\gamma\tau} h(\psi_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}_1(\boldsymbol{d}_1),\boldsymbol{d}_1}(\tau)) \\
&\big\} + \epsilon_3
\end{aligned}$$

with $0 < \epsilon_3 < \min\{\frac{\epsilon_1}{2}, \frac{\epsilon_1}{2\gamma}(1 - e^{-\delta\gamma})\}$. Consequently, V^- is a subsolution to Eq. (10).

In what follows we prove that V^- is a super-solution to Eq. (10). Let $v \in C^\infty(\mathbb{R}^n)$ such that $V^- - v$ attains a local minimum at $\boldsymbol{x}_0$. Without loss of generality, assume that this minimum is zero, i.e., $V^-(\boldsymbol{x}_0) = v(\boldsymbol{x}_0)$. Therefore, there exists a positive value $\overline{\delta}$ such that $V^-(\boldsymbol{x}) - v(\boldsymbol{x}) \geq 0$ for $\boldsymbol{x}$ satisfying $\|\boldsymbol{x} - \boldsymbol{x}_0\| \leq \overline{\delta}$. Assume that (12) is false. Since $V^-(\boldsymbol{x}) \geq h(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathbb{R}^n$ according to (15), $v(\boldsymbol{x}_0) \geq h(\boldsymbol{x}_0)$ holds. Therefore,

$$\gamma v(\boldsymbol{x}_0) - H^-(\boldsymbol{x}_0, \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\boldsymbol{x}_0}) < 0$$

holds, i.e., there exists a positive value $\theta > 0$ such that

$$\gamma v(\boldsymbol{x}_0) - H^-(\boldsymbol{x}_0, \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\boldsymbol{x}_0}) \leq -\theta.$$

According to Lemma 4, we have that for sufficiently small $\delta > 0$, there exists $\boldsymbol{d}_1(\cdot) \in \mathcal{D}$ such that for all strategies $\boldsymbol{\alpha}(\cdot) \in \Gamma$ and all $s \in [0, \delta]$,

$$\gamma v(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}_1),\boldsymbol{d}_1}(s)) - \tilde{v}(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}_1),\boldsymbol{d}_1}(s), \boldsymbol{\alpha}(\boldsymbol{d}_1)(s), \boldsymbol{d}_1(s)) \leq -\frac{\theta}{2}, \tag{24}$$

where $\tilde{v}(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}_1),\boldsymbol{d}_1}(s), \boldsymbol{\alpha}(\boldsymbol{d}_1)(s), \boldsymbol{d}_1(s)) = \frac{\partial v(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}_1),\boldsymbol{d}_1}(s)} \cdot \boldsymbol{F}(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}_1),\boldsymbol{d}_1}(s), \boldsymbol{\alpha}(\boldsymbol{d}_1)(s), \boldsymbol{d}_1(s))$. δ can be chosen such that $\|\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}_1),\boldsymbol{d}_1}(s) - \boldsymbol{x}_0\| \leq \overline{\delta}, \forall s \in [0,\delta], \forall \boldsymbol{\alpha}(\cdot) \in \Gamma$.

By applying Grönwall's inequality [20] to (24) with the time interval [0, δ], we obtain

$$v(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}_1),\boldsymbol{d}_1}(\delta)) \geq e^{\delta\gamma} v(\boldsymbol{x}_0) - \frac{\theta}{2\gamma}(1 - e^{\delta\gamma}).$$

Therefore,

$$e^{-\delta\gamma} v(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}_1),\boldsymbol{d}_1}(\delta)) \geq v(\boldsymbol{x}_0) + \frac{\theta}{2\gamma}(1 - e^{-\delta\gamma}).$$

Furthermore, since $V^- \geq v$ for $\boldsymbol{x} \in \{\boldsymbol{x} \mid \|\boldsymbol{x} - \boldsymbol{x}_0\| \leq \overline{\delta}\}$ with $V^-(\boldsymbol{x}_0) = v(\boldsymbol{x}_0)$ as well as $V^-(\boldsymbol{x}) \geq 0$ over $\boldsymbol{x} \in \mathbb{R}^n$, we have

$$e^{-\delta\gamma} V^-(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}_1),\boldsymbol{d}_1}(\delta)) \geq V^-(\boldsymbol{x}_0) + \frac{\theta}{2\gamma}(1 - e^{-\delta\gamma}).$$

Therefore, according to (15), we finally have

$$\begin{aligned}
&V^-(\boldsymbol{x}_0)\\
&= \inf_{\boldsymbol{\alpha}(\cdot)\in\Gamma} \sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} \max\big\{\\
&e^{-\gamma\delta} V^-(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(\delta)), \sup_{\tau\in[0,\delta]} e^{-\gamma\tau} h(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(\tau))\\
&\big\}\\
&\geq \sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} \max\big\{\\
&e^{-\gamma\delta} V^-(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}_1(\boldsymbol{d}),\boldsymbol{d}}(\delta)), \sup_{\tau\in[0,\delta]} e^{-\gamma\tau} h(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}_1(\boldsymbol{d}),\boldsymbol{d}}(\tau))\\
&\big\} - \epsilon_1\\
&\geq \max\big\{\\
&e^{-\gamma\delta} V^-(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}_1(\boldsymbol{d}_1),\boldsymbol{d}_1}(\delta)), \sup_{\tau\in[0,\delta]} e^{-\gamma\tau} h(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}_1(\boldsymbol{d}_1),\boldsymbol{d}_1}(\tau))\\
&\big\} - \epsilon_1\\
&\geq V^-(\boldsymbol{x}_0) + \frac{\theta}{2\gamma}(1 - e^{-\delta\gamma}) - \epsilon_1\\
&> V^-(\boldsymbol{x}_0),
\end{aligned} \tag{25}$$

which is a contradiction. In (25), $\boldsymbol{\alpha}_1(\cdot) \in \Gamma$ satisfies

$$\begin{aligned}
&\inf_{\boldsymbol{\alpha}(\cdot)\in\Gamma} \sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} \max\big\{\\
&e^{-\gamma\delta} V^-(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(\delta)), \sup_{\tau\in[0,\delta]} e^{-\gamma\tau} h(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}(\boldsymbol{d}),\boldsymbol{d}}(\tau))\\
&\big\}\\
&\geq \sup_{\boldsymbol{d}(\cdot)\in\mathcal{D}} \max\big\{\\
&e^{-\gamma\delta} V^-(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}_1(\boldsymbol{d}),\boldsymbol{d}}(\delta)), \sup_{\tau\in[0,\delta]} e^{-\gamma\tau} h(\boldsymbol{\psi}_{\boldsymbol{x}_0}^{\boldsymbol{\alpha}_1(\boldsymbol{d}),\boldsymbol{d}}(\tau))\\
&\big\} - \epsilon_1
\end{aligned}$$

with $0 < \epsilon_1 < \frac{\theta}{2\gamma}(1 - e^{-\delta\gamma})$. Thus, V^- is a supersolution to Eq. (10).

Therefore, V^- is a viscosity solution to Eq. (10). ■

The proof of Theorem 2:

Proof: We just show the uniqueness of the continuous and bounded viscosity solution to Eq. (10). We first prove a comparison principle: If V_1 and V_2 are bounded continuous functions over $\boldsymbol{x} \in \mathbb{R}^n$, and they are respectively a viscosity sub and supersolution to Eq. (10), then $V_1 \leq V_2$ in $\mathbb{R}^n$. Obviously, if such comparison principle holds, the uniqueness of bounded continuous solutions to Eq. (10) is guaranteed. For ease of exposition, we define $H^-(\overline{\boldsymbol{x}}) = H^-(\overline{\boldsymbol{x}}, \frac{\partial \psi(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\overline{\boldsymbol{x}}})$ and $H^-(\overline{\boldsymbol{y}}) = H^-(\overline{\boldsymbol{y}}, \frac{\partial \psi(\boldsymbol{y})}{\partial \boldsymbol{y}} \mid_{\boldsymbol{y}=\overline{\boldsymbol{y}}})$.

Let

$$\psi(\boldsymbol{x},\boldsymbol{y}) = V_1(\boldsymbol{x}) - V_2(\boldsymbol{y}) - \frac{\|\boldsymbol{x}-\boldsymbol{y}\|^2}{2\epsilon} - \delta(\langle \boldsymbol{x}\rangle^m + \langle \boldsymbol{y}\rangle^m),$$

where $\langle \boldsymbol{x}\rangle = (1 + \|\boldsymbol{x}\|^2)^{\frac{1}{2}}$, and ϵ, δ, m are positive parameters. Assume that there are $\beta > 0$ and $\boldsymbol{z}$ such that $V_1(\boldsymbol{z}) - V_2(\boldsymbol{z}) = \beta$. We choose $\delta > 0$ such that $2\delta\langle \boldsymbol{z}\rangle \leq \frac{\beta}{2}$. Then, for $0 < m \leq 1$,

$$\frac{\beta}{2} < \beta - 2\delta\langle \boldsymbol{z}\rangle^m = \psi(\boldsymbol{z},\boldsymbol{z}) \leq \sup \psi(\boldsymbol{x},\boldsymbol{y}). \tag{26}$$

Since ψ is continuous and $\lim_{\|\boldsymbol{x}\|+\|\boldsymbol{y}\|\to\infty} \psi(\boldsymbol{x},\boldsymbol{y}) = -\infty$, there exist $\overline{\boldsymbol{x}}$, $\overline{\boldsymbol{y}}$ such that

$$\psi(\overline{\boldsymbol{x}},\overline{\boldsymbol{y}}) = \sup \psi(\boldsymbol{x},\boldsymbol{y}). \tag{27}$$

From the inequality $\psi(\overline{\boldsymbol{x}},\overline{\boldsymbol{x}}) + \psi(\overline{\boldsymbol{y}},\overline{\boldsymbol{y}}) \leq 2\psi(\overline{\boldsymbol{x}},\overline{\boldsymbol{y}})$ we easily get

$$\frac{\|\overline{\boldsymbol{x}}-\overline{\boldsymbol{y}}\|^2}{\epsilon} \leq V_1(\overline{\boldsymbol{x}}) - V_1(\overline{\boldsymbol{y}}) + V_2(\overline{\boldsymbol{x}}) - V_2(\overline{\boldsymbol{y}}). \tag{28}$$

Then the boundedness of V_1 and V_2 implies that

$$\|\overline{\boldsymbol{x}}-\overline{\boldsymbol{y}}\| \leq c\sqrt{\epsilon} \tag{29}$$

for a suitable constant c. By plugging (29) into (28) and using the uniform continuity of V_1 and V_2 we get

$$\frac{\|\overline{\boldsymbol{x}}-\overline{\boldsymbol{y}}\|^2}{\epsilon} \leq w(\sqrt{\epsilon})$$

for some modulus w.

Next, define the continuously differentiable functions

$$\begin{aligned}
\psi(\boldsymbol{x}) &:= V_2(\overline{\boldsymbol{y}}) + \frac{\|\boldsymbol{x}-\overline{\boldsymbol{y}}\|^2}{2\epsilon} + \delta(\langle \boldsymbol{x}\rangle^m + \langle \overline{\boldsymbol{y}}\rangle^m),\\
\psi(\boldsymbol{y}) &:= V_1(\overline{\boldsymbol{x}}) - \frac{\|\overline{\boldsymbol{x}}-\boldsymbol{y}\|^2}{2\epsilon} - \delta(\langle \overline{\boldsymbol{x}}\rangle^m + \langle \boldsymbol{y}\rangle^m),
\end{aligned}$$

and observe that $V_1 - \psi$ attains its maximum at $\overline{\boldsymbol{x}}$ and $V_2 - \psi$ attains its minimum at $\overline{\boldsymbol{y}}$. It is easy to compute

$$\begin{aligned}
\frac{\partial \psi(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\overline{\boldsymbol{x}}} &= \frac{\overline{\boldsymbol{x}}-\overline{\boldsymbol{y}}}{\epsilon} + \lambda\overline{\boldsymbol{x}}, \lambda = \delta m \langle \overline{\boldsymbol{x}}\rangle^{m-2},\\
\frac{\partial \psi(\boldsymbol{y})}{\partial \boldsymbol{y}} \mid_{\boldsymbol{y}=\overline{\boldsymbol{y}}} &= \frac{\overline{\boldsymbol{x}}-\overline{\boldsymbol{y}}}{\epsilon} - \tau\overline{\boldsymbol{y}}, \tau = \delta m \langle \overline{\boldsymbol{y}}\rangle^{m-2}.
\end{aligned}$$

According to the definition of sub and super solution, we have that

$$\begin{aligned}
&\min\big\{\gamma V_1(\overline{\boldsymbol{x}}) - H^-(\overline{\boldsymbol{x}}), V_1(\overline{\boldsymbol{x}}) - h(\overline{\boldsymbol{x}})\big\}\\
&\leq \min\big\{\gamma V_2(\overline{\boldsymbol{y}}) - H^-(\overline{\boldsymbol{y}}), V_2(\overline{\boldsymbol{y}}) - h(\overline{\boldsymbol{y}})\big\}.
\end{aligned}$$

Further, we have that

$$\begin{aligned}&\min\big\{\\&\gamma V_1(\overline{\boldsymbol{x}})-H^-(\overline{\boldsymbol{x}})-(\gamma V_2(\overline{\boldsymbol{y}})-H^-(\overline{\boldsymbol{y}})),\\&V_1(\overline{\boldsymbol{x}})-h(\overline{\boldsymbol{x}})-(V_2(\overline{\boldsymbol{y}})-h(\overline{\boldsymbol{y}}))\\&\big\}\leq 0.\end{aligned}$$

Obviously, either

$$\gamma V_1(\overline{\boldsymbol{x}})-H^-(\overline{\boldsymbol{x}})-(\gamma V_2(\overline{\boldsymbol{y}})-H^-(\overline{\boldsymbol{y}}))\leq 0 \tag{30}$$

or

$$V_1(\overline{\boldsymbol{x}})-h(\overline{\boldsymbol{x}})-(V_2(\overline{\boldsymbol{y}})-h(\overline{\boldsymbol{y}}))\leq 0 \tag{31}$$

holds. We will obtain a contradiction separately.

If (30) holds,

$$\begin{aligned}&V_1(\overline{\boldsymbol{x}})-V_2(\overline{\boldsymbol{y}})\\&\leq\frac{1}{\gamma}(H^-(\overline{\boldsymbol{x}})-H^-(\overline{\boldsymbol{y}}))\\&\leq\frac{1}{\gamma}(L_f w(\sqrt{\epsilon})+\delta m K(\langle\overline{\boldsymbol{y}}\rangle^m+\langle\overline{\boldsymbol{x}}\rangle^m+\epsilon)\end{aligned}$$

where $K=L_f+\sup_{\boldsymbol{u}\in U,\boldsymbol{d}\in D}\{\|\boldsymbol{F}(\boldsymbol{0},\boldsymbol{u},\boldsymbol{d})\|\}$ and the last inequality can be obtained as follows:

$$\begin{aligned}&H^-(\overline{\boldsymbol{x}})-H^-(\overline{\boldsymbol{y}})\\&\leq\sup_{\boldsymbol{d}\in D}\big(\inf_{\boldsymbol{u}\in U}v(\overline{\boldsymbol{x}},\boldsymbol{u},\boldsymbol{d})-\inf_{\boldsymbol{u}\in U}v(\overline{\boldsymbol{y}},\boldsymbol{u},\boldsymbol{d})\big)\\&\leq\inf_{\boldsymbol{u}\in U}v(\overline{\boldsymbol{x}},\boldsymbol{u},\boldsymbol{d}_1)-\inf_{\boldsymbol{u}\in U}v(\overline{\boldsymbol{y}},\boldsymbol{u},\boldsymbol{d}_1)+\frac{\epsilon}{2}\\&\leq v(\overline{\boldsymbol{x}},\boldsymbol{u}_2,\boldsymbol{d}_1)-v(\overline{\boldsymbol{y}},\boldsymbol{u}_2,\boldsymbol{d}_1)+\epsilon\\&=(\frac{\overline{\boldsymbol{x}}-\overline{\boldsymbol{y}}}{\epsilon}+\lambda\overline{\boldsymbol{x}})\cdot\boldsymbol{F}(\overline{\boldsymbol{x}},\boldsymbol{u}_2,\boldsymbol{d}_1)\\&\qquad-(\frac{\overline{\boldsymbol{x}}-\overline{\boldsymbol{y}}}{\epsilon}-\tau\overline{\boldsymbol{y}})\cdot\boldsymbol{F}(\overline{\boldsymbol{y}},\boldsymbol{u}_2,\boldsymbol{d}_1)+\epsilon\\&\leq\frac{\|\overline{\boldsymbol{x}}-\overline{\boldsymbol{y}}\|^2}{\epsilon}L_f+\lambda\overline{\boldsymbol{x}}\cdot\boldsymbol{F}(\overline{\boldsymbol{x}},\boldsymbol{u}_2,\boldsymbol{d}_1)+\tau\overline{\boldsymbol{y}}\cdot\boldsymbol{F}(\overline{\boldsymbol{y}},\boldsymbol{u}_2,\boldsymbol{d}_1)+\epsilon\\&\leq\frac{\|\overline{\boldsymbol{x}}-\overline{\boldsymbol{y}}\|^2}{\epsilon}L_f+\lambda\overline{\boldsymbol{x}}\cdot(\boldsymbol{F}(\overline{\boldsymbol{x}},\boldsymbol{u}_2,\boldsymbol{d}_1)-\boldsymbol{F}(\boldsymbol{0},\boldsymbol{u}_2,\boldsymbol{d}_1)\\&\qquad+\boldsymbol{F}(\boldsymbol{0},\boldsymbol{u}_2,\boldsymbol{d}_1))+\tau\overline{\boldsymbol{y}}\cdot(\boldsymbol{F}(\overline{\boldsymbol{y}},\boldsymbol{u}_2,\boldsymbol{d}_1)\\&\qquad-\boldsymbol{F}(\boldsymbol{0},\boldsymbol{u}_2,\boldsymbol{d}_1)+\boldsymbol{F}(\boldsymbol{0},\boldsymbol{u}_2,\boldsymbol{d}_1))+\epsilon\\&\leq\frac{\|\overline{\boldsymbol{x}}-\overline{\boldsymbol{y}}\|^2}{\epsilon}L_f+\lambda L_f\|\overline{\boldsymbol{x}}\|^2+\lambda\|\overline{\boldsymbol{x}}\|\|\boldsymbol{F}(\boldsymbol{0},\boldsymbol{u}_2,\boldsymbol{d}_1)\|\\&\qquad+\tau L_f\|\overline{\boldsymbol{y}}\|^2+\tau\|\overline{\boldsymbol{y}}\|\|\boldsymbol{F}(\boldsymbol{0},\boldsymbol{u}_2,\boldsymbol{d}_1)\|+\epsilon\\&\leq\frac{\|\overline{\boldsymbol{x}}-\overline{\boldsymbol{y}}\|^2}{\epsilon}L_f+\lambda K(1+\|\overline{\boldsymbol{x}}\|^2)+\tau K(1+\|\overline{\boldsymbol{y}}\|^2)+\epsilon\\&\leq L_f w(\sqrt{\epsilon})+\delta m K(\langle\overline{\boldsymbol{y}}\rangle^m+\langle\overline{\boldsymbol{x}}\rangle^m)+\epsilon,\end{aligned}$$

where $\boldsymbol{d}_1$ satisfies

$$\begin{aligned}&\sup_{\boldsymbol{d}\in D}\big(\inf_{\boldsymbol{u}\in U}v(\overline{\boldsymbol{x}},\boldsymbol{u},\boldsymbol{d})-\inf_{\boldsymbol{u}\in U}v(\overline{\boldsymbol{y}},\boldsymbol{u},\boldsymbol{d})\big)\\&\leq\inf_{\boldsymbol{u}\in U}v(\overline{\boldsymbol{x}},\boldsymbol{u},\boldsymbol{d}_1)-\inf_{\boldsymbol{u}\in U}v(\overline{\boldsymbol{y}},\boldsymbol{u},\boldsymbol{d}_1)+\frac{\epsilon}{2},\end{aligned}$$

$\boldsymbol{u}_2$ satisfies

$$\inf_{\boldsymbol{u}\in U}v(\overline{\boldsymbol{y}},\boldsymbol{u},\boldsymbol{d}_1)\geq v(\overline{\boldsymbol{y}},\boldsymbol{u}_2,\boldsymbol{d}_1)-\frac{\epsilon}{2}$$

and $v(\boldsymbol{x},\boldsymbol{u},\boldsymbol{d})=\frac{\partial\psi(\boldsymbol{x})}{\partial\boldsymbol{x}}\cdot\boldsymbol{F}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{d})$.

Therefore, choosing $0<m\leq\frac{\gamma}{K}$, we obtain

$$\psi(\overline{\boldsymbol{x}},\overline{\boldsymbol{y}})\leq V_1(\overline{\boldsymbol{x}})-V_2(\overline{\boldsymbol{y}})-\delta(\langle\overline{\boldsymbol{x}}\rangle^m+\langle\overline{\boldsymbol{y}}\rangle^m)\leq\frac{1}{\gamma}(L_f w(\sqrt{\epsilon})+\epsilon).$$

$\psi(\overline{\boldsymbol{x}},\overline{\boldsymbol{y}})$ can be smaller than $\frac{\beta}{2}$ for ϵ small enough, contradicting (26) and (27).

If (31) holds,

$$\psi(\overline{\boldsymbol{x}},\overline{\boldsymbol{y}})\leq V_1(\overline{\boldsymbol{x}})-V_2(\overline{\boldsymbol{y}})\leq h(\overline{\boldsymbol{x}})-h(\overline{\boldsymbol{y}})\leq L_h c\sqrt{\epsilon},$$

where L_h is the Lipschitz constant over a local compact region in $\mathbb{R}^n$ covering $\overline{\boldsymbol{x}}$ and $\overline{\boldsymbol{y}}$, $\psi(\overline{\boldsymbol{x}},\overline{\boldsymbol{y}})$ can be smaller than $\frac{\beta}{2}$ for ϵ small enough, contradicting (26).

Above all, $V_1\leq V_2$ over $\boldsymbol{x}\in\mathbb{R}^n$. It is evident that if $U(\boldsymbol{x})$ is a bounded continuous viscosity solution to Eq. (10), then $U(\boldsymbol{x})=V^-(\boldsymbol{x})$ over $\boldsymbol{x}\in\mathbb{R}^n$, due to the fact that $U(\boldsymbol{x})$ and $V^-(\boldsymbol{x})$ are both sub and superviscosity solutions. Therefore, the uniqueness of the bounded continuous solutions to Eq. (10) is guaranteed. ■