Reach-Avoid Analysis for Delay Differential Equations

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Abstract— **Time-delay systems are ubiquitous in nature and occur in connection with various aspects of physical, chemical, biological and economic systems. In this paper we propose a semi-definite programming method to address reach-avoid problems for time-delay systems modeled by polynomial delay differential equations (DDEs). The reach-avoid problem of interest is to compute an inner-approximation (i.e., sub-set) of a reach-avoid set, which is the set of initial functions enabling the time-delay system to eventually enter a desirable target set while remaining inside a specified safe set till the target hit. In our approach we first derive an estimate of discrepancies between current states and delayed ones for the time-delay system, and then propose a semi-definite program for inner-approximating the reach-avoid set via incorporating the discrepancy estimate. The incorporation of discrepancy estimates facilitates reduction of conservativeness in computing inner-approximations. Finally, three examples with comparisons are presented to demonstrate the performance of our approach.**

I. Introduction

Cyber-physical systems, which are mechanisms representing the tight integration and interaction of multiple physical and software components, are becoming an integral part of our daily lives [12]. Examples of such systems range from robots, avionic systems and medical devices to smart grids and automotive networks. Many of these are safety-critical and require essential guarantees of safe operations before deployment. This raises the question of whether the system's behavior satisfies given safety specifications. Reachability analysis, i.e., computation of reach states, is a popular tool for addressing this problem [5]. Because of insurmountable difficulties in computing exact reachable sets, especially for nonlinear systems, their outer (i.e., super-sets) and inner (i.e., sub-sets) approximations are often resorted to in the formal methods community, e.g., [2], [20]. Generally, outer approximations are the main tool in justifying the separation of reach states of systems originating inside a defined set of initial states from a set of unsafe states, and inner-approximations are used to determine a set of initial states rendering systems safe.

Particularly, reach-avoid analysis is able to guarantee both safety and performance of systems, and can formalize many important engineering problems such as collision avoidance [14] and target surveillance [8], consequently attracting wide attention from both academic and industrial communities. It is mainly concerned with the computation of reach-avoid sets, which are the set of initial configurations such that the system originating inside it will eventually hit a desired target set over either given finite time horizons or open, i.e. not bounded a priori, time horizons while remaining inside a specified safe set prior to hitting the target set.

Existing works involving reach-avoid analysis for continuous-time systems pay high attention to ordinary differential equations (ODEs) without accounting for time delays [7], which are mathematical models widely used in various areas of engineering and sciences such as mechanical and aerospace engineering. However, ODEs fail to capture dynamics induced by time delays, which are ubiquitous in nature and especially man-made control systems. Time delays are involved in sensing or actuating by physical devices, in data forwarding to or from the controller, in signal processing in the controller, etc. Unfortunately, in various applications they often cause oscillations, affect the damping reduction and may even cause instability, thus resulting in system performance deterioration [1]. Consequently, ODEs may be an idealized model of feedback dynamics in control systems. As a generalization of ODEs, DDEs, which are originally suggested by Bellman and Cooke for modeling physical and chemical processes involving delayed dynamics [3], are able to incorporate time delays within the framework of differential equations. DDEs are a class of differential equations where the derivative at the current time depends on values of the state in the past, leading to a mapping in a function space which is infinite dimensional and thus rendering the reach-avoid analysis more challenging than their ODE counterparts. Most of the available analysis results in existing literature (e.g., [18]) for DDEs have placed emphasis on stability, robustness, or input-output properties rather than reachability, let alone reach-avoid analysis.

In this paper we propose a semi-definite programming method for inner-approximating reach-avoid sets of time-delay systems modeled by polynomial DDEs over open time horizons. *To the best of our knowledge, this is the first work on studying the reach-avoid problem over open time horizons akin to DDEs.* Different from traditional methods (e.g., [17]) of simply regarding delayed states as disturbances, which are independent of current states and thus introduce significant conservativeness in conducting reachability analysis for DDEs, we explore the relationship between delayed states and current ones, and derive an estimate of discrepancies between them. This estimate is important and will facilitate reduction of the conservativeness in inner-approximating reach-avoid sets. Then, via incorporating the estimate, a computationally efficient semi-definite programming method, which is inspired by the one in our previous work [22] for discrete-time systems, is proposed for inner-approximating the reach-avoid set. Finally, we demonstrate the proposed method on three examples.

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Related Work

As surveyed in [7], the research community has over the past three decades witnessed the rapid development of various formal methods for addressing automatic verification of (hybrid) systems in a safety-critical context. Most of these methods are specific to systems, whose continuous dynamics are described by ODEs without accounting for time delays.

Stimulated by actual engineering problems, the interest in safety verification of continuous systems featuring delayed coupling is increasing recently. The contributions are mainly falling into two categories. The first one pursues propagation-based verification over bounded time horizons: a method for simulation-based time-bounded invariant verification of nonlinear networked dynamical systems with delayed interconnections was presented in [11], by analyzing sensitivity of trajectories with respect to initial states and inputs of the system. A class of (perturbed) DDEs featuring a local homeomorphism property was identified in [19], [21]. The homeomorphism analysis facilitates construction of over- and under-approximations of reachable sets over bounded time horizons by just performing reachability analysis on the boundaries of initial sets as in [20]. Recently, an approach combining Taylor models and the method of steps was proposed to inner- and outer-approximating flowpipes for DDEs with uncertain initial states and parameters in [9]. The second one in existing literature tackles safety verification problem of DDEs over the unbounded time horizon. The most notable method is based on the computation of barrier certificates [17], which separate the state space of a considered system into safe and unsafe parts. The other method is by taking into account stability properties of the dynamics under investigation. A safe enclosure method using interval-based Taylor over-approximation was proposed in [23] for verifying stability and safety properties of a simple class of DDEs. Recently, a method based on linearization techniques and spectral analysis was proposed in [6] for verifying safety properties of exponentially stable systems described by DDEs.

Many of the aforementioned methods are for the computation of outer-approximations of forward reach sets, which are the set of all reachable states for the time-delay system starting from a given set of initial functions, although some exist for inner-approximations over given bounded time horizons such as [19], [21] and [9]. *In contrast, the method in the present work attempts to compute inner-approximations of reach-avoid sets over open time horizons, which is in contrast to determine a set of initial functions such that the time-delay system starting from them respects specified reach-avoid requirements.*

The structure of this paper is presented below. In Section II time-delay systems and reach-avoid problems of interest are introduced. After detailing our semi-definite programming based method for solving the reach-avoid problem in Section III, we demonstrate it on several examples in Section IV and finally conclude this paper in Section V.

Notations. $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ denote the set of real numbers and nonnegative real numbers, respectively. $\mathbb{R}^n$ is an n-dimensional real Euclidean space with norm $\|\cdot\|$, i.e., $\|\boldsymbol{x}\| = \sqrt{\sum_{i=1}^n x_i^2}$ with $\boldsymbol{x} \in \mathbb{R}^n$. Given two sets Δ and Δ', the sets $\mathcal{C}(\Delta, \Delta')$ and $\mathcal{C}^1(\Delta, \Delta')$ denote the Banach spaces of continuous functions and continuously differentiable functions mapping the set Δ onto Δ', respectively. For $\boldsymbol{\phi} \in \mathcal{C}([a,b], \mathbb{R}^n)$, the norm of $\boldsymbol{\phi}$ is defined as $\|\boldsymbol{\phi}\| = \sup_{a\leq\theta\leq b}\|\boldsymbol{\phi}(\theta)\|$. For a set Δ, the sets Δ^c, $\overline{\Delta}$ and $\partial\Delta$ denote its complement, closure and boundary, respectively. $\mathbb{R}[\cdot]$ denotes the ring of polynomials in variables given by the argument. Vectors are denoted by boldface letters. Besides, we use $\sum[\boldsymbol{x}]$ to represent the set of sum-of-squares polynomials over variables $\boldsymbol{x}$, i.e.,

$$\sum[\boldsymbol{x}] = \{p \in \mathbb{R}[\boldsymbol{x}] \mid p = \sum_{i=1}^{k'} q_i^2, q_i \in \mathbb{R}[\boldsymbol{x}], i = 1, \dots, k'\}.$$

II. Preliminaries

In this section we formally present the concepts of polynomial DDEs and reach-avoid sets of interest in this paper.

A. Time-delay Systems

The system of interest in this paper is a time-delay dynamical system modeled by DDEs of the following form

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{x}(t-r)), \tag{1}$$

where $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}[\boldsymbol{x}, \boldsymbol{y}]$ and $r \in \mathbb{R}_{\geq 0}$.

Unlike ODEs, an initial condition for system (1) is not simply $\boldsymbol{x}(0) \in \mathbb{R}^n$ but rather a whole function $\boldsymbol{x}_0(\cdot) \in \mathcal{C}([-r,0], \mathbb{R}^n)$, i.e., $\boldsymbol{x}_0(\cdot): [-r,0] \to \mathbb{R}^n$, and $\boldsymbol{x}_0(\theta) = \boldsymbol{x}(\theta)$ for $\theta \in [-r,0]$. A trajectory of system (1) starting from a given initial condition $\boldsymbol{x}_0(\cdot) \in \mathcal{C}([-r,0], \mathbb{R}^n)$ over a time horizon $[0, T_{\boldsymbol{x}_0})$, is denoted as $\boldsymbol{\phi}_t(\cdot) \in \mathcal{C}([-r,0], \mathbb{R}^n)$, where $t \in [0, T_{\boldsymbol{x}_0})$ with $T_{\boldsymbol{x}_0}$ being either a positive value or ∞, $\boldsymbol{\phi}_0(\theta) = \boldsymbol{x}_0(\theta)$ and $\boldsymbol{\phi}_t(\theta) = \boldsymbol{\phi}(t+\theta)$ for $\theta \in [-r,0]$.

In addition to the above, two bounded open sets in the n-dimensional Euclidean space are given: a safe state set $\mathcal{X} \subseteq \mathbb{R}^n$ and a target set $\mathcal{T} \subseteq \mathbb{R}^n$ with $\mathcal{T} \subseteq \mathcal{X}$, where both

$$\mathcal{X} = \{\boldsymbol{x} \in \mathbb{R}^n \mid h(\boldsymbol{x}) < 1\}$$

and

$$\mathcal{T} = \{\boldsymbol{x} \in \mathbb{R}^n \mid g(\boldsymbol{x}) < 1\}$$

with $h(\boldsymbol{x}), g(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$.

A reach-avoid set of interest in this paper is formally presented in Definition 1.

Definition 1: A reach-avoid set RA is the set of initial functions in $\mathcal{C}([-r,0], \mathcal{X}\setminus\mathcal{T})$ such that system (1) originating from it will enter the target set $\mathcal{T}$ in finite time while remaining inside the set $\mathcal{X}$ prior to the target hitting time, i.e., RA =

$$\left\{\boldsymbol{\phi}_0(\cdot) \in \mathcal{C}([-r,0], \mathcal{X}\setminus\mathcal{T}) \,\middle|\, \begin{array}{l} \exists t \in \mathbb{R}_{\geq 0}.\boldsymbol{\phi}_t(0) \in \mathcal{T} \bigwedge \\ \forall \tau \in [0,t].\boldsymbol{\phi}_\tau(0) \in \mathcal{X} \end{array}\right\}.$$

An inner-approximation is a subset of the set RA.

It is worth remarking here that trajectories starting from the reach-avoid set RA are allowed to leave the target set $\mathcal{T}$

after reaching it. In addition, the reason that we require the codomain of an initial function $\phi_0(\cdot)$ in the set RA to be the set $\mathcal{X} \setminus \mathcal{T}$ rather than the set $\mathcal{X}$ is to exclude the trivial set $\{\phi_0(\cdot) \in \mathcal{C}([0,T],\mathcal{X}) \mid \phi_0(0) \in \mathcal{T}\}$.

III. Inner-approximating Reach-avoid Sets

In this section we present our semi-definite programming approach for inner-approximating the reach-avoid set RA of system (1). An estimate of discrepancies between current states and delayed ones for the system (1) is presented in Subsection III-A. Based on the estimate and the semi-definite programming formulation in [22], Subsection III-B presents a semi-definite programming approach for synthesizing inner-approximations of the reach-avoid set RA.

A. Discrepancy Estimation

This subsection gives an estimate of the discrepancy between the current state $\boldsymbol{x}(t)$ in $\mathbb{R}^n$ and its delayed one $\boldsymbol{x}(t-r)$ for system (1) starting from an initial function $\boldsymbol{x}_0(\cdot) \in \mathcal{C}([-r,0],\mathcal{X})$. This estimate will facilitate the conservativeness reduction of inner-approximating the reach-avoid set RA, which is reflected in Section IV.

Proposition 1: Given $\boldsymbol{x}_0(\cdot) \in \mathcal{C}([-r,0],\mathcal{X})$, and $t_1, t_2 \in \mathbb{R}_{\geq 0}$ satisfying that

$$\phi_t(0) \in \overline{\mathcal{X}}, \forall t \in [0, \max\{t_1,t_2\}],$$

where $\phi_0(\theta) = \boldsymbol{x}_0(\theta)$ for $\theta \in [-r,0]$, then

$$\|\phi_{t_1}(0) - \phi_{t_2}(0)\| \leq M|t_1 - t_2|, \tag{2}$$

where $M \geq \sup_{\boldsymbol{x},\boldsymbol{y} \in \overline{\mathcal{X}}} \|\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})\|$.

Proof: Let $\tau = \min\{t_1,t_2\}$ and $\tau' = \max\{t_1,t_2\}$. Since

$$\phi_{t_1}(0) = \phi_0(0) + \int_0^{t_1} \boldsymbol{f}(\phi_t(0), \phi_t(-r))dt$$

and

$$\phi_{t_2}(0) = \phi_0(0) + \int_0^{t_2} \boldsymbol{f}(\phi_t(0), \phi_t(-r))dt$$

according to Lemma 1 in [1], we have that

$$\|\phi_{t_1}(0) - \phi_{t_2}(0)\| = \|\int_{\tau}^{\tau'} \boldsymbol{f}(\phi_t(0), \phi_t(-r))dt\|.$$

Also, since $\phi_t(\theta) \in \overline{\mathcal{X}}$ for $t \in [0,\tau']$ and $\theta \in [-r,0]$,

$$\|\boldsymbol{f}(\phi_t(0), \phi_t(-r))\| \leq M, \forall t \in [0,\tau']$$

holds and therefore we have that

$$\|\phi_{t_1}(0) - \phi_{t_2}(0)\| \leq M|t_2 - t_1|.$$

This completes the proof. ■

Therefore, according to Proposition 1, if

$$\|\boldsymbol{x}_0(\theta) - \boldsymbol{x}_0(0)\| \leq -M\theta$$

for $\theta \in [-r,0]$, we have that

$$\|\phi_t(0) - \phi_t(-r)\| \leq Mr, \forall t \in [0, \max\{t_1,t_2\}]. \tag{3}$$

An estimate of the bound M satisfying inequality (2) in Proposition 1 can be computed via solving the semi-definite program (4). M can take $\sqrt{\hat{M}}$.

B. Inner-Approximating Reach-Avoid Sets

Based on the discrepancy estimate (3) in Proposition 1 and constraints addressing the reach-avoid problem for discrete-time systems in [22], we construct constraints for inner-approximating the reach-avoid set RA of system (1).

The functional structure mainly used for illustration in this paper depends only on the "head" of the functional $\boldsymbol{x}_t$, and is of the following form:

$$v(\boldsymbol{x}_t) = v_0(\boldsymbol{x}(t)),$$

where $v_0 \in \mathcal{C}^1(\overline{\mathcal{X}},\mathbb{R})$. It is the kind of functions used in the Lyapunov-Razumikhin theorem for proving delay-independent stability of time-delay systems [10], [16]. In [17] such functions were used to construct delay-independent conditions guaranteeing the satisfaction of specified safety properties. Delayed states in delay-independent conditions are regarded as disturbance inputs which are independent of current states. This introduces significant conservativeness and complicates the (already difficult) problem solvability, resulting in empty resets for many cases. This statement is further confirmed in Section IV. In order to reduce this conservativeness, we formulate delay-dependent conditions for inner-approximating the reach-avoid set RA via incorporating constraint (2). Our delay-dependent conditions are presented formally in Lemma 1.

Lemma 1: Given system (1), the safe state set $\mathcal{X} \subseteq \mathbb{R}^n$ and the target set $\mathcal{T} \subseteq \mathcal{X}$ be given, if there exist functions $v_0 \in \mathcal{C}^1(\overline{\mathcal{X}},\mathbb{R})$ and $u_0 \in \mathcal{C}^1(\overline{\mathcal{X}},\mathbb{R})$ that satisfy the following delay-dependent conditions:

$$-\frac{\partial v_0(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) \geq 0, \forall \boldsymbol{x} \in \mathcal{X}, \forall \boldsymbol{y} \in \mathcal{B}(\boldsymbol{x},Mr) \cap \mathcal{X}, \tag{6}$$

$$v_0(\boldsymbol{x}) \geq g(\boldsymbol{x}) + \frac{\partial u_0(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}), \quad \forall \boldsymbol{x} \in \mathcal{X}, \forall \boldsymbol{y} \in \mathcal{B}(\boldsymbol{x},Mr) \cap \mathcal{X}, \tag{7}$$

$$v_0(\boldsymbol{x}) \geq 1, \forall \boldsymbol{x} \in \partial\mathcal{X}, \tag{8}$$

where $\mathcal{B}(\boldsymbol{x},Mr)$ is an n-ball of radius Mr and center $\boldsymbol{x}$, i.e., $\mathcal{B}(\boldsymbol{x},Mr) = \{\boldsymbol{y} \in \mathbb{R}^n \mid \|\boldsymbol{x}-\boldsymbol{y}\|^2 \leq M^2r^2\}$, then the set IN =

$$\left\{ \begin{array}{c|c} \boldsymbol{x}_0(\cdot) \in & v_0(\boldsymbol{x}_0(0)) < 1 \bigwedge \forall \theta \in [-r,0]. \\ \mathcal{C}([-r,0],\mathcal{X} \setminus \mathcal{T}) & \|\boldsymbol{x}_0(\theta) - \boldsymbol{x}_0(0)\| \leq -M\theta \end{array} \right\}$$

is an inner-approximation of the reach-avoid set RA.

Proof: Assume that

1) $\boldsymbol{x}_0'(\cdot) \in$ IN,
2) there does not exist $t \in \mathbb{R}_{\geq 0}$ such that

$$\phi_t(0) \in \mathcal{T} \bigwedge \forall \tau \in [0,t].\phi_\tau(0) \in \mathcal{X},$$

where $\phi_0(\theta) = \boldsymbol{x}_0'(\theta)$ for $\theta \in [-r,0]$.

Then, we have that either

$$\exists t \in \mathbb{R}_{\geq 0}.[\phi_t(0) \in \partial\mathcal{X} \bigwedge \forall \tau \in [0,t).\phi_\tau(0) \in \mathcal{Y}]$$

or

$$\forall t \in \mathbb{R}_{\geq 0}.\phi_t(0) \in \mathcal{Y},$$

where $\mathcal{Y} = \mathcal{X} \setminus \mathcal{T}$.

$$
\begin{aligned}
\min \quad & \hat{M} \\
\text{s.t.} \quad & \hat{M} - \|\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})\|^2 + s_1(\boldsymbol{x},\boldsymbol{y})(h(\boldsymbol{x})-1) + s_2(\boldsymbol{x},\boldsymbol{y})(h(\boldsymbol{y})-1) \in \sum[\boldsymbol{x},\boldsymbol{y}], \\
& s_1(\boldsymbol{x},\boldsymbol{y}) \in \sum[\boldsymbol{x},\boldsymbol{y}], s_2(\boldsymbol{x},\boldsymbol{y}) \in \sum[\boldsymbol{x},\boldsymbol{y}]
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
&\inf \ \boldsymbol{c}^\top \cdot \boldsymbol{w} \\
&\text{s.t.} \\
&-\frac{\partial v_0(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) + s_1(\boldsymbol{x},\boldsymbol{y})(h(\boldsymbol{x})-1) + s_2(\boldsymbol{x},\boldsymbol{y})(\|\boldsymbol{x}-\boldsymbol{y}\|^2 - M^2r^2) + s_3(\boldsymbol{x},\boldsymbol{y})(h(\boldsymbol{y})-1) \in \sum[\boldsymbol{x},\boldsymbol{y}], \\
&v_0(\boldsymbol{x}) - g(\boldsymbol{x}) - \frac{\partial u_0(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) + s_4(\boldsymbol{x},\boldsymbol{y})(h(\boldsymbol{x})-1) + s_5(\boldsymbol{x},\boldsymbol{y})(\|\boldsymbol{x}-\boldsymbol{y}\|^2 - M^2r^2) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + s_6(\boldsymbol{x},\boldsymbol{y})(h(\boldsymbol{y})-1) \in \sum[\boldsymbol{x},\boldsymbol{y}], \\
&v_0(\boldsymbol{x}) - 1 + p(\boldsymbol{x})(h(\boldsymbol{x})-1) \in \sum[\boldsymbol{x}],
\end{aligned} \tag{5}
$$

where $\boldsymbol{c}^\top \cdot \boldsymbol{w} = \int_{\mathcal{X}} v_0(\boldsymbol{x})d\boldsymbol{x}$, $\boldsymbol{w}$ is the constant vector computed by integrating the monomials in $v_0(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$ over $\mathcal{X}$, $\boldsymbol{c}$ is the vector composed of unknown coffecients in $v_0(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$, and $u_0(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$, $p(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$ and $s_i(\boldsymbol{x},\boldsymbol{y}) \in \sum[\boldsymbol{x},\boldsymbol{y}]$, $i = 1, \ldots, 6$.

Due to constraints (6), (8) and $v_0(\boldsymbol{\phi}_0(0)) < 1$, we conclude that there does not exist $t \in \mathbb{R}_{\geq 0}$ such that

$$\boldsymbol{\phi}_t(0) \in \partial\mathcal{X} \bigwedge \forall \tau \in [0,t).\boldsymbol{\phi}_t(0) \in \mathcal{Y}.$$

Consequently, we have the only one choice that $\boldsymbol{\phi}_t(0) \in \mathcal{Y}$ for $t \in \mathbb{R}_{\geq 0}$, i.e.,

$$g(\boldsymbol{\phi}_t(0)) \geq 1$$

for $t \in \mathbb{R}_{\geq 0}$. However, constraint (7) indicates that for $t \in \mathbb{R}_{\geq 0}$,

$$v_0(\boldsymbol{\phi}_t(0)) \geq g(\boldsymbol{\phi}_t(0)) + \frac{\partial u_0(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\boldsymbol{\phi}_t(0)} \boldsymbol{f}(\boldsymbol{\phi}_t(0), \boldsymbol{\phi}_t(-r)),$$

which implies that

$$
\begin{aligned}
\int_0^\tau v_0(\boldsymbol{\phi}_t(0))dt \geq & \int_0^\tau g(\boldsymbol{\phi}_t(0))dt \\
& + \int_0^\tau \frac{\partial u_0(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\boldsymbol{\phi}_t(0)} \boldsymbol{f}(\boldsymbol{\phi}_t(0), \boldsymbol{\phi}_t(-r))dt
\end{aligned}
$$

and further, together with (6), that for $\tau \in \mathbb{R}_{\geq 0}$,

$$v_0(\boldsymbol{\phi}_0(0)) \geq \frac{\int_0^\tau g(\boldsymbol{\phi}_t(0))dt}{\tau} + \frac{u_0(\boldsymbol{\phi}_\tau(0)) - u_0(\boldsymbol{\phi}_0(0))}{\tau}.$$

Because of $u_0 \in \mathcal{C}^1(\overline{\mathcal{X}}, \mathbb{R})$, $u_0(\boldsymbol{x})$ is bounded over $\boldsymbol{x} \in \overline{\mathcal{X}}$. Therefore,

$$v_0(\boldsymbol{\phi}_0(0)) \geq \lim \inf_{\tau \to \infty} \frac{\int_0^\tau g(\boldsymbol{\phi}_t(0))dt}{\tau} \geq 1,$$

contradicting $v_0(\boldsymbol{\phi}_0(0)) < 1$.

Consequently, there exists $t \in \mathbb{R}_{\geq 0}$ such that

$$\boldsymbol{\phi}_t(0) \in \mathcal{T} \bigwedge \forall \tau \in [0,t].\boldsymbol{\phi}_\tau(0) \in \mathcal{X}$$

and thus $\boldsymbol{x}'_0(\cdot) \in \mathrm{RA}$, which implies that $\mathrm{IN} \subseteq \mathrm{RA}$. ■

According to Lemma 1, if we obtain a pair of continuously differentiable functions $v_0(\boldsymbol{x}), u_0(\boldsymbol{x}) \in \mathcal{C}^1(\overline{\mathcal{X}}, \mathbb{R})$ satisfying constraints (6)-(8), an inner-approximation of the reach-avoid set RA follows. General nonlinear functions $v_0(\boldsymbol{x}), u_0(\boldsymbol{x}) \in \mathcal{C}^1(\overline{\mathcal{X}}, \mathbb{R})$ are challenging to obtain. However, when $v_0(\boldsymbol{x})$ and $u_0(\boldsymbol{x})$ are searched in the space of polynomials, i.e., $v_0(\boldsymbol{x}), u_0(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$, the problem of solving constraints (6)-(8) can be reduced to the semi-definite programming problem (5) via the sum-of-squares decomposition for multivariate polynomials. The semi-definite program is a kind of convex programs, which can be solved efficiently in polynomial time via interior-point methods.

Theorem 1: Given system (1), sets $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{T} \subseteq \mathcal{X}$, if there exist functions $v_0(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$ and $u_0(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$ that satisfy the semi-definite program (5), then

$$\left\{ \begin{array}{c} \boldsymbol{x}_0(\cdot) \in \\ \mathcal{C}([-r,0], \mathcal{X} \setminus \mathcal{T}) \end{array} \middle| \begin{array}{c} v_0(\boldsymbol{x}_0(0)) < 1 \bigwedge \forall \theta \in [-r,0]. \\ \|\boldsymbol{x}_0(\theta) - \boldsymbol{x}_0(0)\| \leq -M\theta \end{array} \right\}$$

is an inner-approximation of the reach-avoid set RA.

Proof: The conclusion follows from Lemma 1 and $\mathcal{S}-$procedure presented in [4]. ■

It follows from Theorem 1 that an inner-approximation of the reach-avoid set RA can be computed via solving the semi-definite program (5).

IV. Examples

In this section we demonstrate our semi-definite programming approach on three examples. All computations on solving (4) and (5) were performed on an i7-7500U 2.70GHz CPU with 32GB RAM running Windows 10. In the computations the sum-of-squares module of YALMIP [13] was first used to transform the sum-of-squares optimization

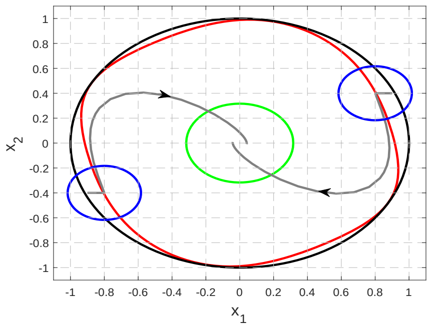

Fig. 1. Illustration of the computed inner-approximation of the set RA for Example 1. Black and green curves denote the boundaries of safe state set $\mathcal{X}$ and target set $\mathcal{T}$, respectively. The red curve denotes the boundary of the set $\{\boldsymbol{x} \in \mathcal{X} \mid v_0(\boldsymbol{x}) < 1\}$. The gray curves denote the trajectories of system (9) starting from initial conditions $\boldsymbol{x}_0(\theta) = (0.8-1.0\theta, 0.4)^\top$ and $\boldsymbol{x}_0(\theta) = (-0.8-1.0\theta, -0.4)^\top$ for $\theta \in [-0.1, 0]$, respectively. The blue curves denote the boundaries of sets $\{\boldsymbol{x} \mid (x_1-0.8)^2 + (x_2-0.4)^2 \leq M^2r^2\}$ and $\{\boldsymbol{x} \mid (x_1+0.8)^2 + (x_2+0.4)^2 \leq M^2r^2\}$, respectively.

problems (4) and (5) into semi-definite programs, and the solver Mosek [15] was then used to solve them.

Example 1: Consider a linear damped oscillator with delays in [17],

$$\begin{cases} \dot{x}_1(t) = \gamma x_2(t) + (1-\gamma)x_2(t-r) \\ \dot{x}_2(t) = -\gamma x_1(t) - (1-\gamma)x_1(t-r) - 1.5x_2(t) \end{cases}. \quad (9)$$

In this system, $r > 0$ is the delay, and $\gamma \in [0, 1]$ is a parameter.

In this example we assume that $r = 0.1$, $\gamma = 0.5$, the target set

$$\mathcal{T} = \{\boldsymbol{x} \in \mathbb{R}^2 \mid g(\boldsymbol{x}) < 1\}$$

with $g(\boldsymbol{x}) = 10x_1^2 + 10x_2^2$, and the safe set

$$\mathcal{X} = \{\boldsymbol{x} \in \mathbb{R}^2 \mid h(\boldsymbol{x}) < 1\}$$

with $h(\boldsymbol{x}) = x_1^2 + x_2^2$. Two cases, with and without considering discrepancy characterization (3), are discussed below.

1) If discrepancy characterization (3) is not considered, via the solving semi-definite program (5) without terms pertinent to sum-of-squares polynomials $s_2(\boldsymbol{x}, \boldsymbol{y})$ and $s_5(\boldsymbol{x}, \boldsymbol{y})$, we obtain an empty set, which is a correct but useless inner-approximation. The corresponding parameters for solving the semi-definite program (5) are listed in Table II.
2) If discrepancy characterization (3) is considered, $M = \sqrt{4.7}$ is computed via solving the semi-definite program (4) with parameters in Table I, and then a non-empty inner-approximation of the reach-avoid set RA is obtained via solving the semi-definite program (5). The related parameters in solving semi-definite program (5) are presented in Table II and the computed set $\{\boldsymbol{x} \in \mathcal{X} \mid v_0(\boldsymbol{x}) < 1\}$ is illustrated in Fig. 1.

Example 2: Consider a PD-controller with linear dynamics in [9] described by

$$\begin{cases} \dot{y}(t) = v(t) \\ \dot{v}(t) = -K_p(y(t-r) - y^*) - K_d v(t-r) \end{cases}, \quad (10)$$

which controls the position y and velocity v of an autonomous vehicle by adjusting its acceleration according to the current distance to a reference position y^*. A constant time delay r is introduced to model the time lag due to sensing, computation, transmission, and/or actuation. We instantiate the parameters as $K_p = K_d = 0.5$, $y^* = 1$ and $r = 0.1$. The system described by Eq. (10) then has one equilibrium at $(1, 0)^\top$, which shares equivalent stability with the zero equilibrium of the following system, with $x_1 = y-1$ and $x_2 = v$:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -0.5(x_2(t-r) - x_2^*) - 0.5x_1(t-r) \end{cases}. \quad (11)$$

Suppose the safe state set and the target set are

$$\mathcal{X} = \{\boldsymbol{x} \mid x_1^2 + x_2^2 < 1\}$$

and

$$\mathcal{T} = \{\boldsymbol{x} \mid 10x_1^2 + 10(x_2+0.15)^2 < 1\},$$

respectively. Similarly, we demonstrate our approach on the following two cases.

1) If discrepancy characterization (3) is not taken into account, an empty set $\{\boldsymbol{x} \in \mathcal{X} \mid v_0(\boldsymbol{x}) < 1\}$ is computed via solving the semi-definite program (5) without terms pertinent to sum-of-squares polynomials $s_2(\boldsymbol{x}, \boldsymbol{y})$ and $s_5(\boldsymbol{x}, \boldsymbol{y})$. Thus, an empty inner-approximation of the reach-avoid set RA is obtained. The corresponding parameters in solving the semi-definite program (5) are listed in Table II.
2) If discrepancy characterization (2) is considered, we obtain $M = \sqrt{1.26}$ via solving the semi-definite program (4) with parameters listed in Table I and then compute an inner-approximation of the reach-avoid set RA via solving the semi-definite program (5). The corresponding parameters in solving (5) are listed in Table II. The computed set $\{\boldsymbol{x} \in \mathcal{X} \mid v_0(\boldsymbol{x}) < 1\}$ is illustrated in Fig. 2. Two trajectories are also showcased in Fig. 2, where one starting from the initial condition $\boldsymbol{x}_0(\theta) = (0.2-0.8\theta, 0.85-0.8\theta)^\top$ for $\theta \in [-0.1, 0]$ satisfies the reach-avoid specification while the other starting from $\boldsymbol{x}_0(\theta) = (0.75, 0.6)^\top$ for $\theta \in [-0.1, 0]$ fails.

Example 3: In this example we consider a two-dimensional example of the following form,

$$\begin{cases} \dot{x}_1(t) = x_1(t)[b - ax_1(t) - k_1x_2(t)] \\ \dot{x}_2(t) = -\sigma x_2(t) + k_2x_1(t-r)x_2(t-r) \end{cases}. \quad (12)$$

This delay differential equation was a model for predator-prey populations in [16].

Let $a = -0.01$, $b = -1$, $k_1 = -2$, $\sigma = 1$ and $k_2 = 2$. In this example we assume that $r = 0.0001$, the target set

$$\mathcal{T} = \{\boldsymbol{x} \in \mathbb{R}^2 \mid g(\boldsymbol{x}) < 1\}$$

with $g(\boldsymbol{x}) = 10x_1^2 + 10x_2^2$ and the safe set

$$\mathcal{X} = \{\boldsymbol{x} \in \mathbb{R}^2 \mid h(\boldsymbol{x}) < 1\}$$

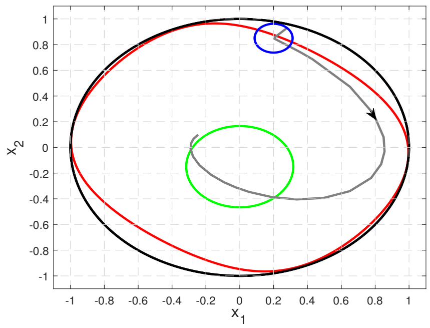

Fig. 2. Illustration of the computed inner-approximation of the set RA for Example 2. Black and green curves denote the boundaries of safe set $\mathcal{X}$ and target set $\mathcal{T}$. The red curve denotes the boundary of the set $\{\boldsymbol{x} \in \mathcal{X} \mid v_0(\boldsymbol{x}) < 1\}$. The blue curve denotes the boundary of $\{\boldsymbol{x} \mid (x_1 - 0.2)^2 + (x_2 - 0.85)^2 \leq M^2 r^2\}$. The gray curve denotes the trajectory of system (11) starting from the initial condition $\boldsymbol{x}_0(\theta) = (0.2-0.8\theta, 0.85-0.8\theta)^\top$.

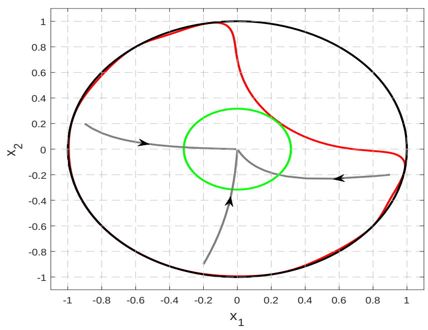

Fig. 3. Illustration of the computed inner-approximation of the set RA for Example 3. Black and green curves denote the boundaries of the safe set $\mathcal{X}$ and target set $\mathcal{T}$. The red curve denotes the boundary of the set $\{\boldsymbol{x} \in \mathcal{X} \mid v_0(\boldsymbol{x}) < 1\}$. The gray curves denote the trajectories of system (12) starting from initial conditions $\boldsymbol{x}_0(\theta) = (-0.9, 0.2)^\top$, $\boldsymbol{x}_0(\theta) = (0.9, -0.2)^\top$ and $\boldsymbol{x}_0(\theta) = (-0.2, -0.9)^\top$ for $\theta \in [-0.001, 0]$ respectively.

with $h(\boldsymbol{x}) = x_1^2 + x_2^2$.

Similar to the first example, we discuss our approach on the following two cases.

1) if discrepancy characterization (3) is not considered, we solve the semi-definite program (5) without terms pertinent to sum-of-squares polynomials $s_2(\boldsymbol{x}, \boldsymbol{y})$ and $s_5(\boldsymbol{x}, \boldsymbol{y})$, an empty set $\{\boldsymbol{x} \in \mathcal{X} \mid v_0(\boldsymbol{x}) < 1\}$ is computed. Thus, an empty set, which is a correct but useless inner-approximation, is computed. The parameters in solving (5) are listed in Table II.
2) If taking discrepancy characterization (3) into account, we first compute $M = \sqrt{5.9}$ via solving the semi-definite program (4) with parameters listed in Table I, and then obtain a non-empty inner-approximation of the reach-avoid set RA via solving the semi-definite program (5) with parameters listed in Table II. The computed set $\{\boldsymbol{x} \in \mathcal{X} \mid v_0(\boldsymbol{x}) < 1\}$ via solving the semi-definite program (5) is illustrated in Fig. 3.

It is concluded from the above three examples that discrepancy characterization (2) indeed facilitates conservativeness reduction of inner-approximating reach-avoid sets for some time-delay systems, and we can compute an (non-empty) inner-approximation of the reach-void set via solving the semi-definite program (5).

Examples	d_{s_1}	d_{s_2}	Time(s)
Ex.1	4	4	0.89
Ex.2	4	4	0.86
Ex.3	4	4	1.28

TABLE I

Parameters for solving the semi-definite program (4) *in three Examples.* d_{s_1} *and* d_{s_2} *denote the degree of the polynomial* $s_1(\boldsymbol{x}, \boldsymbol{y})$ *and* $s_2(\boldsymbol{x}, \boldsymbol{y})$ *respectively.* Time *denotes the computation time (in seconds).*

Examples	Cases	d_{v_0}	d_{u_0}	d_{s_i}	d_p	Time(s)
Ex.1	1)	6	6	8	8	12.04
	2)	6	6	8	8	19.59
Ex.2	1)	6	6	6	6	3.03
	2)	6	6	6	6	6.24
Ex.3	1)	10	10	10	10	227.36
	2)	10	10	10	10	125.55

TABLE II

Parameters for solving the semi-definite program (5) *in three Examples.* d_{v_0}, d_{u_0}, d_{s_i} *and* d_p *denote the degree of the polynomial* $v_0(\boldsymbol{x})$, $u_0(\boldsymbol{x})$, $s_i(\boldsymbol{x}, \boldsymbol{y})$ *and* $p(\boldsymbol{x})$ *respectively.* Time *denotes the computation time.*

V. Conclusion

In this paper we proposed a semi-definite programming method for inner-approximating reach-avoid sets of time-delay systems modeled by polynomial DDEs over open time horizons. The reach-avoid set is a set of initial functions enabling the system to reach a desired target set in finite time while remaining inside certain legal state set preceding the target hit. We for the first study the reach-avoid problem over open time horizons akin to DDEs. Besides, unlike traditional methods of simply regarding delayed states as disturbances and formulating delay-independent constraints in studying reachability for DDEs, we formulated delay-dependent constraints for computing reach-avoid sets from the inner. The constraints were efficiently solved within the semi-definite programming framework. Finally, three examples demonstrated the proposed method.

In our future work we would extend our approach to reach-avoid analysis for (hybrid) time-delay systems with competing inputs (i.e., disturbance and control inputs).

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