A Generalized Gelfond-Lifschitz Transformation for Logic Programs with Abstract Constraints

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Background

- Answer Set Programming (ASP)
  - Logic programming with the stable model semantics; an effective formalism for solving combinatorial search problems
- Logic Programs with Abstract Constraints
  - Extensions of ASP with means to model aggregate constraints in particular, and abstract constraints on sets in general
  - Represent and reason with sets of atoms, in contrast with traditional logic programs primarily for reasoning with individuals (Marek & Remmel 2004; Marek & Truszczynski 2004)

Background

- Abstract Constraint Atoms (C-Atoms)
  - A *c-atom* $\boldsymbol{A = (A_d, A_c)}$, where $\boldsymbol{A_d}$ is a finite set of atoms and $\boldsymbol{A_c \subseteq 2^{A_d}}$ (Marek & Remmel 2004; Marek & Truszczynski 2004)
  - Represent any constraints with a finite set $\boldsymbol{A_c}$ of admissible *solutions* over a finite *domain* $\boldsymbol{A_d}$
- Logic Programs with C-atoms
  - Consist of clauses of the form

    $\boldsymbol{H_1 \vee \ldots \vee H_k \leftarrow A_1, \ldots, A_m, not\ B_1, \ldots, not\ B_n}$

  where H_i, A_i and B_i are either atoms or c-atoms

Issues of Semantics

- The Standard Gelfond-Lifschitz Transformation
  - For logic programs without c-atoms (Gelfond & Lifschitz 1988; 1991)
  - Not applicable to logic programs with c-atoms
- A Challenging Question:
  - What is an appropriate semantics for logic programs with c-atoms?

Existing Proposals

- Unfolding (Translation) Approaches
  - Transform ***P*** with c-atoms to ***P′*** without c-atoms and define an interpretation ***I*** as a stable model of ***P*** if it is a stable model of ***P′*** (Pelov et al. 2003; Son et al. 2006)
- Fixpoint (Operator-Based) Approaches
  - Apply some immediate consequence operator to construct a fixed point ***lfp(P)*** and define *I* as a stable model if ***I = lfp(P)*** (Marek & Truszczynski 2004; Pelov 2004; Son et al. 2006)
- Minimal Model Approaches
  - Define a stable model to be a minimal model (Faber et al. 2004)

Our Proposal

- Define the stable model semantics for logic programs with abstract constraints by developing

  A generalized Gelfond-Lifschitz transformation

Our Contributions

- A Formal Definition of the Semantics of C-Atoms
  - Currently, the meaning of a c-atom is interpreted by means of propositional interpretations (truth assignments)
- A Succinct Abstract Representation of C-Atoms
  - A c-atom is coded with a substantially smaller size than using the current power set form representation
- A Generalization of the Gelfond-Lifschitz Transformation
  - Used to define the stable model semantics for disjunctive logic programs with arbitrary c-atoms appearing anywhere in a clause

1. Semantics of C-Atoms

- Marek & Truszczynski's Definition
  - The meaning of a c-atom $\boldsymbol{A}$ is interpreted by means of propositional interpretations (truth assignments)
  - An interpretation $\boldsymbol{I}$ *satisfies* $\boldsymbol{A} = (\boldsymbol{A_d}, \boldsymbol{A_c})$, written as $\boldsymbol{I} \models \boldsymbol{A}$, if $\boldsymbol{I} \cap \boldsymbol{A_d} \in \boldsymbol{A_c}$; $\boldsymbol{I}$ *satisfies* **not** $\boldsymbol{A}$ if $\boldsymbol{I} \cap \boldsymbol{A_d} \notin \boldsymbol{A_c}$
- Our Observation
  - Marek & Truszczynski's truth assignment-based interpretation can be concisely formalized using a logic expression, thus leading to a formal definition of the semantics of c-atoms

1. Semantics of C-Atoms

- Our Formalization

Definition 1 Let $A = (A_d, A_c)$ be a c-atom. Its semantics is defined by

$$A \equiv \bigvee_{S \in A_c} S \wedge not\ (A_d \setminus S) \tag{1}$$

$$A = (\{a, b\}, \{\{a\}, \{b\}, \{a, b\}\})$$

↓ semantic definition

$$A \equiv (a \wedge not\ b) \vee (b \wedge not\ a) \vee (a \wedge b)$$

1. Semantics of C-Atoms

- Justification of Our Formalization

Theorem 1 *An interpretation* I *satisfies* A *iff* I *satisfies*

$$\bigvee_{S \in A_c} S \wedge not\ (A_d \setminus S)$$

I *satisfies* $not\ A$ *iff* I *satisfies*

$$not\ (\bigvee_{S \in A_c} S \wedge not\ (A_d \setminus S))$$

Logical Equivalence Simplification

For any S_1 and S_2,

$$(S_1 \wedge \underline{L} \wedge S_2) \vee (S_1 \wedge \underline{not\ L} \wedge S_2) \equiv S_1 \wedge S_2$$

$$A = (\{a, b\}, \{\{a\}, \{b\}, \{a, b\}\})$$

↓ semantic definition

$$A \equiv (a \wedge not\ b) \vee (b \wedge not\ a) \vee (a \wedge b)$$

↓ logically simplified

$$A \equiv a \vee b$$

2. Abstract Representation of C-Atoms

- Current Power Set Form Representation
  - $A = (A_d, A_c)$
  - $A_c \subseteq 2^{A_d}$ would be extremely large
- Our Power Set Free Abstract Representation
  - $A = (A_d, A_c^*)$
  - $W \uplus V$ in A_c^* covers all W-prefixed power sets of V in A_c

    i.e., $W \uplus V = \{W \cup S \mid S \in 2^V\}$

2. Abstract Representation of C-Atoms

$$A_c = \{\emptyset, \{b\}, \{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}\}$$

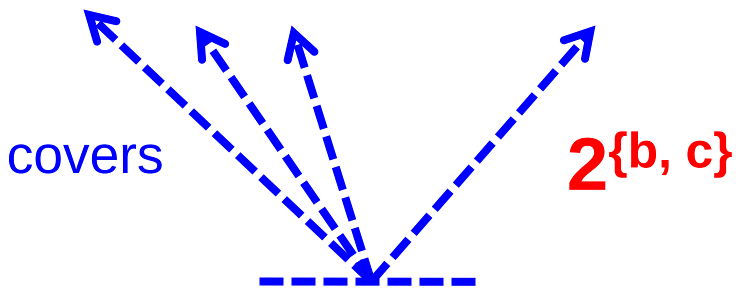
Power set form representation

$$A_c^* = \{\emptyset \uplus \{b, c\}, \{c\} \uplus \{a, b\}, \{c\} \uplus \{b, d\}\}$$

Abstract representation

2. Abstract Representation of C-Atoms

$$A_c = \{\emptyset, \{b\}, \{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}\}$$



$$A_c^* = \{\emptyset \uplus \{b, c\}, \{c\} \uplus \{a, b\}, \{c\} \uplus \{b, d\}\}$$

** $W \uplus V$ covers a set S if $W \subseteq S$ *and* $S \subseteq (W \cup V)$

2. Abstract Representation of C-Atoms

$$A_c = \{\emptyset, \{b\}, \{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}\}$$

$\{c\} \cup S, \quad S \in 2^{\{a, b\}}$

$$A_c^* = \{\emptyset \uplus \{b, c\}, \{c\} \uplus \{a, b\}, \{c\} \uplus \{b, d\}\}$$

2. Abstract Representation of C-Atoms

Theorem 2 *Let* $A = (A_d, A_c)$ *be a c-atom.*

1. A *has a unique abstract form* (A_d, A_c^*).
2. *An interpretation* $I \models A$ *iff* A_c^* *contains* $W \uplus V$ *covering* $I \cap A_d$.
3. A_c^* *is power set free.*

***** In many cases, $|A_c^*| \ll |A_c|$; in an extreme case, $|A_c| = \mathbf{2}^{|A_d|}$, but $|A_c^*| = 1$ ($A_c = \mathbf{2}^{A_d}$, $A_c^* = \{\emptyset \uplus A_d\}$)

2. Abstract Representation of C-Atoms

- C-atoms can be characterized in terms of the abstract representation

Theorem 3 *Let A be a c-atom. Then*

$$A \equiv \bigvee_{W \uplus V \in A_c^*} W \wedge not\ (A_d \backslash (W \cup V)) \qquad (2)$$

***** This theorem lays a solid basis for the development of the semantics of logic programs with c-atoms

2. Abstract Representation of C-Atoms

- Abstract Satisfiable Sets

Definition 2 Let A be a c-atom and I an interpretation.

1. $W \uplus V \in A_c^*$ is an *abstract satisfiable set* if $W \uplus V$ covers $I \cap A_d$.
2. W is called a *satisfiable set* if there is an abstract satisfiable set $W \uplus V$.

2. Abstract Representation of C-Atoms

- Characterizing C-Atoms in terms of Abstract Satisfiable Sets

Theorem 4 *Let A be a c-atom and I an interpretation. $I \models A$ iff I satisfies*

$$\bigvee_{\text{each abs. sat. set } W \uplus V} W \wedge not\ (A_d \backslash (W \cup V))$$

3. A Generalization of the Gelfond-Lifschitz Transformation

- Key Ideas (1): for each c-atom A in the body of a clause

$$\ldots \leftarrow \ldots, A, \ \ldots$$

↓ replaced by

$$\theta_A$$

↓ defined by

$$\theta_A \leftarrow W,$$ for each abstract satisfiable set $W \uplus V$

3. A Generalization of the Gelfond-Lifschitz Transformation

- Key Ideas (2): for each c-atom $\boldsymbol{A}$ in the head of a clause

$$\ldots\ \boldsymbol{A}\ \ldots \leftarrow \ldots$$

↓ replaced by

$$\beta_A$$

↓ defined by

$$\boldsymbol{B} \leftarrow \beta_A, \qquad \text{for each } \boldsymbol{B} \text{ in } \boldsymbol{I} \cap \boldsymbol{A_d}$$

$$\bot \leftarrow \boldsymbol{B}, \beta_A, \qquad \text{for each } \boldsymbol{B} \text{ in } \boldsymbol{A_d} \setminus (\boldsymbol{I} \cap \boldsymbol{A_d})$$

$$\beta_A \leftarrow \boldsymbol{I} \cap \boldsymbol{A_d}$$

** **These new clauses define that** β_A ***iff*** $\boldsymbol{I} \cap \boldsymbol{A_d}$

3. A Generalization of the Gelfond-Lifschitz Transformation

- Key Ideas (3): for a c-atom $A = (A_d, A_c)$, its negation ***not A*** is treated as the complement of A; i.e.,

$$not\ A = (A_d,\ \underline{2^{A_d} \setminus A_c})$$

the complement of A_c

Definition 3 Given a logic program P and an interpretation I, the *generalized Gelfond-Lifschitz transformation* of P w.r.t. I, written as P^I, is obtained from P by performing the following four operations:

1. Remove from P all clauses whose bodies contain either a negative literal $not\ A$ such that $I \not\models not\ A$ or a c-atom A such that $I \not\models A$.
2. Remove from the remaining clauses all negative literals, and then
3. Replace each c-atom A in the body of a clause with a special atom θ_A and introduce a new clause $\theta_A \leftarrow A_1, ..., A_m$ for each satisfiable set $\{A_1, ..., A_m\}$ of A w.r.t. $I \cap A_d$.
4. Replace each c-atom A in the head of a clause with $\bot$ if $I \not\models A$, or replace it with a special atom β_A and introduce a new clause $B \leftarrow \beta_A$ for each $B \in I \cap A_d$, a new clause $\bot \leftarrow B, \beta_A$ for each $B \in A_d \setminus (I \cap A_d)$, and a new clause $\beta_A \leftarrow I \cap A_d$.

Stable Models under the Generalized Gelfond-Lifschitz Transformation

Definition 4 For any logic program P, an interpretation I is a *stable model* of P if $I = M \setminus \{\theta_X, \beta_X\}$, where M is a minimal model of the generalized Gelfond-Lifschitz transformation P^I.

Main Properties (1)

Theorem 5 *Let P be a logic program such that c-atoms appearing in the heads of its clauses are all elementary. Any stable model of P is a minimal model of P.*

** An elementary c-atom is of the form ({***a***}, {{***a***}}), where ***a*** is an atom.

Main Properties (2)

Theorem 6 *Let P be a non-disjunctive logic program. An interpretation I is a stable model if and only if it is a stable model under Son et al.'s fixpoint definition.*

** T. C. Son, E. Pontelli and P. H. Tu. Answer sets for logic programs with arbitrary abstract constraint atoms. In *AAAI-06*, 2006.

Complexity

Theorem 8 *Let P be a logic program with n different c-atoms.*

1. *The time complexity of computing all satisfiable sets of A is linear in the size of A_c^*.*
2. *The time complexity of the generalized Gelfond-Lifschitz transformation is bounded by $O(|P| + n * (2M_{A_c^*} + M_{A_d} + 1))$, where $M_{A_c^*}$ and M_{A_d} are the maximum sizes of A_c^* and A_d of a c-atom in P, respectively.*
3. *The size of P^I is bounded by $O(|P| + n * (M_{A_c^*} + M_{A_d} + 1))$.*
4. *The time to compute A_c^* from A_c is bounded by $O(|A_c|^3 * |A_d|)$.*

Relationship to Existing Approaches

- Essentially different from the existing approaches in that we define the stable model semantics for logic programs with c-atoms by developing a generalized Gelfond-Lifschitz transformation based on the formal semantics and abstract representation of c-atoms.

Relationship to Existing Approaches (1)

- Let r be a clause $B \leftarrow A_1, ..., A_m$. An unfolding approach (Pelov et al. 2003; Son and Pontelli 2006) will transform r into $n_1 * ... * n_m$ new clauses of the form $B \leftarrow \overline{A}_1, ..., \overline{A}_m$, where each $\overline{A}_i$ is built from an aggregate solution of A_i. Our approach transforms r into $1 + n'_1 + ... + n'_m$ clauses, where n'_i is the number of satisfiable sets of A_i. In general, for each i we have $n_i \gg n'_i$.

** n_i is the number of aggregate solutions of A_i

Relationship to Existing Approaches (2)

- Stable models defined using our approach coincide with those applying Son et al.'s fixpoint approach (Son et al. 2006; 2007) for non-disjunctive logic programs with arbitrary c-atoms.

** Son et al. show that their fixpoint semantics coincides with that of Marek and Truszczynski (2004) for non-disjunctive logic programs with monotone c-atoms; with that of Faber et al. (2004) and Ferraris (2005) for positive basic logic programs with monotone c-atoms; with that of Denecker et al. (2001; 2003) for positive basic logic programs with arbitrary c-atoms.

Relationship to Existing Approaches (3)

- Our approach has the minimality property for the class of logic programs in which c-atoms appearing in clause heads are all elementary. It is different from the minimal model approach by Faber et al. (2004).

Summary

- We introduced a formal characterization of the semantics of c-atoms
- We created an abstract representation of c-atoms
- We developed a generalized Gelfond-Lifschitz transformation based on the formal semantics and abstract representation of c-atoms
- Stable models coincide with Son et al.'s fixpoint approach for non-disjunctive logic programs with arbitrary c-atoms

Thanks !

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http://lcs.ios.ac.cn/~ydshen/Shen-AAAI07.pdf