Bisimulation and Logic
Lecture 3

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Modal logic and bisimulation

- Behavioural equivalence between concurrent processes (Park, Hennessy + Milner)
- Model theory of modal logic (van Benthem)

Modal characterisation of bisimulation and some model theory

Modal (Hennessy-Milner) logic: syntax

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We define when a process E satisfies a formula Φ. Either E satisfies Φ, denoted by $E \models \Phi$, or it doesn't, denoted by $E \not\models \Phi$.

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- $E \models [\mathtt{tick}] \mathtt{tt}$
  This is equivalent to `true`!

Checking satisfaction

$$\mathtt{Cl} \stackrel{\text{def}}{=} \mathtt{tick}.\mathtt{Cl}$$

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Φ^c is inductively defined as follows:

$$\begin{array}{rcl}
\mathtt{tt}^c & = & \mathtt{ff} \\
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(\Phi_1 \wedge \Phi_2)^c & = & \Phi_1^c \vee \Phi_2^c \\
(\Phi_1 \vee \Phi_2)^c & = & \Phi_1^c \wedge \Phi_2^c \\
([a]\Phi)^c & = & \langle a \rangle \Phi^c \\
(\langle a \rangle \Phi)^c & = & [a]\Phi^c
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Proposition: For every process F and HML-formula Φ:

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Induction step:

Case $\Phi = \Phi_1 \wedge \Phi_2$

$$\begin{array}{lll}
 & F \models (\Phi_1 \wedge \Phi_2)^c & \\
\text{iff} & F \models \Phi_1^c \vee \Phi_2^c & \\
\text{iff} & F \models \Phi_1^c \ \text{ or } \ F \models \Phi_2^c & \text{(by clause for } \vee) \\
\text{iff} & F \not\models \Phi_1 \ \text{ or } \ F \not\models \Phi_2 & \text{(by i.h.)} \\
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\end{array}$$

Case $\Phi = [a]\Phi_1$.

$$
\begin{array}{lll}
 & F \models ([a]\Phi_1)^c & \\
\text{iff} & F \models \langle a \rangle \Phi_1^c & \\
\text{iff} & \exists G.\, F \xrightarrow{a} G \text{ and } G \models \Phi_1^c & \\
\text{iff} & \exists G.\, F \xrightarrow{a} G \text{ and } G \not\models \Phi_1 & \text{(by i.h.)} \\
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- Basis: $\Phi = \mathtt{tt}$ or $\Phi = \mathtt{ff}$. Clear.
- Step: We consider only the case $\Phi = [a]\Psi$. By symmetry, it suffices to show that $G \models [a]\Psi$ implies $H \models [a]\Psi$.
  Assume $G \models [a]\Psi$. For any G' such that $G \xrightarrow{a} G'$, it follows that $G' \models \Psi$.
  Let $H \xrightarrow{a} H'$. Since $G \sim H$, there is a G' such that $G \xrightarrow{a} G'$ and $G' \sim H'$. By the induction hypothesis $H' \models \Psi$, and therefore $H \models \Phi$.

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  So $\{H' : H \xrightarrow{a} H'\} = \{H_1, \ldots, H_n\}$ is non-empty and finite by image-finiteness.

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  $G \models \langle a \rangle \Psi$ but $H \not\models \langle a \rangle \Psi$ because each H_i fails to have property Ψ. Contradicts $G \equiv_{\mathrm{HM}} H$.

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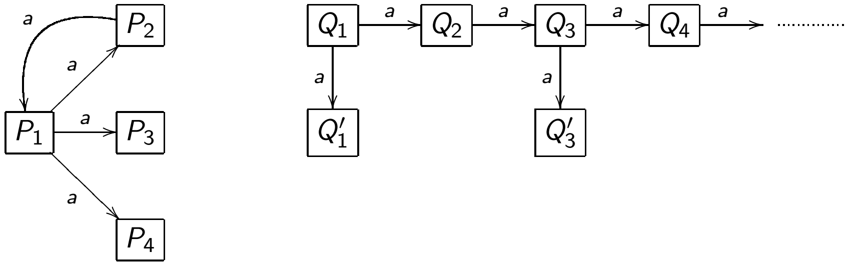
- First theorem equivalent to properties expressed by modal formulas are bisimulation invariant: if $E \in \|\phi\|$ and $E \sim F$ then $F \in \|\phi\|$

Bisimulation invariance



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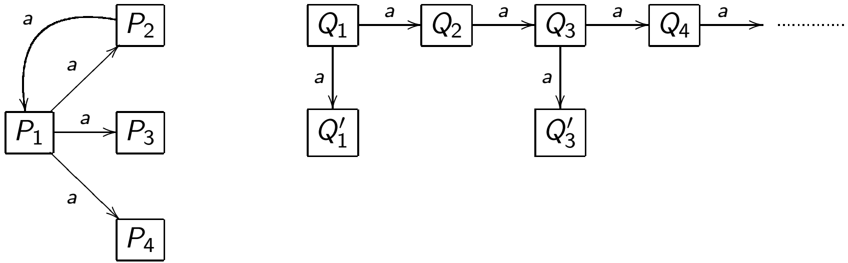

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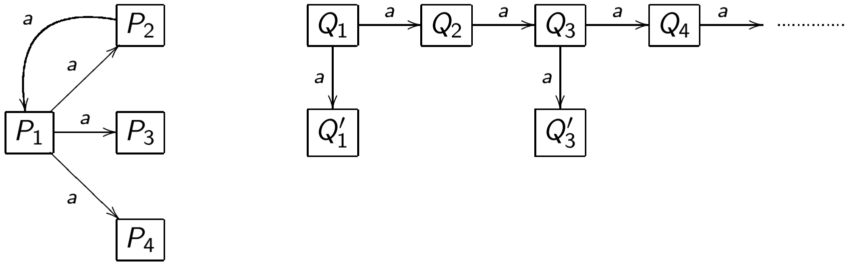

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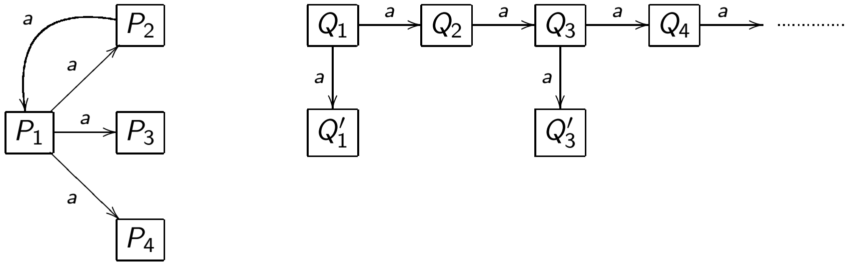

- Many kinds of properties not bisimulation invariant
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First order logic (FOL)

$$\phi \ ::= xE_ay \mid x = y \mid \neg \phi \mid \phi_1 \vee \phi_2 \mid \exists x.\phi$$

- $x, y \in Var$ (variables); E_a is binary transition relation for each action a

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- formulas are interpreted over transition systems
- Valuation $\sigma : Var \rightarrow Pr$ (Pr are the processes)
- $\sigma\{P_1/x_1, \ldots, P_n/x_n\}$ is the valuation that is the same as σ except that its value for x_i is P_i, $1 \leq i \leq n$ (where each x_i is distinct).

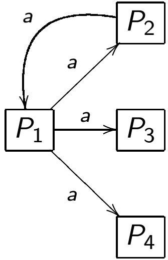
Semantics

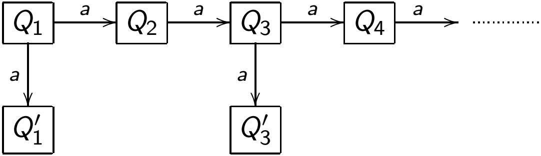
Inductively define when FOL formula ϕ is true on an LTS with respect to a valuation σ as $\sigma \models \phi$

$$
\begin{array}{lll}
\sigma \models xE_a y & \text{iff} & \sigma(x) \xrightarrow{a} \sigma(y) \\
\sigma \models x = y & \text{iff} & \sigma(x) = \sigma(y) \\
\sigma \models \neg\phi & \text{iff} & \sigma \not\models \phi \\
\sigma \models \phi_1 \vee \phi_2 & \text{iff} & \sigma \models \phi_1 \text{ or } \sigma \models \phi_2 \\
\sigma \models \exists x.\phi & \text{iff} & \sigma\{P/x\} \models \phi \text{ for some } P \in Pr
\end{array}
$$

The universal quantifier, $\forall x.\phi = \neg\exists\neg\phi$
$\sigma \models \forall x.\phi$ iff $\sigma\{P/x\} \models \phi$ for all $P \in Pr$.

Example





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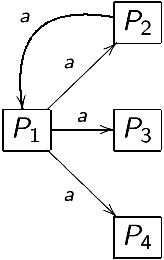

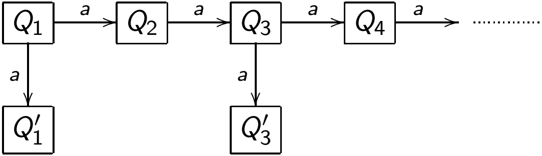

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- $\sigma \models \exists x.\exists y.\exists z.(x_1 E_a x \wedge x_1 E_a y \wedge x_1 E_a z \wedge x \neq y \wedge x \neq z \wedge y \neq z)$

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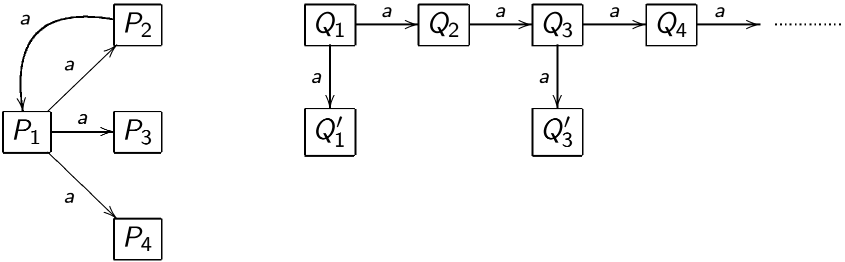

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- $\sigma \models \forall y.\forall z.(x_2 E_a y \wedge y E_a z \to z \neq x_2)$

Translating modal logic into FOL

The FOL translation of modal formula ϕ relative to variable x is $T_x(\phi)$ which is defined inductively

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\begin{array}{lcl}
T_x(\texttt{tt}) & = & x = x \\
T_x(\texttt{ff}) & = & \neg(x = x) \\
T_x(\phi_1 \wedge \phi_2) & = & T_x(\phi_1) \wedge T_x(\phi_2) \\
T_x(\phi_1 \vee \phi_2) & = & T_x(\phi_1) \vee T_x(\phi_2) \\
T_x([a]\phi) & = & \forall y.\neg(xE_a y) \vee T_y(\phi) \\
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Theorem $P \models \phi$ iff $\sigma\{P/x\} \models T_x(\phi)$
Theorem Any first-order formula $T_x(\phi)$ is bisimulation invariant

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Theorem Any first-order formula $T_x(\phi)$ is bisimulation invariant

A FOL formula $\phi(x)$ is equivalent to modal $\phi' \in M$ provided that for any LTS and for any state P, $\sigma\{P/x\} \models \phi$ iff $P \models \phi'$

Van Benthem's theorem

Theorem A FOL formula $\phi(x)$ is equivalent to a modal formula iff $\phi(x)$ is bisimulation invariant.
Proof

Van Benthem's theorem

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For the converse property, assume that $\phi(x)$ is bisimulation invariant.

Let $\Phi = \{T_x(\psi) \mid \psi \in M$ and $\{\phi(x)\} \models T_x(\psi)\}$

We show that $\Phi \models \phi(x)$ and, therefore, by the compactness theorem, $\phi(x)$ is equivalent to a modal formula ψ' such that $T_x(\psi') \in \Phi$.

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Assume $\sigma\{P/x\} \models \psi$ for all $\psi \in \Phi$. We show $\sigma\{P/x\} \models \phi$. We choose a P with the Hennessy-Milner property (that is, if $P' \equiv_M P$ then $P' \sim P$)

Proof Continued

Let $\Psi = \{\, T_x(\psi) \mid P \models \psi \}$.
First, $\Phi \subseteq \Psi$.
Next, $\Psi \cup \{\phi\}$ is satisfiable
Therefore, for some Q, $\sigma\{Q/x\} \models \psi$ for all $\psi \in \Psi$ and $\sigma\{Q/x\} \models \phi$.
However, $Q \sim P$ and because ϕ is bisimulation invariant, $\sigma\{P/x\} \models \phi$ as required.

Alternative Proof

Uses ω-unravelling;
Given a LTS there is a way of unfolding $P \in Pr$ and all its reachable processes into a tree rooted at P which is called *unravelling*.
Theorem If $P \sim Q$. then the ω-unravellings of P and Q are isomorphic