Quantitative Analysis of Systems

– Beyond Markov Models -

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Overview

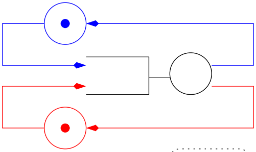
1. Beyond Markov Automata
2. Analysis of an Automaton
3. Equivalence of Automata
4. Some Aspects of Model Checking
5. Conclusions

A Stochastic Automata Model

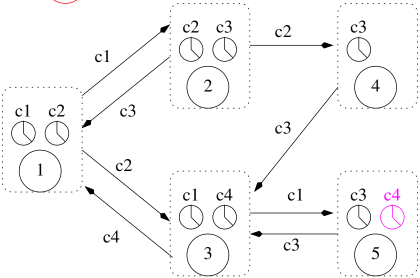
An SA is defined as $(\mathcal{L}, ini, \mathcal{C}, ena, sus, trans, res)$ where

- $\mathcal{L} = \{1, \ldots, N\}$ is a finite set of locations (automata states),
- $ini : \mathcal{L} \rightarrow [0, 1]$ defines the initial distribution over the set of locations
- $\mathcal{C}$ is a finite set of clock processes with classes (defined later) ($\mathcal{K}$ is the set of clock process class pairs),
- $ena : \mathcal{L} \rightarrow \mathcal{C}$ enabled clock processes in a location,
- $sus : \mathcal{L} \rightarrow \mathcal{C}$ suspended clock processes in a location, for $l \in \mathcal{L}$, $ena(l) \cap sus(l) = \emptyset$,
- $trans : \mathcal{L} \times \mathcal{K} \times \mathcal{L} \rightarrow [0, 1]$ is the transition function that assigns to a source location, an enabled clock process class pair and a destination location a probability distribution over the set of locations,
- $res : \mathcal{L} \times \mathcal{K} \times \mathcal{L} \rightarrow \mathcal{C}$ reset function.

Example 1:



Queueing System with
preemptive pritoity resume
class 1 (blue) with priority
(general inter−arrival and
service times)



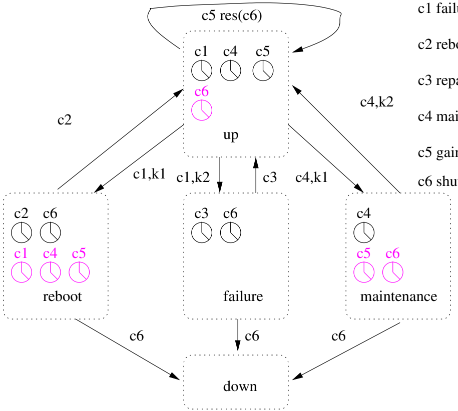
c1 arrival 1

c2 arrival 2

c3 service 1

c4 service 2

Example 2:



c1 failure (2 classes, soft and hard)

c2 reboot

c3 repair

c4 maintenance (2 classes wait and duration)

c5 gain

c6 shut down

How to model clock processes?

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- But we would like to analyze the process numerically/analytically!
- Use some form of phase type process
- Markov processes and beyond ...!?

Markov Automata (MA):

- clock process $c \in \mathcal{C}$ is an MMAP:

$$\left(\pi_0^{(c)}, \mathbf{G}_0^{(c)}, \mathbf{G}_1^{(c)}, \ldots, \mathbf{G}_k^{(c)}\right)$$

  n^c is the size of the state space $\pi_0^{(c)}$ is the initial vector

  $\mathbf{G}_0^{(c)}$ is the generator of an absorbing Markov process

  $\mathbf{G}_k^{(c)}$ $(1 \leq k \leq K)$ are non-negative,

  $\mathbf{G}_0^{(c)} + \sum_{k=1}^{K} \mathbf{G}_k^{(c)}$ is an irreducible generator matrix

  Probabilistic interpretation of the behavior

Phase type processes beyond Markov processes:

Matrix-exponential (ME) distributions: $(\pi_0, \mathbf{G}_0)$ such that

$$F_{(\pi_0,\mathbf{G}_0)} = 1 - \pi_0 e^{\mathbf{G}_o t} \mathbb{I}$$

is a valid distribution function.
Example:

$$\pi_0 = (2.63479, -1.22850, -0.406283)$$

$$\mathbf{G}_0 = \begin{pmatrix} -2.25709 & 0 & 0 \\ 0 & -2.25709 & -2.338187 \\ 0 & 2.338187 & -2.25709 \end{pmatrix}$$

distribution with $CV^2 = 0.2009$

Specification of processes $c \in \mathcal{C}$ (general case):

$(\pi_0^{(c)}, \mathbf{G}_0^{(c)}, \mathbf{G}_1^{(c)}, \ldots, \mathbf{G}_K^{(c)})$

without probabilistic interpretation, but

- $F_{\pi_0^{(c)}, \mathbf{G}_0^{(c)}}(t) = 1 - \pi_0^{(p)} e^{\mathbf{G}_0^{(c)} t} \mathbb{I}$ is a valid distribution function,
- $f_{\pi_0^{(c)}, \mathbf{G}_0^{(c)}, \mathbf{G}_1^{(c)}, \ldots, \mathbf{G}_K^{(c)}}(t_1, k_1, \ldots, t_j, k_j) =$
  $\pi_0^{(c)} e^{\mathbf{G}_0^{(c)} t_1} \mathbf{G}_{k_1}^{(c)} e^{\mathbf{G}_0^{(c)} t_2} \mathbf{G}_{k_2}^{(c)} \ldots e^{\mathbf{G}_0^{(c)} t_j} \mathbf{G}_{k_j}^{(c)} \mathbb{I}$
  is a valid density for $t_i \geq 0$ and $k_i \in \{1, \ldots, K\}$,
- $\mathbf{G}^{(c)} \mathbb{I} = \sum_{k=0}^{K} \mathbf{G}_k^{(c)} \mathbb{I} = \mathbf{0}$ and $\mathbf{G}^{(c)}$ is irreducible.

Marked Rational Arrival Process (MRAP)

Behavior of MRAPs:

- process behaves deterministically according to the ODE $\dot{\pi} = \pi \mathbf{G}_0$ (i.e., $\pi_t = \pi_0 e^{\mathbf{G}_0 t}$)
- at time t an event of type/class k occurs with density $\pi_t \mathbf{G}_k \mathbb{1}$
- if event k occurs at time t, the state changes from $\frac{\pi_t}{\pi_t \mathbb{1}}$ to $\frac{\pi_t \mathbf{G}_k}{\pi_t \mathbf{G}_k \mathbb{1}}$

state is given by the whole vector π_t

Piecewise Deterministic Markov Process

Behavior is deterministic for a given history
$\mathcal{H} = (t_0, k_1, t_1, k_2, \ldots, t_{H-1}, k_H, t_H)$
(removing the stochastic part)

Some properties of MRAP $(\pi_0, \mathbf{G}_0, \mathbf{G}_1, \ldots, \mathbf{G}_K)$:

- the MRAP is *class random* if for every state π that is reached after an arbitrary history $\mathcal{H}$ $\pi e^{\mathbf{G}_0 t'} \mathbf{G}_k$ holds for every $k \in \{1, \ldots, K\}$ and some $t' \geq 0$
- the MRAP has an *equivalent Markovian representation* if an equivalent MMAP exists (definition of equivalence later)
- MRAPs without a finite MMAP representation exist
- the MRAP is *minimal* if no equivalent MRAP with a state space of a smaller dimension exists.

Behavior of an SA (with MRAPs):

- the initial vector $\pi_0 = \left(\pi_0^1, \ldots, \pi_0^N\right)$ where $\pi_0^l = \nu_l \otimes_{p \in ena(l) \cup sus(l)} \pi_0^{(p)}$ and ν_l is the probability to start in location l
- in a location l MRAPs from $ena(l)$ behave deterministically according to the ODE $\dot{\pi} = \pi \oplus_{c \in ena(l)} \mathbf{G}_0^{(c)}$
- location changes occur according to events defined by the rates of enabled MRAPs
- state changes occur at event times with respect to
  - state change in the MRAP that causes the event
  - enabling/suspending/disabling of events in the destination location
  - resetting of event due to function $res(.)$

An SA describes an MRAP!

Analysis of an SA with representation $(\pi_0, \mathbf{G}_0, \mathbf{G}_1, \ldots, \mathbf{G}_K)$:

- expectation of the state at time t:

$$\mathbf{p}_t = E[\pi_t] = \pi_0 e^{\left(\mathbf{G}_0 + \sum_{k=1}^{K} \mathbf{G}_k\right)t}$$

- state at time t for history $\mathcal{H} = (t_0, k_1, t_1, \ldots, k_H, t_H)$ with $t = \sum_{h=1}^{H} t_h$:

$$\pi_t = \frac{\pi_0 e^{\mathbf{G}_o t_0} \prod_{h=1}^{H} \mathbf{G}_{k_h} e^{\mathbf{G}_0 t_h}}{\pi_0 e^{\mathbf{G}_o t_0} \prod_{h=1}^{H} \mathbf{G}_{k_h} e^{\mathbf{G}_0 t_h} \mathbb{1}}$$

- if values of the state components in π_t or $\mathbf{p}_t$ are added, then a probability distribution over the locations of the SA is defined

Equivalence of SAs:

1. Locations of the SA are observable
   equivalent SAs have isomorphic locations but possibly different processes in $\mathcal{C}$

2. Locations observe predicates
   sets of locations with different predicates have to be distinguished, but not locations with the same predicates

3. Locations are not observable
   locations need not be distinguished, only events are observable

1. and 2. can be transformed in 3. by defining *pseudo* events for states that have to be distinguished, e.g., event e is state l with $trans(l, e, l) = 1$ and corresponding process with $\mathbf{G}_0 = (-\mu)$, $\mathbf{G}_1 = (\mu)$.

Equivalence of MRAPs

$(\pi_0, \mathbf{G}_0, \mathbf{G}_1, \ldots, \mathbf{G}_K)$ and $(\phi_0, \mathbf{H}_0, \mathbf{H}_1, \ldots, \mathbf{H}_K)$ are equivalent

if and only if for all histories $\mathcal{H} = (t_0, k_1, t_1, \ldots, k_H, 0)$:

$$\pi_0 \left(\prod_{h=1}^{H} e^{\mathbf{G}_0 t_{h-1}} \mathbf{G}_{k_h} \right) \mathbb{1} = \phi_0 \left(\prod_{h=1}^{H} e^{\mathbf{H}_0 t_{h-1}} \mathbf{H}_{k_h} \right) \mathbb{1}$$

i.e., the conditional density of observing a sequence of events is identical for both MRAPs

Let $(\pi_0, \mathbf{G}_0, \mathbf{G}_1, \ldots, \mathbf{G}_K)$ and $(\phi_0, \mathbf{H}_0, \mathbf{H}_1, \ldots, \mathbf{H}_K)$ be two equivalent MRAPs of size m and n $(m \geq n)$, respectively.

Then one of the following two relations hold:

1. there exists a $m \times n$ matrix $\mathbf{V}$ with:

   $\mathbf{V}\mathbb{I} = \mathbb{I}$, $\pi_0\mathbf{V} = \phi_0$ and $\mathbf{G}_k\mathbf{V} = \mathbf{V}\mathbf{H}_k$ for all $k = 0, \ldots, K$

2. there exists an $n \times m$ matrix $\mathbf{W}$ with:

   $\mathbf{W}\mathbb{I} = \mathbb{I}$, $\pi_0 = \mathbf{W}\phi_0$ and $\mathbf{W}\mathbf{G}_k = \mathbf{H}_k\mathbf{W}$ for all $k = 0, \ldots, K$

$\mathbf{V}$ or $\mathbf{W}$ can be efficiently computed to find a minimal equivalent representation for a given MRAP

An Example:

$$\pi_0 = (0.5, 0, 0, 0.5)\,, \qquad \mathbf{G}_0 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & -4 \end{pmatrix},$$

$$\mathbf{G}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1.5 & 0 & 0 & 1.5 \end{pmatrix}, \qquad \mathbf{G}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 \end{pmatrix}$$

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An equivalent MRAP of size 3

$$\phi_0 = (0, 0, 1), \mathbf{H}_0 = \begin{pmatrix} -1.36364 & 4.13365 & -6.65777 \\ -1.14992 & -1.46376 & 4.02489 \\ 0 & 1.1726 & -3.1726 \end{pmatrix},$$

$$\mathbf{H}_1 = \begin{pmatrix} 0 & 0 & 2.91582 \\ 0 & 0 & -1.05841 \\ 0 & 0 & 1.5 \end{pmatrix}, \mathbf{H}_2 = \begin{pmatrix} 0 & 0 & 0.97194 \\ 0 & 0 & -0.3528 \\ 0 & 0 & 0.5 \end{pmatrix}.$$

Computation of minimal equivalent representations

- compute minimal representations for all processes in $\mathcal{C}$
- compute minimal representation for the whole SA

Steps for larger state spaces:

1. compute stochastic bisimulation (ordinary and inverse)
2. compute the minimal representation from the minimized processes according to bisimulation

Model Checking of SAs

CSL formulas for SAs

- tt is a location formula
- an atomic proposition $a \in \mathtt{AP}$ is a location formula
- if Φ and Ψ are location formulas, so are $\neg\Phi$ and $\Phi \vee \Psi$,
- if Φ is a location formula, then so is $\mathcal{S}_{\bowtie p}$,
- if φ is a path formula, the $\mathcal{P}_{\bowtie p}(\varphi)$ is a location formula,
- if Φ and Ψ are location formulas, then $X_{int}\Phi$ and $\Phi\mathcal{U}_{int}\Psi$ are path formulas.

$int \subseteq \mathbb{R}_{\geq 0}$, $\bowtie \in \{<, \leq, \geq, >\}$ and $p \in [0, 1]$

Model Checking Approaches:

- formulas with atomic propositions for locations are evaluated as usual
- steady state analysis
  - for irreducible SAs solve $\mathbf{p}\left(\sum_{k=0}^{K}\mathbf{G}_k\right) = \mathbf{0}$ or $\mathbf{p}\left(\sum_{k=1}^{K}\mathbf{G}_k\right)\left(-\mathbf{G}_0\right)^{-1} = \mathbf{0}$ subject to $\mathbf{p}\mathbb{I} = 1$
  - otherwise determine the strongly connected components and compute the stationary vector for strongly connected components (locations may belong to more than one strongly connected component)

  add the values in π belonging to locations to obtain a probability distribution

Model Checking Approaches:

Compute probabilities for path formulas:

- for $\Phi\mathcal{U}_{[t_0,t_1]}\Psi$ $(0 \leq t_0 \leq t_1)$:
  - make all locations that do not observe Φ or Ψ absorbing and compute $\mathbf{b}_1 = e^{\sum_{k=0}^{K} \mathbf{G}_k[\neg\Phi\vee\neg\Psi](t_1-t_0)}\mathbb{I}$
  - make all locations that do not observe Φ absorbing and compute $\mathbf{b}_0 = e^{\sum_{k=0}^{K} \mathbf{G}_k[\neg\Phi]t_0}\mathbf{b}_1$
  - for each $l \in \mathcal{L}$
    $Prob_l(\Phi\mathcal{U}_{[t_0,t_1]}\Psi) = \otimes_{p\in ena(l)\cup sus(l)}\pi_0^{(p)} \cdot \mathbf{b}_0^l$ or
    $Prob_l(\Phi\mathcal{U}_{[t_0,t_1]}\Psi) = \mathbf{p}^l \cdot \mathbf{b}_0^l$
    for some vector $\mathbf{p}$ reached during an execution of the SA
  - check $Prob_l(\Phi\mathcal{U}_{[t_0,t_1]}\Psi) \bowtie p$

Model Checking Approaches:

- for $X_{[t_0,t_1]}\Phi$ $(0 \leq t_0 \leq t_1)$ with initial vector $\otimes_{p \in ena(l) \cup sus(l)} \pi_0^{(p)}$: for each location $l \in \mathcal{L}$
  - define

$$\mathbf{F}_{l,\Phi}^{(p)} = \sum_{J \in \{1,\ldots,N\}, \Phi(J)=tt} \sum_{k=1}^{K_p} trans(l,(p,k),J)\mathbf{G}_k^{(p)} \quad \text{and}$$

$$\mathbf{F}_l^{(p)} = \begin{pmatrix} \mathbf{G}_0^{(p)} & \mathbf{F}_{l,\Phi}^{(p)}\mathbb{I} \\ \mathbf{0} & 0 \end{pmatrix}$$

compute $\left(\mathbf{b}_l^{(p)}, \beta_l^{(p)}\right) = \left(\pi_0^{(p)} e^{\mathbf{G}_0^{(p)} t_0}, 0\right) e^{\mathbf{F}_l^{(p)}(t_1 - t_0)}$

  - check $\sum_{p \in ena(l)} \beta_l^{(p)} \bowtie p$

- similar approach for initial vector $\mathbf{p}^l$

Conclusions

- new class of automata
- interpretation as a piecewise deterministic Markov process
- numerical analysis
- equivalence relations
- first ideas for model checking state labels/rewards
- composition of SAs can be defined and preserves equivalence (not presented here)
- model checking path labels (not presented here)

Open issues

- complete characterization of equivalent automata by CSL formulas
- introduction of indeterminism
- decision whether an automaton is valid SA
  (vector matrices describe a valid stochastic process)
- decision whether an SA has an equivalent representation as an MA

Thank you!