Late Weak Bisimilarity for Markov Automata

Christian Eisentraut\textsuperscript{1} \quad Jens Chr. Godskesen\textsuperscript{2}  
Holger Hermanns\textsuperscript{1} \quad Lei Song\textsuperscript{3,1} \quad Lijun Zhang\textsuperscript{4,1}

Saarland University, Germany  
IT University of Copenhagen, Denmark  
Max-Planck-Institut für Informatik, Germany  
Institute of Software, Chinese Academy of Sciences, China

September 27, 2013  
ISCAS, Beijing
Markov Automata

An MA is a tuple $(S, Act_\tau, \rightarrow, \rightarrow\rightarrow, \bar{s})$ where

- \(\bar{s} \in S\) is the initial state,
- \(S\) is a finite but non-empty set of states,
- \(Act_\tau = Act \cup \{\tau\}\) is a set of actions including the internal action \(\tau\),
- \(\rightarrow\subset S \times Act_\tau \times Dist(S)\) is a finite set of probabilistic transitions,
- \(\rightarrow\rightarrow\subset S \times \mathbb{R}_{>0} \times S\) is a finite set of Markovian transitions.
A Probabilistic Automaton $M$ is a tuple $(S, Act_\tau, \xrightarrow{\cdot}, \xrightarrow{\cdot\cdot}, \bar{s})$ where

- $\bar{s} \in S$ is the initial state,
- $S$ is a finite but non-empty set of states,
- $Act_\tau = Act \cup \{\tau\}$ is a set of actions including the internal action $\tau$,
- $\xrightarrow{\cdot} \subseteq S \times Act_\tau \times Dist(S)$ is a finite set of probabilistic transitions,
- $\xrightarrow{\cdot\cdot} = \emptyset$. 

Problabilistic Automata
Interactive Markov Chain

An Interactive Markov Chain $M$ is a tuple $(S, Act_\tau, \rightarrow, \rightarrow\rightarrow, \bar{s})$ where

- $\bar{s} \in S$ is the initial state,
- $S$ is a finite but non-empty set of states,
- $Act_\tau = Act \cup \{\tau\}$ is a set of actions including the internal action $\tau$,
- $\rightarrow \subset S \times Act_\tau \times S$ is a finite set of transitions,
- $\rightarrow\rightarrow \subset S \times \mathbb{R}_{>0} \times S$ is a finite set of Markovian transitions.
Early Weak Bisimilarity

(a) \( s_1 \)  

\( s_3 \)

\[ \tau \]

\[ \alpha \]

\[ \frac{1}{3} \quad \frac{2}{3} \]

\( r_1 \)

\( r_2 \)

(b) \( s_0 \)  

\[ \alpha \]

\[ \frac{1}{3} \quad \frac{2}{3} \]

\( r_1 \)

\( r_2 \)

\( s_3 \)

\[ \tau \]

\[ \alpha \]

\( s_4 \)

\( s_5 \)

(c) \( s_2 \)  

\[ \tau \]

\[ \alpha \]

\( s_4 \)

\( s_5 \)

\( r_1 \)

\( r_2 \)
A relation $R \subseteq \text{Dist}(S) \times \text{Dist}(S)$ is an early weak bisimulation over $\mathcal{M}$ iff $\mu \mathrel{R} \nu$ implies:

- whenever $\mu \xrightarrow{\theta} \mu'$, there exists a $\nu \xrightarrow{\theta} \nu'$ such that $\mu' \mathrel{R} \nu'$;
- whenever $\mu = \sum_{0 \leq i \leq n} p_i \cdot \mu_i$, there exists
  $\nu \xrightarrow{\tau} \sum_{0 \leq i \leq n} p_i \cdot \nu_i$ such that $\mu_i \mathrel{R} \nu_i$ for each $0 \leq i \leq n$
  where $\sum_{0 \leq i \leq n} p_i = 1$;
- symmetrically for $\nu$.

$s \simeq r$ iff $\delta_s \simeq \delta_r$
A relation $\mathcal{R} \subseteq \text{Dist}(S) \times \text{Dist}(S)$ is an early weak bisimulation over $\mathcal{M}$ iff $\mu \mathcal{R} \nu$ implies:

- whenever $\mu \xrightarrow{\theta} \mu'$, there exists a $\nu \xrightarrow{\theta} \nu'$ such that $\mu' \mathcal{R} \nu'$;
- whenever $\mu = \sum_{0 \leq i \leq n} p_i \cdot \mu_i$, there exists $\nu \xrightarrow{\tau} \sum_{0 \leq i \leq n} p_i \cdot \nu_i$ such that $\mu_i \mathcal{R} \nu_i$ for each $0 \leq i \leq n$ where $\sum_{0 \leq i \leq n} p_i = 1$;
- symmetrically for $\nu$.

$s \approx r$ iff $\delta_s \approx \delta_r$

$\mu \xrightarrow{\theta} \mu'$ iff $\mu' = \sum_{s \in \text{Supp}(\mu)} \mu(s) \cdot \mu_s$ where $s \xrightarrow{\theta} \mu_s$
Early Weak Bisimilarity

A relation $\mathcal{R} \subseteq \text{Dist}(S) \times \text{Dist}(S)$ is an early weak bisimulation over $\mathcal{M}$ iff $\mu \mathcal{R} \nu$ implies:

- whenever $\mu \xrightarrow{\theta} \mu'$, there exists a $\nu \xrightarrow{\theta} \nu'$ such that $\mu' \mathcal{R} \nu'$;
- whenever $\mu = \sum_{0 \leq i \leq n} p_i \cdot \mu_i$, there exists $\nu \xrightarrow{\tau} \sum_{0 \leq i \leq n} p_i \cdot \nu_i$ such that $\mu_i \mathcal{R} \nu_i$ for each $0 \leq i \leq n$ where $\sum_{0 \leq i \leq n} p_i = 1$;
- symmetrically for $\nu$.

\[
\left\{ \frac{1}{2} : s_1, \frac{1}{2} : s_2 \right\} = \frac{1}{2} \delta_{s_1} + \frac{1}{2} \delta_{s_2}
\]

\[
= \frac{2}{3} \left\{ \frac{1}{4} : s_1, \frac{3}{4} : s_2 \right\} + \frac{1}{3} \delta_{s_1}
\]
Properties of $\equiv$

- Relation on distributions.
- $\equiv$ is strictly coarser than Weak Probabilistic Bisimulation by Segala.
- $\equiv$ is compositional.
- $\equiv$ is the coarsest compositional equivalence preserving trace distribution equivalence.
A piece of probabilistic program

```plaintext
print("I am going to toss");

r = rand();

if r ≥ \frac{1}{2} then
    print("head");
else
    print("tail");
end
```
A piece of probabilistic program

print("I am going to toss");

\[ r = \text{rand}(); \]

if \( r \geq \frac{1}{2} \) then
\[
\text{print("head");}
\]
else
\[
\text{print("tail");}
\]
end

![Diagram](image-url)
Another piece of probabilistic program

```
r = rand();
if r ≥ \frac{1}{2} then
    print(“I am going to toss”);
    print(“head”);
else
    print(“I am going to toss”);
    print(“tail”);
end
```
Another piece of probabilistic program

```plaintext
r = rand();
if r ≥ 1/2 then
    print("I am going to toss");
    print("head");
else
    print("I am going to toss");
    print("tail");
end
```
The guesser
Two Less Powerful Schedulers

Partial Information Schedulers

Two Less Powerful Schedulers

Partial Information Schedulers


Distributed Schedulers

Partial Information Schedulers
Distributed Schedulers

\[ S \parallel_A r_1 \]

\[ \alpha \]

\[ \mu \]
Distributed Schedulers

\[ S \parallel_A r_1 \]

\[ S \parallel_A r_2 \]
Late Weak Bisimilarity

A relation \( \mathcal{R} \subseteq \text{Dist}(S) \times \text{Dist}(S) \) is a late weak bisimulation over \( \mathcal{M} \) iff \( \mu \mathcal{R} \nu \) implies:

1. Whenever \( \mu \xrightarrow{\theta} \mu' \), there exists a \( \nu \xrightarrow{\theta} \nu' \) such that \( \mu' \mathcal{R} \nu' \);
2. If not \( \xrightarrow{\cdot} \mu \), then there exists \( \mu = \sum_{0 \leq i \leq n} p_i \cdot \mu_i \) and 
   \( \nu \xrightarrow{\tau} \sum_{0 \leq i \leq n} p_i \cdot \nu_i \) such that \( \overrightarrow{\mu_i} \) and \( \mu_i \mathcal{R} \nu_i \) for each \( 0 \leq i \leq n \) where \( \sum_{0 \leq i \leq n} p_i = 1 \);
3. Symmetrically for \( \nu \).

where \( \overrightarrow{\mu} \) if all states in \( \mu \) have the same observable actions.
A relation $\mathcal{R} \subseteq \text{Dist}(S) \times \text{Dist}(S)$ is a late weak bisimulation over $\mathcal{M}$ iff $\mu \mathcal{R} \nu$ implies:

- whenever $\mu \xrightarrow{\theta} \mu'$, there exists a $\nu \xrightarrow{\theta} \nu'$ such that $\mu' \mathcal{R} \nu'$;
- if not, then there exists $\mu = \sum_{0 \leq i \leq n} p_i \cdot \mu_i$ and $\nu \xrightarrow{\tau} \sum_{0 \leq i \leq n} p_i \cdot \nu_i$ such that $\mu_i$ and $\mu_i \mathcal{R} \nu_i$ for each $0 \leq i \leq n$ where $\sum_{0 \leq i \leq n} p_i = 1$;
- symmetrically for $\nu$.

where $\mu$ if all states in $\mu$ have the same observable actions.

$s \approx^\bullet r$ iff $\delta_s \approx^\bullet \delta_r$
Examples
Examples

\[
\begin{aligned}
S_0 & \xrightarrow{\beta} r_1 \\
S_1 & \xrightarrow{\tau} S_2 \\
S_2 & \xrightarrow{\beta} S_3 \\
S_3 & \xrightarrow{\alpha} S_0
\end{aligned}
\]
Properties of $\approx$

- Relation on distributions.
- $\approx$ is strictly coarser than $\approx$.
- $\approx$ is compositional w.r.t. to partial information and distributed schedulers.
- $\approx$ is the coarsest compositional equivalence preserving trace distribution equivalence w.r.t. partial information and distributed schedulers.
Conclusion and Future Work

- Efficient Decision Algorithm (Currently exponential).
- Logical Characterization.
- ....
Thank You

Q & A