Floyd-Hoare Logic for Quantum Programs

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Outline

Introduction

Syntax of Quantum Programs

Operational Semantics

Denotational Semantics

Correctness Formulas

Proof System for Quantum Programs

Conclusion
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Conclusion
Quantum Programming

- Quantum Random Access Machine (QRAM) model

Quantum Programming

- Quantum Random Access Machine (QRAM) model
- A set of conventions for writing quantum pseudocode

Quantum Programming Languages

- qGCL: quantum extension of Dijkstra’s Guarded Command Language [1]

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Quantum Programming Languages

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- QCL: high-level, architecture independent, with a syntax derived from classical procedural languages like C or Pascal [2]
- QPL: functional in nature, with high-level features (loops, recursive procedures, structured data types) [3]

Quantum Programming Languages

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Quantum Programming Languages

- Scaffold: Quantum programming language (Princeton, UCS, UCSB) [1]
- Quipper: A Scalable Quantum Programming Language [2]

Floyd-Hoare Logic

Floyd-Hoare Logic for Quantum Programs


*This talk is based on:*

Syntax
A “core” language for imperative quantum programming
- A countably infinite set $Var$ of quantum variables
Syntax
A “core” language for imperative quantum programming
  ▶ A countably infinite set $\text{Var}$ of quantum variables
  ▶ Two basic data types: $\text{Boolean}$, $\text{integer}$
Syntax, Continued

Hilbert spaces denoted by **Boolean** and **integer**:

\[ \mathcal{H}_{\text{Boolean}} = \mathcal{H}_2, \]
\[ \mathcal{H}_{\text{integer}} = \mathcal{H}_\infty. \]

Space \( l_2 \) of square summable sequences

\[ \mathcal{H}_\infty = \left\{ \sum_{n=-\infty}^{\infty} \alpha_n |n\rangle : \alpha_n \in \mathbb{C} \text{ for all } n \in \mathbb{Z} \text{ and } \sum_{n=-\infty}^{\infty} |\alpha_n|^2 < \infty \right\}, \]

where \( \mathbb{Z} \) is the set of integers.
Syntax, Continued

A quantum register is a finite sequence of distinct quantum variables.

State space of a quantum register $\bar{q} = q_1, ..., q_n$:

$$\mathcal{H}_{\bar{q}} = \bigotimes_{i=1}^{n} \mathcal{H}_{q_i}.$$
Syntax, Continued

Quantum extension of classical \textbf{while}-programs:

\[ S ::= \text{skip} \mid q := 0 \mid \bar{q} := U\bar{q} \mid S_1 ; S_2 \mid \text{measure } M[\bar{q}] : S \]
\[ \mid \text{while } M[\bar{q}] = 1 \text{ do } S \]

- \( q \) is a quantum variable and \( \bar{q} \) a quantum register
Syntax, Continued

Quantum extension of classical **while**-programs:

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- \( q \) is a quantum variable and \( \bar{q} \) a quantum register
- \( U \) in the statement “\( \bar{q} := U\bar{q} \)” is a unitary operator on \( \mathcal{H}_{\bar{q}} \)
Syntax, Continued

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\]

- $q$ is a quantum variable and $\bar{q}$ a quantum register
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- statement \textbf{measure}:
Syntax, Continued

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- statement measure:
  - \( M = \{M_m\} \) is a measurement on the state space \( \mathcal{H}_{\overline{q}} \) of \( \overline{q} \)
Syntax, Continued

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- statement measure:
  - \( M = \{ M_m \} \) is a measurement on the state space \( \mathcal{H}_{\overline{q}} \) of \( \overline{q} \)
  - \( S = \{ S_m \} \) is a set of quantum programs such that each outcome \( m \) of measurement \( M \) corresponds to \( S_m \)
Syntax, Continued

Quantum extension of classical while-programs:

\[
S ::= \text{skip} \mid q := 0 \mid \overline{q} := Uq \mid S_1; S_2 \mid \text{measure } M[\overline{q}] : S \\
\mid \text{while } M[\overline{q}] = 1 \text{ do } S
\]

- \( q \) is a quantum variable and \( \overline{q} \) a quantum register
- \( U \) in the statement “\( \overline{q} := U\overline{q} \)” is a unitary operator on \( \mathcal{H}_{\overline{q}} \)
- statement measure:
  - \( M = \{M_m\} \) is a measurement on the state space \( \mathcal{H}_{\overline{q}} \) of \( \overline{q} \)
  - \( S = \{S_m\} \) is a set of quantum programs such that each outcome \( m \) of measurement \( M \) corresponds to \( S_m \)
- statement while: \( M = \{M_0, M_1\} \) is a yes-no measurement on \( \mathcal{H}_{\overline{q}} \)
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Operational Semantics

Denotational Semantics

Correctness Formulas

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Notation

- A quantum configuration is a pair

\[ \langle S, \rho \rangle \]
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- \( \rho \in \mathcal{D}^- (\mathcal{H}_{\text{all}}) \) is a partial density operator on \( \mathcal{H}_{\text{all}} \) — (global) state of quantum variables
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- \( \rho \in \mathcal{D}^{-}(\mathcal{H}_{\text{all}}) \) is a partial density operator on \( \mathcal{H}_{\text{all}} \) — (global) state of quantum variables
- Tensor product of the state spaces of all quantum variables:

\[ \mathcal{H}_{\text{all}} = \bigotimes_{\text{all } q} \mathcal{H}_{q} \]
Notation

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- \( \rho \in \mathcal{D}^- (\mathcal{H}_{\text{all}}) \) is a partial density operator on \( \mathcal{H}_{\text{all}} \) — (global) state of quantum variables
- Tensor product of the state spaces of all quantum variables:
  \[ \mathcal{H}_{\text{all}} = \bigotimes_{\text{all } q} \mathcal{H}_q \]

- Transitions between configurations:
  \[ \langle S, \rho \rangle \rightarrow \langle S', \rho' \rangle \]
Operational Semantics

(Skip) \[
\langle \text{skip}, \rho \rangle \rightarrow \langle E, \rho \rangle
\]

(Initialization) \[
\langle q := 0, \rho \rangle \rightarrow \langle E, \rho^q_0 \rangle
\]

- \textbf{type}(q) = \text{Boolean:}

\[
\rho^q_0 = |0\rangle_q \langle 0|_q \rho |0\rangle_q \langle 0| + |0\rangle_q \langle 1|_q \rho |1\rangle_q \langle 0|
\]
Operational Semantics

\[(\text{Skip})\quad \langle \text{skip}, \rho \rangle \rightarrow \langle E, \rho \rangle\]

\[(\text{Initialization})\quad \langle q := 0, \rho \rangle \rightarrow \langle E, \rho_q^0 \rangle\]

- \textit{type}(q) = \text{Boolean}: 
  \[\rho_q^0 = |0\rangle_q |0\rangle_q |0\rangle_q |0\rangle + |0\rangle_q |1\rangle_q |1\rangle_q |0\rangle\]

- \textit{type}(q) = \text{integer}: 
  \[\rho_q^0 = \sum_{n=-\infty}^{\infty} |0\rangle_q |n\rangle_q |n\rangle_q |0\rangle\]
Operational Semantics, Continued

(Unitary Transformation) \[ \langle \bar{q} := Uq, \rho \rangle \rightarrow \langle E, U\rho U^\dagger \rangle \]

(Sequential Composition) \[ \langle S_1, \rho \rangle \rightarrow \langle S'_1, \rho' \rangle \]
\[ \langle S_1; S_2, \rho \rangle \rightarrow \langle S'_1; S_2, \rho' \rangle \]

Convention: \( E; S_2 = S_2 \).

(Measurement) \[ \langle \text{measure } M[\bar{q}] : \bar{S}, \rho \rangle \rightarrow \langle S_m, M_m\rho M_m^\dagger \rangle \]

for each outcome \( m \)
Operational Semantics, Continued

(Loop 0) \[ \langle \text{while } M[\bar{q}] = 1 \text{ do } S, \rho \rangle \rightarrow \langle E, M_0 \rho M_0^\dagger \rangle \]

(Loop 1) \[ \langle \text{while } M[\bar{q}] = 1 \text{ do } S, \rho \rangle \rightarrow \langle S; \text{while } M[\bar{q}] = 1 \text{ do } S, M_1 \rho M_1^\dagger \rangle \]
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Operational Semantics

**Denotational Semantics**

Correctness Formulas

Proof System for Quantum Programs

Conclusion
**Definition**

Semantic function of quantum program $S$:

$$\llbracket S \rrbracket : \mathcal{D}^{-}(\mathcal{H}_{\text{all}}) \rightarrow \mathcal{D}^{-}(\mathcal{H}_{\text{all}})$$

is defined by

$$\llbracket S \rrbracket (\rho) = \sum \{|\rho' : \langle S, \rho \rangle \rightarrow^* \langle E, \rho' \rangle| \}$$

for all $\rho \in \mathcal{D}^{-}(\mathcal{H}_{\text{all}})$. 
Representation of Semantic Function

1. $\llbracket \text{skip} \rrbracket (\rho) = \rho$. 
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2. $\triangleright type(q) = \text{Boolean}$:

   $\llbracket q := 0 \rrbracket (\rho) = |0\rangle_q \langle 0| \rho |0\rangle_q \langle 0| + |0\rangle_q \langle 1| \rho |1\rangle_q \langle 0|$.

   $\triangleright type(q) = \text{integer}$:

   $\llbracket q := 0 \rrbracket (\rho) = \sum_{n=-\infty}^{\infty} |0\rangle_q \langle n| \rho |n\rangle_q \langle 0|$. 
Representation of Semantic Function

1. $⟦\text{skip}⟧(\rho) = \rho$.

2. $\text{type}(q) = \text{Boolean}$:
   
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   $\text{type}(q) = \text{integer}$:
   
   $⟦q := 0⟧(\rho) \sum_{n=-\infty}^{\infty} |0\rangle_q\langle n|\rho|n\rangle_q\langle 0|$. 

3. $⟦\bar{q} := U\bar{q}⟧(\rho) = U\rho U^\dagger$. 

**Representation of Semantic Function**

1. $\llbracket \text{skip} \rrbracket(\rho) = \rho$.

2. $\triangleright type(q) = \text{Boolean}$:
   
   $\llbracket q := 0 \rrbracket(\rho) = |0\rangle_q\langle 0| \rho |0\rangle_q\langle 0| + |0\rangle_q\langle 1| \rho |1\rangle_q\langle 0|$. 

   $type(q) = \text{integer}$:

   $\llbracket q := 0 \rrbracket(\rho) = \sum_{n=-\infty}^{\infty} |0\rangle_q\langle n| \rho |n\rangle_q\langle 0|$. 

3. $\llbracket \overline{q} := U\overline{q} \rrbracket(\rho) = U \rho U^\dagger$.

4. $\llbracket S_1; S_2 \rrbracket(\rho) = \llbracket S_2 \rrbracket(\llbracket S_1 \rrbracket(\rho))$. 
Representation of Semantic Function

1. \([\text{skip}] (\rho) = \rho\).

2. ▶ \(\text{type}(q) = \text{Boolean}:\)

\[
[q := 0](\rho) = |0\rangle_q |0\rangle_\rho |0\rangle_q \langle 0| + |0\rangle_q |1\rangle_\rho |1\rangle_q \langle 0|.
\]

\(\text{type}(q) = \text{integer:}\)

\[
[q := 0](\rho) \sum_{n=-\infty}^{\infty} |0\rangle_q |n\rangle_\rho |n\rangle_q \langle 0|.
\]

3. \([\bar{q} := U\bar{q}] (\rho) = U\rho U^\dagger\).

4. \([S_1; S_2] (\rho) = [S_2] ([S_1] (\rho))\).

5. \([\text{measure } M[\bar{q}] : S] (\rho) = \sum_m [S_m] (M_m \rho M_m^\dagger)\).
Representation of Semantic Function

1. $\llbracket \text{skip} \rrbracket (\rho) = \rho$.
2. $\triangleright$ *type*(\(q\)) = *Boolean*:

   \[
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   *type*(\(q\)) = *integer*:

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   \llbracket q := 0 \rrbracket (\rho) \sum_{n=-\infty}^{\infty} |0\rangle_q \langle n| \rho |n\rangle_q \langle 0|.
   \]

3. $\llbracket \overline{q} := U\overline{q} \rrbracket (\rho) = U\rho U^\dagger$.
4. $\llbracket S_1; S_2 \rrbracket (\rho) = \llbracket S_2 \rrbracket (\llbracket S_1 \rrbracket (\rho))$.
5. $\llbracket \text{measure } M[\overline{q}]: \overline{S} \rrbracket (\rho) = \sum_{m} \llbracket S_m \rrbracket (M_m \rho M_m^\dagger)$.
6. $\llbracket \text{while } M[\overline{q}] = 1 \text{ do } S \rrbracket (\rho) = \bigvee_{n=0}^{\infty} \llbracket (\text{while } M[\overline{q}] = 1 \text{ do } S)^n \rrbracket (\rho)$.
Notation

\((\text{while } M[\overline{q}] = 1 \text{ do } S)^0 = \Omega,\)

\((\text{while } M[\overline{q}] = 1 \text{ do } S)^{n+1} = \text{measure } M[\overline{q}] : \overline{S},\)

where:

- \(\Omega\) is a program such that \([\Omega] = 0\text{ for all } \rho \in \mathcal{D}(\mathcal{H})\)
Notation

\[(\text{while } M[\bar{q}] = 1 \text{ do } S)^0 = \Omega,\]
\[(\text{while } M[\bar{q}] = 1 \text{ do } S)^{n+1} = \text{measure } M[\bar{q}] : \bar{S},\]

where:

- \(\Omega\) is a program such that \([\Omega] = 0_{\mathcal{H}}\) for all \(\rho \in \mathcal{D}(\mathcal{H})\)
- \(\bar{S} = S_0, S_1\),
Notation

\[(\text{while } M[q] = 1 \text{ do } S)^0 = \Omega,\]
\[(\text{while } M[q] = 1 \text{ do } S)^{n+1} = \text{measure } M[q] : \bar{S},\]

where:

- \(\Omega\) is a program such that \([\Omega] = 0_{\forall}\) for all \(\rho \in D(H)\)
- \(\bar{S} = S_0, S_1,\)
  - \(S_0 = \text{skip},\)
  - \(S_1 = S; (\text{while } M[q] = 1 \text{ do } S)^n\)

for all \(n \geq 0.\)
Recursion

\[
[\texttt{while}] (\rho) = M_0 \rho M_0^\dagger + [\texttt{while}] ([S](M_1 \rho M_1^\dagger))
\]
for all \( \rho \in \mathcal{D}^- (\mathcal{H}_{all}) \), where:

- \texttt{while} is the quantum loop “\texttt{while} \ M[\bar{q}] = 1 \ \texttt{do} \ S”.
Observation:

\[ \text{tr}(\|S\|(\rho)) \leq \text{tr}(\rho) \]

for any quantum program \( S \) and all \( \rho \in \mathcal{D}^{-}(\mathcal{H}_{\text{all}}) \).

- \( \text{tr}(\rho) - \text{tr}(\|S\|(\rho)) \) is the probability that program \( S \) diverges from input state \( \rho \).
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Operational Semantics

Denotational Semantics

Correctness Formulas

Proof System for Quantum Programs

Conclusion
Definition


- For any $X \subseteq \text{Var}$, a quantum predicate on $\mathcal{H}_X$ is a Hermitian operator $P$:

\[ 0_{\mathcal{H}_X} \sqsubseteq P \sqsubseteq I_{\mathcal{H}_X}. \]
Definition


- For any $X \subseteq \text{Var}$, a quantum predicate on $\mathcal{H}_X$ is a Hermitian operator $P$:
  
  $$0_{\mathcal{H}_X} \subseteq P \subseteq I_{\mathcal{H}_X}.$$ 

- $\mathcal{P}(\mathcal{H}_X)$ denotes the set of quantum predicates on $\mathcal{H}_X$. 

Definition


▶ For any $X \subseteq \text{Var}$, a quantum predicate on $\mathcal{H}_X$ is a Hermitian operator $P$:

$$0_{\mathcal{H}_X} \subseteq P \subseteq I_{\mathcal{H}_X}.$$  

▶ $\mathcal{P}(\mathcal{H}_X)$ denotes the set of quantum predicates on $\mathcal{H}_X$.

▶ For any $\rho \in \mathcal{D}^-(\mathcal{H}_X)$, $tr(P\rho)$ stands for the probability that predicate $P$ is satisfied in state $\rho$. 
Definition

A correctness formula (Hoare triple) is a statement of the form:

{P}S{Q}

where:

- S is a quantum program
Definition

A correctness formula (Hoare triple) is a statement of the form:

\[ \{P\} S \{Q\} \]

where:
- $S$ is a quantum program
- $P$ and $Q$ are quantum predicates on $\mathcal{H}_{all}$. 
Definition

A correctness formula (*Hoare triple*) is a statement of the form:

\[ \{ P \} S \{ Q \} \]

where:
- $S$ is a quantum program
- $P$ and $Q$ are quantum predicates on $\mathcal{H}_{all}$.
- Operator $P$ is called the *precondition* and $Q$ the *postcondition*. 
Definition

1. The correctness formula $\{P\} S \{Q\}$ is true in the sense of *total correctness*, written

$$\models_{\text{tot}} \{P\} S \{Q\},$$

if

$$tr(P\rho) \leq tr(Q[S](\rho))$$

for all $\rho \in \mathcal{D}^{-}(\mathcal{H}_{all})$. 
Definition

1. The correctness formula \( \{P \} S \{Q \} \) is true in the sense of total correctness, written
   \[ \models_{\text{tot}} \{P \} S \{Q \}, \]
   if
   \[ tr(P\rho) \leq tr(Q[S](\rho)) \]
   for all \( \rho \in \mathcal{D}^{-}(\mathcal{H}_{all}) \).

2. The correctness formula \( \{P \} S \{Q \} \) is true in the sense of partial correctness, written
   \[ \models_{\text{par}} \{P \} S \{Q \}, \]
   if
   \[ tr(P\rho) \leq tr(Q[S](\rho)) + [tr(\rho) - tr([S](\rho))] \]
   for all \( \rho \in \mathcal{D}^{-}(\mathcal{H}_{all}) \).
Outline

Introduction

Syntax of Quantum Programs

Operational Semantics

Denotational Semantics

Correctness Formulas

Proof System for Quantum Programs

Conclusion
Proof System PD for Partial Correctness

(Axiom Skip) \( \{ P \} \text{Skip}\{ P \} \)

(Axiom Initialization)
\[
\text{type}(q) = \text{Boolean} : \n\{ |0\rangle_q \langle 0| P |0\rangle_q \langle 0| + |1\rangle_q \langle 0| P |0\rangle_q \langle 1| \} q := 0\{ P \}
\]
\[
\text{type}(q) = \text{integer} : \n\{ \sum_{n=-\infty}^{\infty} |n\rangle_q \langle 0| P |0\rangle_q \langle n| \} q := 0\{ P \}
\]

(Axiom Unitary Transformation) \( \{ U^\dagger PU \} q := Uq\{ P \} \)
Proof System $PD$ for Partial Correctness, Continued

(Rule Sequential Composition) \[
\frac{\{P\} S_1 \{Q\} \quad \{Q\} S_2 \{R\}}{\{P\} S_1; S_2 \{R\}}
\]

(Rule Measurement) \[
\frac{\{P_m\} S_m \{Q\} \text{ for all } m}{\{\sum_m M^+_m P_m M_m\} \text{measure } M[\bar{q}] : \bar{S}\{Q\}}
\]

(Rule Loop Partial) \[
\frac{\{Q\} S\{M^+_0 PM_0 + M^+_1 QM_1\}}{\{M^+_0 PM_0 + M^+_1 QM_1\}} \text{while } M[\bar{q}] = 1 \text{ do } S\{P\}
\]

(Rule Order) \[
\frac{P \sqsubseteq P' \quad \{P'\} S\{Q'\} \quad Q' \sqsubseteq Q}{\{P\} S\{Q\}}
\]
Soundness Theorem for PD

Proof system PD is sound for partial correctness of quantum programs.

- For any quantum program $S$ and quantum predicates $P, Q \in \mathcal{P}(\mathcal{H}_{\text{all}})$, we have:

$$
\vdash_{PD} \{P\} S\{Q\} \text{ implies } \models_{\text{par}} \{P\} S\{Q\}.
$$
Completeness Theorem for PD

Proof system PD is complete for partial correctness of quantum programs.

- For any quantum program $S$ and quantum predicates $P, Q \in \mathcal{P}(\mathcal{H}_{all})$, we have:

$$\models_{\text{par}} \{P\}S\{Q\} \text{ implies } \vdash_{PD} \{P\}S\{Q\}.$$
Proof System \textit{TD} for Total Correctness

Let \( P \in \mathcal{P}(\mathcal{H}_{\text{all}}) \) and \( \epsilon > 0 \). A function

\[ t : \mathcal{D}^- (\mathcal{H}_{\text{all}}) \rightarrow \mathbb{N} \]

is called a \((P, \epsilon)\)-bound function of quantum loop:

\[
\text{while } M[\tilde{q}] = 1 \text{ do } S
\]

if:

1. \( t(\| S \| (M_1 \rho M_1^\dagger)) \leq t(\rho) \);

for all \( \rho \in \mathcal{D}^- (\mathcal{H}_{\text{all}}) \).
Proof System $TD$ for Total Correctness

Let $P \in \mathcal{P}(\mathcal{H}_{\text{all}})$ and $\epsilon > 0$. A function

$$t : \mathcal{D}^{-}(\mathcal{H}_{\text{all}}) \rightarrow \mathbb{N}$$

is called a $(P, \epsilon)$–bound function of quantum loop:

$$\textbf{while } M[\bar{q}] = 1 \textbf{ do } S$$

if:

1. $t(\|S\| (M_1 \rho M_1^\dagger)) \leq t(\rho)$;
2. $\text{tr}(P \rho) \geq \epsilon$ implies $t(\|S\| (M_1 \rho M_1^\dagger)) < t(\rho)$

for all $\rho \in \mathcal{D}^{-}(\mathcal{H}_{\text{all}})$. 
Proof System $TD$ for Total Correctness

Proof System $TD = (\text{Proof System } PD - \text{Rule Loop Partial})$

$+ \text{Rule Loop Total}$
Proof System $TD$ for Total Correctness

Proof System $TD = (\text{Proof System } PD - \text{Rule Loop Partial})$

$+ \text{Rule Loop Total}$

Rule: Total Correctness for Loop

$$\{Q\} S\{M_0^\dagger PM_0 + M_1^\dagger QM_1\}$$

(2) for any $\epsilon > 0$, $t_\epsilon$ is a $(M_1^\dagger QM_1, \epsilon) - \text{bound}$ function of loop $\textbf{while } M[\bar{q}] = 1 \textbf{ do } S$

$$\{M_0^\dagger PM_0 + M_1^\dagger QM_1\} \textbf{while } M[\bar{q}] = 1 \textbf{ do } S\{P\}$$
Soundness Theorem for TD

Proof system TD is sound for total correctness of quantum programs.

For any quantum program $S$ and quantum predicates $P, Q \in \mathcal{P}(\mathcal{H}_{all})$, we have:

$$\vdash_{TD} \{P\}S\{Q\} \text{ implies } \models_{tot} \{P\}S\{Q\}.$$
Completeness Theorem

The proof system $TD$ is complete for total correctness of quantum programs.

- For any quantum program $S$ and quantum predicates $P, Q \in \mathcal{P}(\mathcal{H}_{all})$, we have:

$$\models_{\text{tot}} \{P\}S\{Q\} \text{ implies } \vdash_{TD} \{P\}S\{Q\}.$$
Proof Outline

- Claim: \( \vdash_{PD} \{ wp.S.Q \} S \{ Q \} \) for any quantum program \( S \) and quantum predicate \( P \in \mathcal{P}(\mathcal{H}_{all}) \).

Induction on the structure of \( S \).

\[
wp.\text{while}.Q = M_0^\dagger Q M_0 + M_1^\dagger (wp.S.(wp.\text{while}.Q)) M_1.
\]

Our aim is to derive:

\[
\{ M_0^\dagger Q M_0 + M_1^\dagger (wp.S.(wp.\text{while}.Q)) M_1 \} \text{while}\{ Q \}.
\]
Proof Outline

▶ Claim: $\vdash_{PD} \{wlps.Q\}S\{Q\}$ for any quantum program $S$ and quantum predicate $P \in \mathcal{P}(H_{all})$.

Induction on the structure of $S$.

▶ Example case: $S = \textbf{while } M[\bar{q}] = 1 \textbf{ do } S'$.

\[
wp.\textbf{while}.Q = M^+_0QM_0 + M^+_1(wp.S.(wp.\textbf{while}.Q))M_1.
\]

Our aim is to derive:

\[
\{M^+_0QM_0 + M^+_1(wp.S.(wp.\textbf{while}.Q))M_1\}\textbf{while}\{Q\}.
\]
Proof Outline, Continued

- Induction hypothesis on $S'$:

$$\{wp.S'.(wp.while.Q)\}S\{wp.while.Q\}.$$
Proof Outline, Continued

- Induction hypothesis on $S'$:
  \[
  \{wp.S'.(wp.\textbf{while}.Q)\}S\{wp.\textbf{while}.Q\}.
  \]

- Rule Loop Total: It suffices to show that for any $\epsilon > 0$, there exists a $(M_1^\dagger(wp.S'.(wp.S.Q))M_1, \epsilon)$—bound function of quantum loop \textbf{while}.
Proof Outline, Continued

- Induction hypothesis on \( S' \):

\[
\{wp.S'.(wp.\text{while}.Q)\}S\{wp.\text{while}.Q\}.
\]

- Rule Loop Total: It suffices to show that for any \( \epsilon > 0 \), there exists a \((\hat{M}_1^+(wp.S'. (wp.S.Q))M_1, \epsilon)\)—bound function of quantum loop \text{while}.

- Bound Function Lemma: We only need to prove:

\[
\lim_{n \to \infty} tr(\hat{M}_1^+(wp.S'.(wp.\text{while}.Q))M_1([S'] \circ \mathcal{E}_1)^n(\rho)) = 0.
\]
Proof Outline, Continued

We observe:

\[
\text{tr}(M_1^\dagger(wp.S'.(wp.\text{while}.Q))M_1([S'] \circ E_1)^n(\rho)) = \text{tr}(wp.S'.(wp.\text{while}.Q)M_1([S'] \circ E_1)^n(\rho)M_1^\dagger) = \text{tr}(wp.\text{while}.Q[S'](M_1([S'] \circ E_1)^n(\rho)M_1^\dagger)) = \text{tr}(wp.\text{while}.Q([S'] \circ E_1)^{n+1}(\rho)) = \text{tr}(Q[\text{while}][S'] \circ E_1)^{n+1}(\rho)) = \sum_{k=n+1}^{\infty} \text{tr}(Q[E_0 \circ ([S'] \circ E_1)^k](\rho)).
\]
Proof Outline, Continued

We consider the infinite series of nonnegative real numbers:

$$\sum_{n=0}^{\infty} tr(Q[\mathcal{E}_0 \circ (\llbracket S' \rrbracket \circ \mathcal{E}_1)^k](\rho)) = tr(Q \sum_{n=0}^{\infty} [\mathcal{E}_0 \circ (\llbracket S' \rrbracket \circ \mathcal{E}_1)^k](\rho)).$$

Since $Q \sqsubseteq I_{\mathcal{H}_{all}}$, it follows that

$$tr(Q \sum_{n=0}^{\infty} [\mathcal{E}_0 \circ (\llbracket S' \rrbracket \circ \mathcal{E}_1)^k](\rho)) = tr(Q[\textbf{while}](\rho))$$

$$\leq tr([\textbf{while}](\rho)) \leq tr(\rho) \leq 1.$$
Outline

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Syntax of Quantum Programs

Operational Semantics

Denotational Semantics

Correctness Formulas

Proof System for Quantum Programs

Conclusion
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Floyd-Hoare logic for deterministic quantum programs!

Topics for further studies:

- Connection to probabilistic programming:
Conclusion

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- Concurrent quantum programs:
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Floyd-Hoare logic for deterministic quantum programs!

Topics for further studies:

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▶ Classical control flow $\Rightarrow$ quantum control flow?
Thank You!