Exercise sheet 3 on Discrete Mathematics

Naijun Zhan Yangjia Li

http://lcs.ios.ac.cn/~znj/DM2016

Let A be a finite alphabet, such as the English alphabet, and consider the following simple programming language **WHILE** whose well-formed programs are those obtained by applying the rules of the following grammar:

Numbers:

$$Num ::= d \mid dNum \qquad \text{where } d \in \{0, 1, \ldots, 9\}$$

Identifiers:

$$\begin{aligned} Id &::= aId' \qquad \text{where } a \in A \\ Id' &::= \lambda \mid aId' \mid NumId' \end{aligned}$$

Numeric expressions:

$$\begin{aligned} Exp ::= &\ Num \mid Id \\ &\mid Exp + Exp \mid Exp - Exp \mid Exp * Exp \mid Exp/Exp \end{aligned}$$

Boolean expressions:

$$\begin{aligned} BExp ::= &\ \mathbf{true} \mid \mathbf{false} \mid Exp = Exp \\ &\mid \neg BExp \mid BExp \wedge BExp \mid BExp \vee BExp \end{aligned}$$

Programs:

$$\begin{aligned} P ::= &\ \mathbf{skip} \mid Id := Exp \mid P; P \\ &\mid \mathbf{if}\ BExp\ \mathbf{then}\ P\ \mathbf{else}\ P\ \mathbf{fi} \\ &\mid \mathbf{while}\ BExp\ \mathbf{do}\ P\ \mathbf{done} \end{aligned}$$

Exercise 3.1. *Show, by providing the appropriate functions, that the set of identifiers has the same cardinality as* $\mathbb{N}$.

Exercise 3.2. *Show that the set of all well-formed* **WHILE** *programs is countable.*

Exercise 3.3. *Show the following statements:*

1. *If A is an uncountable set and B is a countable set, then $|A \setminus B| = |A|$.*
2. *Let $\mathbb{I}$ be the set of irrational numbers, i.e., $\mathbb{I} = \{ r \in \mathbb{R} \mid r \notin \mathbb{Q} \}$. Then $|\{0,1\}^\omega \setminus \{0,1\}^* 0^\omega| = |\mathbb{I} \cap [0,1]|$.*
3. *$|\{0,1\}^\omega| = |[0,1]|$.*
4. *$|[0,1]| = |[0,1] \times [0,1]| = |[0,1]^n|$ for each $n \geq 1$.*

Exercise 3.4. *Consider the semantic bracket operator presented in Definition 2.1.6 of the lecture notes. Show that $\models \varphi \rightarrow \psi$ iff $[\![\varphi]\!] \subseteq [\![\psi]\!]$.*

Exercise 3.5. *Let $\mathfrak{Q}_1 \subseteq FOL$ be the set of FOL formulas φ such that each quantifier operator Qx (with $Q \in \{\forall, \exists\}$ and $x \in VS$) appears at most once in φ.*

Provide a function Scope, including its type, such that, given a quantifier operator Qx (with $Q \in \{\forall, \exists\}$ and $x \in VS$) and a formula $\varphi \in \mathfrak{Q}_1$, it returns the formula corresponding to the scope of Qx.

Exercise 3.6. *Given $\varphi, \psi, \eta \in FOL$,*

1. *provide a function $\#$, including its type, that returns how many times ψ occurs in φ as sub-formula;*
2. *provide a function R_∞, including its type, that replaces each occurrence of φ in η with ψ;*
3. *provide a function R_1, including its type, that replaces exactly one occurrence of φ in η with ψ if φ occurs in η, and that returns η if φ does not occur in η. If φ occurs in η multiple times, then there is no requirement on the particular instance to be replaced.*

Exercise 3.7. *Prove the following:*

1. *For any predicate P with arity 2, $\forall x \forall y P(x, y) \vdash \forall y \forall z P(y, z)$.*
2. *Assume x is not free in φ, then $\varphi \to \forall x \psi$ and $\forall x(\varphi \to \psi)$ are syntactically equivalent.*
3. *We say a formula φ has* repeated occurrences *of a bound variable x, if Qx appears more than once in the sub-formulas of φ (recall $Q \in \{\forall, \exists\}$). Prove that there exists a formula φ' which has no repeated occurrences of any bound variable such that φ and φ' are syntactically equivalent.*

Exercise 3.8. *Prove the following:*

Q1. $\neg\forall x\varphi \vdash\dashv \exists x\neg\varphi$; *and* $\neg\exists x\varphi \vdash\dashv \forall x\neg\varphi$.

Q2. $\forall x\varphi \wedge \psi \vdash\dashv \forall x(\varphi \wedge \psi$, *if* x *does not occur in* ψ.

Q3. $\exists x\varphi \vee \exists x\psi \vdash\dashv \exists x(\varphi \vee \psi)$.

Exercise 3.9. *Let* S *be a binary predicate symbol,* P *and* Q *unary predicate symbols. Prove the following:*

Q1 $\exists x\exists y(S(x,y) \vee S(y,x)) \vdash \exists x\exists y S(x,y)$.

Q2 $\forall x\forall y\forall z(S(x,y) \wedge S(y,z) \to S(x,z))$, $\forall x\neg S(x,x) \vdash \forall x\forall y(S(x,y) \to \neg S(y,x))$.

Q3 $\exists x\exists y(S(x,y) \vee S(y,x))$, $\neg\exists x S(x,x) \vdash \exists x\exists y(x \neq y)$.

Q4 $\forall x(P(x) \vee Q(x)) \vdash \forall x P(x) \vee \forall Q(x)$ *is not provable.*

Exercise 3.10. *Prove Lemma 3.4.3: a Hintikka set* Γ *is consistent, and moreover, for each formula* φ*, either* $\varphi \notin \Gamma$*, or* $\neg\varphi \notin \Gamma$*.*

Exercise 3.11. *Given a formula* φ*, let* $\mathfrak{H}(\varphi)$ *be the set of Hintikka sets containing* φ*, that is,* $\mathfrak{H}(\varphi) = \{\, \Gamma \subseteq FOF \mid \varphi \in \Gamma \text{ and } \Gamma \text{ is a Hintikka set} \,\}$*. We say that* $\Gamma \in \mathfrak{H}(\varphi)$ *is minimal if, for each* $\Gamma' \in \mathfrak{H}(\varphi)$*, it holds that* $\Gamma' \subseteq \Gamma$ *implies* $\Gamma' = \Gamma$*; we denote by* $\mathfrak{m}(\varphi)$ *the set of minimal Hintikka sets in* $\mathfrak{H}(\varphi)$*, that is,* $\mathfrak{m}(\varphi) = \{\, \Gamma \in \mathfrak{H}(\varphi) \mid \Gamma \text{ is minimal} \,\}$*.*

1. *Provide a minimal Hintikka set* $\Gamma_\varphi \in \mathfrak{m}(\varphi)$ *for the formula*

$$\varphi = \forall x \forall y (\neg(x \approx y) \to (R(x,y) \to \neg R(y,x)))$$

   under the assumption that $VS = \{x, y\}$*,* $FS = CS = \{a, b\}$*,* $PS = \{R\}$*, and* $ES = \{\approx\}$*.*

2. *Prove that* $\mathfrak{H}(\varphi) \cap \mathfrak{H}(\neg\varphi) = \emptyset$ *for each* $\varphi \in FOF$*.*

3. *Let* $PL \subseteq FOF$ *be the set of FOL formulas in which each predicate appears at most once and in which no quantifier* $Q \in \{\forall, \exists\}$ *occurs. Define a function* $c \colon PL \to \mathbb{N}$ *such that, for each* $\varphi \in PL$*, returns the number of different minimal Hintikka sets containing* φ*.*