A Characterization of Robust Regions of Attraction for Discrete-Time Systems Based on Bellman Equations

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Abstract: In this paper we present a Bellman equation for computing robust regions of attraction for state-constrained perturbed discrete-time systems. The robust region of attraction of interest is a set of states such that every trajectory initialized in it will approach an equilibrium while never violating the specified state constraint, regardless of the actual perturbation. The interior of the maximal robust region of attraction is characterized as the strict one sub-level set of the unique bounded and continuous solution to a Bellman equation.

Keywords: Robust Regions of Attraction; State-Constrained Perturbed Discrete-Time Systems; Bellman Equations.

1. INTRODUCTION

A fundamental problem in control engineering consists of determining the robust region of attraction of an equilibrium, which is a set of states such that every trajectory starting from it will move towards this equilibrium while never leaving a specified state-constraint set irrespective of the actual perturbation. Its applications include biology systems (Merola et al., 2008), ecology systems (Ludwig et al., 1997) and among others. Computing robust regions of attraction has been the subject of extensive research over the past several decades, resulting in the emergence of a number of theories and corresponding computational approaches, e.g., Lyapunov function-based methods (Salle and Lefschetz, 1961; Coutinho and de Souza, 2013; Chesi, 2004; Giesl, 2007; Valmorbida and Anderson, 2014; Giesl and Hafstein, 2014), trajectory reversing methods (Genesio et al., 1985), moment-based optimization methods (Korda et al., 2013) and so on.

Another attractive means in computing robust regions of attraction is by exploiting the link to optimal control. When the system is continuous-time, the link is established through viscosity solutions of Hamilton-Jacobi type equations, e.g., (Margellos and Lygeros, 2011; Mitchell et al., 2005; Bokanowski et al., 2010; Xue et al., 2019, 2020). It extends the use of Hamilton-Jacobi equations, which are widely used in optimal control theory (e.g., (Bardi and Capuzzo-Dolcettae, 1997)), to perform reachability analysis. While computationally intensive, Hamilton-Jacobi reachability approaches are appealing nowadays due to the availability of modern numerical tools such as (Mitchell, 2007; Bokanowski et al., 2011), which allow solving associated problems conveniently for appropriate numbers of state variables. Recently, Zubov's equation (Zubov, 1964), which was originally inferred to describe the maximal region of attraction for continuous-time dynamical systems free of state constraints and perturbation inputs, was extended to perturbed systems in (Camilli et al., 2001) and further to state-constrained perturbed systems in (Grüne and Zidani, 2015). When the system is discrete-time, Bellman equations, which are widely used in discrete-time optimal control (Bardi and Capuzzo-Dolcettae, 1997), have also been studied for performing reachability analysis (Xue and Zhan, 2018). However, to the best of our knowledge, there is no previous work on the use of Bellman equations to characterize the maximal robust region of attraction for state-constrained perturbed discrete-time systems. This motivates the study in this paper.

In this paper we present a modified Bellman equation for computing robust regions of attraction for state-constrained perturbed discrete-time systems with an equilibrium state, which is uniformly locally exponentially stable. The interior of the maximal robust region of attraction is characterized as the strict one sub-level set of the unique bounded and continuous solution to the derived Bellman equation. The derivation of the Bellman equation follows the reasoning in (Grüne and Zidani, 2015), which presented a modified Zubov's equation for computing robust regions of attraction for state-constrained perturbed continuous systems. One example is used to illustrate the computation of the interior of the maximal robust region of attraction via solving the Bellman equation.

The main contribution of this paper is summarized as follows. We for the first time infer a Bellman equation, to which the strict one sub-level set of the unique bounded and continuous solution characterizes the interior of the maximal robust region of attraction for state-constrained perturbed discrete-time polynomial systems. To the best of our knowledge, this is the first possibility to estimate the maximal robust region of attraction for state-constrained perturbed discrete-time polynomial systems.

This paper is structured as follows. In Section 2 basic notions and the problem of interest are introduced. After presenting the Bellman equation in Section 3, we estimate the maximal robust region of attraction for one example via solving the Bellman equation in Section 4. Finally, we conclude this paper in Section 5.

2. PRELIMINARIES

In this section we describe the system of interest and the concept of robust regions of attraction.

The notions will be used in this paper: $\mathbb{R}^n$ denotes the set of $n-$dimensional real vectors. Δ°, $\partial\Delta$, $\overline{\Delta}$ and Δ^c denote the interior, boundary, closure and complement of a set Δ, respectively. The space of continuous functions on a set Δ is denoted by $C(\Delta)$. The difference of two sets A and B is denoted by $A \setminus B$. $\mu(A)$ denotes the Lebesgue measure on $A \subset \mathbb{R}^n$. $\mathbb{N}$ denotes the set of non-negative integers. $\|\boldsymbol{x}\|$ denotes the 2-norm, i.e., $\|\boldsymbol{x}\| = \sqrt{\sum_{i=1}^n x_i^2}$, where $\boldsymbol{x} = (x_1, \dots, x_n)^\top$. $B(\mathbf{0}, r)$ denotes a ball of radius $r > 0$ and center $\mathbf{0}$, i.e., $B(\mathbf{0}, r) = \{\boldsymbol{x} \mid \|\boldsymbol{x}\|^2 \leq r\}$. Vectors are denoted by boldface letters.

The perturbed discrete-time system of interest in this paper is of the following form

$$\boldsymbol{x}(k+1) = \boldsymbol{f}(\boldsymbol{x}(k), \boldsymbol{d}(k)), k \in \mathbb{N}, \quad (1)$$

where $\boldsymbol{x}(\cdot) : \mathbb{N} \to \mathbb{R}^n$, $\boldsymbol{d}(\cdot) : \mathbb{N} \to D$, $D = \{\boldsymbol{d} \in \mathbb{R}^m \mid \wedge_{i=1}^{m_d} [h_i^D(\boldsymbol{d}) \leq 0]\}$ is a compact subset in $\mathbb{R}^m$ with $h_i^D \in C(\mathbb{R}^m)$, $\boldsymbol{f} \in C(\mathbb{R}^n \times \mathbb{R}^m)$ is locally Lipschitz continuous over $\boldsymbol{x} \in \mathbb{R}^n$ uniformly over $\boldsymbol{d} \in D$, $\boldsymbol{f}(\mathbf{0}, \boldsymbol{d}) = \mathbf{0}$ for $\boldsymbol{d} \in D$.

In order to define our problem succinctly, we present the definition of a perturbation input policy π.

Definition 1. A perturbation input policy, denoted by π, refers to a function $\pi(k) : \mathbb{N} \to D$. In addition, we denote the set of all perturbation policies by $\mathcal{D}$.

Given a perturbation input policy π, a trajectory to system (1) is presented in Definition 2.

Definition 2. Given a perturbation input policy $\pi \in \mathcal{D}$, a trajectory of system (1) initialized in $\boldsymbol{x}_0 \in \mathbb{R}^n$ is defined as $\boldsymbol{\phi}_{\boldsymbol{x}_0}^\pi(\cdot) : \mathbb{N} \to \mathbb{R}^n$, where $\boldsymbol{\phi}_{\boldsymbol{x}_0}^\pi(0) = \boldsymbol{x}_0$, and

$$\boldsymbol{\phi}_{\boldsymbol{x}_0}^\pi(k+1) = \boldsymbol{f}(\boldsymbol{\phi}_{\boldsymbol{x}_0}^\pi(k), \pi(k)), \forall k \in \mathbb{N}.$$

We assume that $\mathbf{0}$ is uniformly locally exponentially stable for system (1).

Assumption 1. The equilibrium state $\mathbf{0}$ is uniformly locally exponentially stable for (1), i.e., there exist positive constants $M > 0$, $r > 0$ and $0 < \lambda < 1$ such that

$$\|\boldsymbol{\phi}_{\boldsymbol{x}_0}^\pi(k)\| \leq \lambda^k M \|\boldsymbol{x}_0\|, \forall \boldsymbol{x}_0 \in B(\mathbf{0}, r), \forall \pi \in \mathcal{D}, \forall k \in \mathbb{N},$$

where $B(\mathbf{0}, r) \subset X$.

Assumption 1 implies the existence of a positive constant $\overline{\epsilon}$ such that $B(\mathbf{0}, \overline{\epsilon}) \subseteq X$ and

$$\boldsymbol{\phi}_{\boldsymbol{x}_0}^\pi(k) \in B(\mathbf{0}, \frac{r}{2}), \forall \boldsymbol{x}_0 \in B(\mathbf{0}, \overline{\epsilon}), \forall k \in \mathbb{N}, \forall \pi \in \mathcal{D}. \quad (2)$$

Suppose that the state constraint set

$$X = \{\boldsymbol{x} \in \mathbb{R}^n \mid \wedge_{i=1}^{n_X} [h_i^X(\boldsymbol{x}) < 1]\}$$

is a bounded open set with $h_i^X(\boldsymbol{x}) \in C(\mathbb{R}^n)$ being locally Lipschitz continuous over $\boldsymbol{x}$. Also, $h_i^X(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \neq \mathbf{0}$ and $h_i^X(\mathbf{0}) = 0$, $i = 1, \dots, n_X$. We present the concept of robust regions of attraction.

Definition 3. (Robust Regions of Attraction). The maximal robust region of attraction $\mathcal{R}$ is the set of states such that every possible trajectory of system (1) starting from it will approach the equilibrium state $\mathbf{0}$ while never leaving the state constraint set X, i.e.

$$\mathcal{R} = \left\{ \boldsymbol{x}_0 \;\middle|\; \begin{array}{l} \boldsymbol{\phi}_{\boldsymbol{x}_0}^\pi(k) \in X, \forall k \in \mathbb{N}, \forall \pi \in \mathcal{D}, \\ \text{and } \lim_{k\to\infty} \boldsymbol{\phi}_{\boldsymbol{x}_0}^\pi(k) = \mathbf{0}, \forall \pi \in \mathcal{D} \end{array} \right\}.$$

Correspondingly, a robust region of attraction is a subset of the maximal robust region of attraction $\mathcal{R}$.

3. BELLMAN EQUATIONS

In this section we characterize the interior of the maximal region of attraction $\mathcal{R}$ as the strict one sub-level set of the unique bounded and continuous solution to a modified Bellman equation. The derivation process follows the reasoning in Grüne and Zidani (2015). In Subsection 3.1 we introduce the maximal robust region of uniform attraction, which is equal to the interior of the maximal robust region of attraction. In Subsection 3.2 we reduce the maximal robust region of uniform attraction to the strict one sub-level set of the unique bounded and continuous solution to a Bellman equation.

3.1 Robust Regions of Uniform Attraction

In this subsection we introduce the maximal robust region of uniform attraction, which is equal to the interior of the maximal robust region of attraction. The maximal robust region of uniform attraction was first proposed in Grüne and Zidani (2015) for state-constrained perturbed continuous-time systems.

Denote the first hitting time $k'(\boldsymbol{x}_0, \pi)$, induced by the initial state $\boldsymbol{x}_0$ and the input policy $\pi\mathcal{D}$, of $B(\mathbf{0}, \overline{\epsilon})$ as

$$k'(\boldsymbol{x}_0, \pi) := \inf\{k > 0 \mid \boldsymbol{\phi}_{\boldsymbol{x}_0}^\pi(k) \in B(\mathbf{0}, \overline{\epsilon})\}, \quad (3)$$

where $B(\mathbf{0}, \overline{\epsilon})$ is defined in (2). Also, let the Euclidean distance between a point $\boldsymbol{x} \in \mathbb{R}^n$ and a set $A \subset \mathbb{R}^n$ be $\mathtt{dist}(\boldsymbol{x}, A) := \mathtt{inf}_{\boldsymbol{y} \in A} \|\boldsymbol{x} - \boldsymbol{y}\|$, and the set of δ-admissible perturbation input policies be

$$\mathcal{D}_{ad,\delta}(\boldsymbol{x}_0) := \{\pi \mid \mathtt{dist}(\boldsymbol{\phi}_{\boldsymbol{x}_0}^\pi(k), X^c) > \delta \text{ for } k \in \mathbb{N}\},$$

where $\delta > 0$ and X^c is the complement of the set X. The maximal robust region of uniform attraction $\mathcal{R}_0$ is then defined by

$$\mathcal{R}_0 := \left\{ \boldsymbol{x}_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \text{there exists } \delta > 0 \text{ s.t. } \mathcal{D}_{ad,\delta}(\boldsymbol{x}_0) \\ = \mathcal{D} \text{ and } \sup_{\pi \in \mathcal{D}} k'(\boldsymbol{x}_0, \pi) < \infty \end{array} \right\}.$$

Lemma 1 presents the openness property of the region $\mathcal{R}_0$ and the relationship between $\mathcal{R}_0$ and $\mathcal{R}$.

Lemma 1. Under Assumption 1, then

(a) $\mathcal{R}_0 = \mathcal{R}_0'$, where

$$\mathcal{R}_0' = \left\{ \boldsymbol{x}_0 \;\middle|\; \begin{array}{l} \text{there exists } \delta > 0 \text{ s.t. } \mathcal{D}_{ad,\delta}(\boldsymbol{x}_0) \\ = \mathcal{D} \text{ and there exists} \\ \beta(k) : \mathbb{N} \to [0, \infty) \text{ satisfying} \\ \lim_{k\to\infty} \beta(k) = 0 \text{ s.t.} \\ \|\boldsymbol{\phi}_{\boldsymbol{x}_0}^\pi(k)\| \leq \beta(k) \text{ for } k \in \mathbb{N} \\ \text{and } \pi \in \mathcal{D} \end{array} \right\}.$$

(b) $\mathcal{R}_0$ is open.

(c) $\mathcal{R}_0 = \mathcal{R}^\circ$.

Proof. (a). Let $\boldsymbol{x}_0 \in \mathcal{R}_0$ and $K = \sup_{\pi\in\mathcal{D}} k'(\boldsymbol{x}_0, \pi) < \infty$. Then, for $k \geq K$ we have

$$\|\boldsymbol{\phi}^\pi_{\boldsymbol{x}_0}(k)\| \leq \beta(r,k) = \lambda^k M r,$$

where r is defined in (1). Hence, for $k \geq K$ we can choose $\beta(k) = \beta(r,k)$. Since $\boldsymbol{\phi}^\pi_{\boldsymbol{x}_0}(k) \in X$ for $k \in [0,K]$ and $\pi \in \mathcal{D}$, and X is bounded, there exists $M' \geq 0$ such that

$$\|\boldsymbol{\phi}^\pi_{\boldsymbol{x}_0}(k)\| \leq M', \forall k \in [0,K], \forall \pi \in \mathcal{D}.$$

Choosing $\beta(k) = M'$ for $k \in [0,K]$ then yields the function $\beta(k)$ with the desired properties. Thus,

$$\boldsymbol{x}_0 \in \mathcal{R}'_0,$$

implying that $\mathcal{R}_0 \subseteq \mathcal{R}'_0$.

Conversely, let $\boldsymbol{x}_0 \in \mathcal{R}'_0$ and pick the corresponding $\delta > 0$ and $\beta(k)$. Then there exists $K > 0$ such that

$$\beta(k) < \bar{\epsilon}, \forall k \geq K$$

(K exists since $\lim_{k\to\infty} \beta(k) = 0$), where $\bar{\epsilon}$ is defined in (2). Then we have

$$\|\boldsymbol{\phi}^\pi_{\boldsymbol{x}_0}(k)\| \leq \beta(k) < \bar{\epsilon}, \forall k \geq K, \forall \pi \in \mathcal{D},$$

which implies

$$\boldsymbol{\phi}^\pi_{\boldsymbol{x}_0}(k) \in B(\boldsymbol{0}, \bar{\epsilon}), \forall k \geq K, \forall \pi \in \mathcal{D}.$$

Hence,

$$k'(\boldsymbol{x}_0, \pi) \leq K, \forall \pi \in \mathcal{D}$$

and thus

$$\sup_{\pi\in\mathcal{D}} k'(\boldsymbol{x}_0, \pi) \leq K < \infty.$$

Also, since $\mathcal{D}_{ad,\delta} = \mathcal{D}$, we have that $\boldsymbol{x}_0 \in \mathcal{R}_0$, implying that $\mathcal{R}'_0 \subseteq \mathcal{R}_0$.

(b). Since $\mathcal{R}_0 = \mathcal{R}'_0$, we prove the openness of $\mathcal{R}'_0$ instead. Let $\boldsymbol{x}_0 \in \mathcal{R}'_0$ with corresponding $\delta > 0$ and $\beta(\cdot) : \mathbb{N} \to [0,\infty)$, and $K > 0$ be such that $\beta(k) < \frac{\bar{\epsilon}}{2}$ for $k \geq K$, where $\bar{\epsilon}$ is defined in (3).

Since $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{d})$ is Lipschitz continuous over $\boldsymbol{x} \in X$ uniformly over $\boldsymbol{d} \in D$, implying that there exists $B(\boldsymbol{x}_0, \epsilon)$ such that for $\boldsymbol{y}_0 \in B(\boldsymbol{x}_0, \epsilon), \pi \in \mathcal{D}$ and $k \in [0,K]$,

$$\|\boldsymbol{\phi}^\pi_{\boldsymbol{x}_0}(k) - \boldsymbol{\phi}^\pi_{\boldsymbol{y}_0}(k)\| < \min\{\frac{\delta}{2}, \frac{\bar{\epsilon}}{2}\}.$$

This further implies that for $\boldsymbol{y}_0 \in B(\boldsymbol{x}_0, \epsilon)$, $\pi \in \mathcal{D}$ and $k \in [0,K]$,

$$\texttt{dist}(\boldsymbol{\phi}^\pi_{\boldsymbol{y}_0}(k), X^c) > \frac{\delta}{2}$$

holds. Thus, $\boldsymbol{\phi}^\pi_{\boldsymbol{y}_0}(K) \in B(\boldsymbol{0}, \bar{\epsilon}), \forall \pi \in \mathcal{D}$. Hence

$$\sup_{\pi\in\mathcal{D}} k'(\boldsymbol{y}_0, \pi) \leq K.$$

Together with (2) this implies

$$\mathcal{D}_{ad,\min\{\frac{\delta}{2},\frac{r}{2}\}}(\boldsymbol{y}_0) = \mathcal{D},$$

hence we conclude that $\boldsymbol{y}_0 \in \mathcal{R}'_0$. Thus, $B(\boldsymbol{x}_0, \epsilon) \subset \mathcal{R}'_0$ and consequently $\mathcal{R}'_0$ is open.

(c). Obviously, $\mathcal{R}_0 \subseteq \mathcal{R}$. Therefore, $\mathcal{R}^\circ_0 \subseteq \mathcal{R}^\circ$ and by (b) it implies $\mathcal{R}_0 \subseteq \mathcal{R}^\circ$.

Next we just prove that $\mathcal{R}^\circ \subseteq \mathcal{R}_0$, let $\boldsymbol{x}_0 \in \mathcal{R}^\circ \setminus \mathcal{R}_0$. Since $\boldsymbol{x}_0 \notin \mathcal{R}_0$, either

$$\sup_{\pi\in\mathcal{D}} k'(\boldsymbol{x}_0, \pi) = \infty \tag{4}$$

or

$$\mathcal{D}_{ad,\delta}(\boldsymbol{x}_0) \neq \mathcal{D}, \forall \delta > 0 \tag{5}$$

must hold. If (4) holds, then we obtain $\boldsymbol{x}_0 \in \partial\mathcal{R}$ since in every neighborhood of $\boldsymbol{x}_0$ there exist $\boldsymbol{x}'_0$ and a perturbation input policy π such that $k'(\boldsymbol{x}'_0, \pi) = \infty$, contradicting $\boldsymbol{x}_0 \in \mathcal{R}^\circ$.

Hence assume

$$K = \sup_{\pi\in\mathcal{D}} k'(\boldsymbol{x}_0, \pi) < \infty.$$

Then we have the conclusion that (5) holds and thus there exists a sequence $(\pi_i, k_i)_{i\in\mathbb{N}}$ such that

$$\lim_{i\to\infty} \texttt{dist}(\boldsymbol{\phi}^{\pi_i}_{\boldsymbol{x}_0}(k_i), X^c) = 0.$$

Since (2) and $k'(\boldsymbol{x}_0, \pi_i) \leq K$, we have that

$$k_i \leq K, \forall i \in \mathbb{N}.$$

Also, since $\boldsymbol{x}_0 \in \mathcal{R}$, we have that $\boldsymbol{\phi}^\pi_{\boldsymbol{x}_0}(j) \in X$ for $j \in \mathbb{N}$ and $\pi \in \mathcal{D}$. Thus, $\boldsymbol{x}_i = \boldsymbol{\phi}^{\pi_i}_{\boldsymbol{x}_0}(k_i)$ is bounded. The fact that $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{d})$ is locally Lipschitz continuous over $\mathbb{R}^n$ yields that for every $\epsilon > 0$ the set

$$\{\boldsymbol{\phi}^{\pi_i}_{\boldsymbol{y}}(k_i) \mid \boldsymbol{y} \in B(\boldsymbol{x}_0, \epsilon)\}$$

contains a ball $B(\boldsymbol{x}_i, \rho)$ with $\rho > 0$ independent of i(since $k_i \leq K$, $\forall i \in \mathbb{N}$). For sufficiently large i this implies $B(\boldsymbol{x}_i, \rho) \nsubseteq X$. This means that

$$\pi_i \notin \mathcal{D}_{ad,0}(\boldsymbol{z}_i)$$

for some $\boldsymbol{z}_i \in B(\boldsymbol{x}_0, \epsilon)$ and consequently $\boldsymbol{z}_i \notin \mathcal{R}$. Since $\epsilon > 0$ is arbitrary, this implies $\boldsymbol{x}_0 \in \partial\mathcal{R}$, again contradicting $\boldsymbol{x}_0 \in \mathcal{R}^\circ$. Hence, $\mathcal{R}^\circ \setminus \mathcal{R}_0 = \emptyset$, implying $\mathcal{R}^\circ \subset \mathcal{R}_0$. □

3.2 Bellman Equations

In this section we mainly present a modified Bellman equation, to which the strict one sub-level set of the unique bounded and continuous solution is equal to the maximal robust region of uniform attraction $\mathcal{R}_0$. For this sake we first introduce a value function, whose strict one sub-level set is equal to the maximal robust region of uniform attraction $\mathcal{R}_0$. Then we reduce this value function to the unique continuous and bounded solution to a modified Bellman equation.

We first introduce a semi-definite positive polynomial cost $g : \mathbb{R}^n \to \mathbb{R}$ satisfying that $g(\boldsymbol{x}) = 0$ iff $\boldsymbol{x} = \boldsymbol{0}$. For the sake of simplicity, we denote $\ln(g(\boldsymbol{\phi}^\pi_{\boldsymbol{x}}(i)) + 1)$ and $\ln(l(1 - h^X_j(\boldsymbol{\phi}^\pi_{\boldsymbol{x}}(i))))$ as $g_i(\boldsymbol{x}, \pi)$ and $h_{j,i}(\boldsymbol{x}, \pi)$ respectively, i.e.

$$g_i(\boldsymbol{x}, \pi) = \ln(g(\boldsymbol{\phi}^\pi_{\boldsymbol{x}}(i)) + 1),$$

and

$$h_{j,i}(\boldsymbol{x}, \pi) = \ln(l(1 - h^X_j(\boldsymbol{\phi}^\pi_{\boldsymbol{x}}(i)))), \tag{6}$$

where

$$l(x) = \begin{cases} x, \text{ if } x \geq 0, \\ 0, \text{ otherwise.} \end{cases}$$

Besides, we define $\ln 0 := -\infty$.

We define the value function $V : \mathbb{R}^n \to \mathbb{R}^+ \cup \{\infty\}$ as

$$V(\boldsymbol{x}) := \sup_{\pi\in\mathcal{D}} \sup_{k\in\mathbb{N}} \Big\{ \sum_{i=1}^{k} g_{i-1}(\boldsymbol{x}, \pi) - \min_{j\in\{1,\dots,n_X\}} h_{j,k}(\boldsymbol{x}, \pi) \Big\} \tag{7}$$

and consider the Kruzhkov transformed optimal value function $v : \mathbb{R}^n \to [0,1]$ given by

$$v(\boldsymbol{x}) := 1 - e^{-V(\boldsymbol{x})} = \sup_{\pi\in\mathcal{D}} \sup_{k\in\mathbb{N}} \big\{1 - e^{\tilde{V}}\big\}, \tag{8}$$

where

$$\tilde{V} = -\sum_{i=1}^{k} g_{i-1}(\boldsymbol{x},\pi) + \min_{j\in\{1,\dots,n_X\}} h_{j,k}(\boldsymbol{x},\pi). \quad (9)$$

Theorem 1. Under Assumption 1, then

(a) $\mathcal{R}_0 = \{\boldsymbol{x} \mid V(\boldsymbol{x}) < \infty\} = \{\boldsymbol{x} \mid v(\boldsymbol{x}) < 1\}$.
(b) $V(\boldsymbol{x})$ is continuous over $\mathcal{R}_0$. Also, $V(\boldsymbol{x}) = \infty$ for $\boldsymbol{x} \notin \mathcal{R}_0$.
(c) $v(\boldsymbol{x})$ is continuous over $\mathbb{R}^n$.

Proof. In these proofs, $\Omega(\boldsymbol{x}_0, k)$ denotes the set of states visited by system (1) initialized at $\boldsymbol{x}_0$ within $k \geq 1$ steps, i.e. $\Omega(\boldsymbol{x}_0, k) = \{\boldsymbol{y} \in \mathbb{R}^n \mid \boldsymbol{y} = \boldsymbol{\phi}^{\pi}_{\boldsymbol{x}_0}(i), \forall i \in [0,k] \cap \mathbb{N}, \forall \pi \in \mathcal{D}\}$.

(a). Firstly, by (8), we obtain immediately the equality between the two sets $\{\boldsymbol{x} \in \mathbb{R}^n \mid V(\boldsymbol{x}) < \infty\}$ and $\{\boldsymbol{x} \in \mathbb{R}^n \mid v(\boldsymbol{x}) < 1\}$. It remains to prove the first identity that $\mathcal{R}_0 = \{\boldsymbol{x} \in \mathbb{R}^n \mid V(\boldsymbol{x}) < \infty\}$.

Let $\boldsymbol{x}_0 \in \mathcal{R}_0$. We first prove that

$$\sup_{\pi\in\mathcal{D}} \sum_{i=1}^{\infty} g_{i-1}(\boldsymbol{x}_0,\pi) < \infty.$$

Let $W(\boldsymbol{x}_0) = \sup_{\pi\in\mathcal{D}} \sum_{i=1}^{\infty} g_{i-1}(\boldsymbol{x}_0,\pi)$. According to Assumption 1 and the definition of $\mathcal{R}_0$, there exists $K > 0$ such that $\boldsymbol{\phi}^{\pi}_{\boldsymbol{x}_0}(k) \in B(\boldsymbol{0}, r)$ for $k \geq K$ and $\pi \in \mathcal{D}$. Also, the closure of the reachable set $\overline{\Omega(\boldsymbol{x}_0, K)}$ is compact. Thus for $\pi \in \mathcal{D}$,

$$\begin{aligned} W(\boldsymbol{x}_0) \leq & K \sup_{\pi\in\mathcal{D}, \boldsymbol{x}\in\overline{\Omega(\boldsymbol{x}_0,K)}} \ln(g(\boldsymbol{x})+1) \\ & + \sum_{i=K+1}^{\infty} L_r M r \lambda^{i-K-1} \leq C, \end{aligned}$$

where L_r is the Lipschitz constant of $\ln(g(\boldsymbol{x})+1)$ over $\boldsymbol{x} \in B(\boldsymbol{0}, r)$. Therefore $W(\boldsymbol{x}_0) < \infty$. Next we prove that

$$-\sup_{\pi\in\mathcal{D}, k\in\mathbb{N}} \min_{j\in\{1,\dots,n_X\}} h_{j,k}(\boldsymbol{x}_0,\pi) < \infty.$$

Since $\|\boldsymbol{\phi}^{\pi}_{\boldsymbol{x}_0}(k)\| \leq \beta(k)$ for $\pi \in \mathcal{D}$, the reachable set $\Omega(\boldsymbol{x}_0, \infty)$ is bounded, hence $\overline{\Omega(\boldsymbol{x}_0, \infty)}$ is compact. Moreover, since $\mathcal{D} = \mathcal{D}_{ad,\delta}(\boldsymbol{x}_0)$ for some $\delta > 0$, we have that $\overline{\Omega(\boldsymbol{x}_0, \infty)} \subset X$. Also, since each h^X_j, $j = 1, \dots, n_X$, is continuous over X, it will attain a (finite) maximum on $\overline{\Omega(\boldsymbol{x}_0, \infty)}$ and thus

$$\sup_{\pi\in D, k\in\mathbb{N}} \min_{j\in\{1,\dots,n_X\}} h_{j,k}(\boldsymbol{x}_0,\pi)$$

will attain a finite minimum over $\overline{\Omega(\boldsymbol{x}_0, \infty)}$ according to (6). We prove the claim.

Let $\boldsymbol{x}_0 \notin \mathcal{R}_0$. Then either $\sup_{\pi\in\mathcal{D}} k'(\boldsymbol{x}_0,\pi) = \infty$ or the existence of δ in the definition of $\mathcal{R}_0$ is not satisfied, where $k'(\boldsymbol{x}_0,\pi)$ is defined in (3).

For the first case, there exists a sequence $(\pi_{j'} \in \mathcal{D})_{j'\in\mathbb{N}}$ such that $\lim_{j'\to\infty} k'(\boldsymbol{x}_0, \pi_{j'}) = \infty$. Then for any $j' \in \mathbb{N}$,

$$\begin{aligned} \sum_{i=1}^{\infty} g_{i-1}(\boldsymbol{x}_0,\pi_{j'}) \geq & \sum_{i=1}^{k'(\boldsymbol{x}_0,\pi_{j'})} g_{i-1}(\boldsymbol{x}_0,\pi_{j'}) \\ & \geq \ln(c_0+1) k'(\boldsymbol{x}_0,\pi_{j'}), \end{aligned}$$

where c_0 is a constant such that $\inf_{\boldsymbol{x}\notin B(\boldsymbol{0},r)} g(\boldsymbol{x}) \geq c_0$ (Such c_0 exists since $g(\boldsymbol{x})$ is a polynomial function over $\boldsymbol{x}$ and $g(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \neq \boldsymbol{0}$). It follows that $W(\boldsymbol{x}_0) \geq \lim_{j'\to\infty} \sum_{i=1}^{\infty} g_{i-1}(\boldsymbol{x}_0,\pi_{j'}) = \infty$. Therefore, $V(\boldsymbol{x}_0) = \infty$ since $V(\boldsymbol{x}_0) \geq W(\boldsymbol{x}_0)$. In the second case, the non-existence of δ implies the existence of a sequence $(\pi_{j'}, k_{j'})_{j'\in\mathbb{N}}$ with $\lim_{j'\to\infty} \mathtt{dist}(\boldsymbol{\phi}^{\pi_{j'}}_{\boldsymbol{x}_0}(k_{j'}), X^c) = 0$. Then either there exists $l_0 \in \mathbb{N}$ such that $\boldsymbol{\phi}^{\pi_{l_0}}_{\boldsymbol{x}_0}(k_{l_0}) \in X^c$ or there exists a subsequence $(\boldsymbol{x}_{k_{j'_l}})_{l\in\mathbb{N}}$ converging to some $\boldsymbol{x} \notin X$ (This is due to the fact that the sequence $(\boldsymbol{\phi}^{\pi_{j'}}_{\boldsymbol{x}_0}(k_{j'}))_{j'\in\mathbb{N}}$ lies in the bounded set X.), where $\boldsymbol{x}_{k_{j'_l}} = \boldsymbol{\phi}^{\pi_{j'_l}}_{\boldsymbol{x}_0}(k_{j'_l})$. Both cases imply that

$$\lim_{l\to\infty} \sup_{\pi\in\mathcal{D}} \big(-\min_{j\in\{1,\dots,n_X\}} h_{j,k_{j'_l}}(\boldsymbol{x}_0,\pi)\big) = \infty.$$

Also, since

$$V(\boldsymbol{x}_0) \geq \sup_{\pi\in\mathcal{D}} \sup_{l\in\mathbb{N}} (-\min_{j\in\{1,\dots,n_X\}} h_{j,k_{j'_l}}(\boldsymbol{x}_0,\pi)),$$

we obtain $V(\boldsymbol{x}_0) = \infty$.

(b). Let $\boldsymbol{x}_0, \boldsymbol{y}_0 \in \mathcal{R}_0$,

$$|V(\boldsymbol{x}_0) - V(\boldsymbol{y}_0)| \leq |W(\boldsymbol{x}_0) - W(\boldsymbol{y}_0)| + |W'(\boldsymbol{x}_0) - W'(\boldsymbol{y}_0)|,$$

where $W(\boldsymbol{x}_0) = \sup_{\pi\in\mathcal{D}} \sum_{i=1}^{\infty} g_{i-1}(\boldsymbol{x}_0,\pi)$ and $W'(\boldsymbol{x}_0) = \sup_{\pi\in\mathcal{D}} \sup_{k\in\mathbb{N}} \min_{j\in\{1,\dots,n_X\}} h_{j,k}(\boldsymbol{x}_0,\pi)$. In the following we separately prove the continuity of $W(\boldsymbol{x}_0)$ and $W'(\boldsymbol{x}_0)$. Firstly, we prove that W is continuous on $B(\boldsymbol{0}, \frac{r}{M})$. Assume that $\boldsymbol{x}_0 \in B(\boldsymbol{0}, \frac{r}{M})$. Then

$$\begin{aligned} \sum_{i=0}^{\infty} |\ln(g(\boldsymbol{\phi}^{\pi}_{\boldsymbol{x}_0}(i))+1)| \leq & L_r \sum_{i=0}^{\infty} \|\boldsymbol{\phi}^{\pi}_{\boldsymbol{x}_0}(i)\| \\ & \leq L_r M \sum_{i=0}^{\infty} \lambda^i \|\boldsymbol{x}_0\| \leq M_1 \|\boldsymbol{x}_0\|, \end{aligned}$$

where L_r is the Lipschitz constant of $\ln(g(\boldsymbol{x})+1)$ over $\boldsymbol{x} \in B(\boldsymbol{0}, r)$, r, λ and M are defined in (1).

For arbitrary but fixed $\epsilon > 0$, we can conclude from Assumption 1 that there exists $K > 0$ such that $M_1\|\boldsymbol{\phi}^{\pi}_{\boldsymbol{x}_0}(k)\| \leq \frac{\epsilon}{3}$ for $k \geq K$ and $\boldsymbol{x}_0 \in B(\boldsymbol{0}, \frac{r}{M})$. In addition, by Lipschitz continuity of $\boldsymbol{f}$ there exists $\delta > 0$ such that

$$\|\boldsymbol{\phi}^{\pi}_{\boldsymbol{x}_0}(k) - \boldsymbol{\phi}^{\pi}_{\boldsymbol{y}_0}(k)\| \leq \frac{\epsilon}{3L_r(K+1)}$$

for $k \in [0, K]$ and $\boldsymbol{y}_0 \in \{\boldsymbol{x} \in B(\boldsymbol{0}, \frac{r}{M}) \mid \|\boldsymbol{x} - \boldsymbol{x}_0\| < \delta\}$. Then, we have

$$\begin{aligned} & |W(\boldsymbol{x}_0) - W(\boldsymbol{y}_0)| \\ & \leq \sup_{\pi\in\mathcal{D}} \sum_{i=1}^{\infty} |\ln(g(\boldsymbol{\phi}^{\pi}_{\boldsymbol{x}_0}(i-1))+1) - \ln(g(\boldsymbol{\phi}^{\pi}_{\boldsymbol{y}_0}(i-1))+1)| \\ & \leq \sup_{\pi\in\mathcal{D}} \big(\sum_{i=0}^{K} L_r \|\boldsymbol{\phi}^{\pi}_{\boldsymbol{x}_0}(i) - \boldsymbol{\phi}^{\pi}_{\boldsymbol{y}_0}(i)\| + \\ & \qquad M_1\|\boldsymbol{\phi}^{\pi}_{\boldsymbol{x}_0}(k)\|_{k>K} + M_1\|\boldsymbol{\phi}^{\pi}_{\boldsymbol{y}_0}(k)\|_{k>K}\big) \\ & \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \leq \epsilon. \end{aligned}$$

Therefore, $W(\boldsymbol{x})$ is continuous over $B(\boldsymbol{0}, \frac{r}{M})$.

For $\boldsymbol{x}_0 \in \mathcal{R}_0$, assume L is the Lipschitz constant of $\ln(g(\boldsymbol{x})+1)$ over $\boldsymbol{x} \in \overline{X}$ and $\pi \in \mathcal{D}$. Since $\mathcal{R}_0$ is open and $\boldsymbol{f}$ is Lipschitz continuous, we have that for ϵ satisfying $0 < \epsilon < LK\delta$, there exists an open neighborhood O in $\mathcal{R}_0$ of $\boldsymbol{x}_0$ and $K > 0$ such that

$$\boldsymbol{\phi}^{\pi}_{\boldsymbol{y}_0}(k) \in B(\boldsymbol{0}, \frac{r}{M}), \forall \boldsymbol{y}_0 \in O, \forall \pi \in \mathcal{D}, \forall k \geq K$$

and

$$\|\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi}(k) - \boldsymbol{\phi}_{\boldsymbol{y}_0}^{\pi}(k)\| \leq \frac{\epsilon}{LK}, \forall k \in [0, K],$$

which implies that

$$\|\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi}(K) - \boldsymbol{\phi}_{\boldsymbol{y}_0}^{\pi}(K)\| \leq \frac{\epsilon}{LK} < \delta.$$

Therefore, similar to the deduction in (3.2), we have

$$|W(\boldsymbol{x}_0) - W(\boldsymbol{y}_0)| \leq 2\epsilon.$$

In conclusion, $W(\boldsymbol{x}_0)$ is continuous over $\mathcal{R}_0$.

Next, we prove the continuity of $W'(\boldsymbol{x}_0)$. It is obvious that

$$|W'(\boldsymbol{x}_0) - W'(\boldsymbol{y}_0)| \leq \sup_{\pi\in\mathcal{D}} \sup_{k\in\mathbb{N}} |\min_{j\in\{1,\ldots,n_X\}} h_{j,k}(\boldsymbol{x}_0, \pi) - \min_{j\in\{1,\ldots,n_X\}} h_{j,k}(\boldsymbol{y}_0, \pi)|.$$

As $\boldsymbol{x}_0 \in \mathcal{R}_0$, $\lim_{k\to\infty} \min_{j\in\{1,\ldots,n_X\}} h_{j,k}(\boldsymbol{x}_0, \pi) = 0$. Observing that $h_{j,k}$ is Lipschitz continuous over $\mathcal{R}_0$ and there exists $\beta(k) : \mathbb{N} \to [0, \infty)$, which is independent of $\boldsymbol{x}_0$ and π, such that $\|\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi}(k)\| \leq \beta(k)$ for $k \in \mathbb{N}, \pi \in \mathcal{D}$ and $\boldsymbol{x}_0 \in \mathcal{R}_0$, we can find a neighborhood $B(\boldsymbol{x}_0, \delta)$ and a function $\gamma(k) : \mathbb{N} \to [0, \infty)$ with $\lim_{k\to\infty} \gamma(k) = 0$ such that $|\min_{j\in\{1,\ldots,n_X\}} h_{j,k}(\boldsymbol{y}_0, \pi)| \leq \gamma(k)$ holds for $\boldsymbol{y}_0 \in B(\boldsymbol{x}_0, \delta)$. This implies that the supremum

$$\sup_{\pi\in\mathcal{D}} \sup_{k\in\mathbb{N}} |\min_{j\in\{1,\ldots,n_X\}} h_{j,k}(\boldsymbol{x}_0, \pi) - \min_{j\in\{1,\ldots,n_X\}} h_{j,k}(\boldsymbol{y}_0, \pi)|$$

is attained on a finite interval $[0, K] \cap \mathbb{N}$. On a compact time interval, the map $\boldsymbol{x} \to \min_{j\in\{1,\ldots,n_X\}} h_{j,k}(\boldsymbol{x}, \pi)$ is Lipschitz continuous over $\boldsymbol{x} \in \mathcal{R}_0$ uniformly over $\pi \in \mathcal{D}$ since $h_j^X(\boldsymbol{x})$ and $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{d})$ are Lipschitz continuous over $\boldsymbol{x} \in \mathcal{R}_0$ uniformly over $\boldsymbol{d} \in D$, implying that

$$\lim_{\boldsymbol{y}_0\to\boldsymbol{x}_0} \sup_{\pi\in\mathcal{D}} \sup_{k\in\mathbb{N}} |\min_{j\in\{1,\ldots,n_X\}} h_{j,k}(\boldsymbol{x}_0, \pi) - \min_{j\in\{1,\ldots,n_X\}} h_{j,k}(\boldsymbol{y}_0, \pi)| = 0.$$

This shows the desired continuity.

The second assertion that $V(\boldsymbol{x}) = \infty$ if $\boldsymbol{x} \notin \mathcal{R}_0$, can be proved by following the proof when $\boldsymbol{x} \notin \mathcal{R}_0$ in the proof of (a).

(c). From (b) we have that $V(\boldsymbol{x}) = \infty$ for $\boldsymbol{x} \in \mathbb{R}^n \setminus \mathcal{R}_0$. Therefore, $v(\boldsymbol{x}) = 1$ for $\boldsymbol{x} \in \mathbb{R}^n \setminus \mathcal{R}_0$ due to the fact that $v(\boldsymbol{x}) = 1 - e^{-V(\boldsymbol{x})}$ over $\mathbb{R}^n$. Therefore, $v(\boldsymbol{x})$ is continuous over $\mathbb{R}^n \setminus \mathcal{R}_0$.

Also since $V(\boldsymbol{x})$ is continuous over $\mathcal{R}_0$, we have that $v(\boldsymbol{x})$ is continuous over $\mathcal{R}_0$.

We just prove that if $\lim_{\boldsymbol{x}\to\boldsymbol{y}} v(\boldsymbol{x}) = v(\boldsymbol{y})$ for $\boldsymbol{x} \in \mathcal{R}_0$ and $\boldsymbol{y} \in \mathbb{R}^n\setminus\mathcal{R}_0$. According to (b) we have that $\lim_{\boldsymbol{x}\to\boldsymbol{y}} V(\boldsymbol{x}) = \infty$ and consequently $\lim_{\boldsymbol{x}\to\boldsymbol{y}} v(\boldsymbol{x}) = 1 = v(\boldsymbol{y})$.

Above all, we have that $v(\boldsymbol{x})$ is continuous over $\mathbb{R}^n$. □

Theorem 1 indicates that the interior of the maximal robust region of attraction can be obtained by computing either the value function $V(\boldsymbol{x})$ in (7) or the value function $v(\boldsymbol{x})$ in (8). Below we show that they can be computed by solving modified Bellman equations. For this sake, we first show that $V(\boldsymbol{x})$ and $v(\boldsymbol{x})$ satisfy the dynamic programming principle.

Lemma 2. Under Assumption 1, the following assertions are satisfied:

(a) For $\boldsymbol{x} \in \mathbb{R}^n$ and $k \in \mathbb{N}$, we have:

$$\begin{aligned} V(\boldsymbol{x}) = \sup_{\pi\in\mathcal{D}} \max\{ & \\ & \sum_{i=1}^{k} g_{i-1}(\boldsymbol{x}, \pi) + V(\boldsymbol{\phi}_{\boldsymbol{x}}^{\pi}(k)), \\ & \sup_{i\in[0,k-1]\cap\mathbb{N}} \{\sum_{j_1=1}^{i} g_{j_1-1}(\boldsymbol{x}, \pi) - \min_{j\in\{1,\ldots,n_X\}} h_{j,i}(\boldsymbol{x}, \pi)\} \\ \}. & \end{aligned} \tag{10}$$

(b) For $\boldsymbol{x} \in \mathbb{R}^n$ and $k \in \mathbb{N}$, we have:

$$\begin{aligned} v(\boldsymbol{x}) = \sup_{\pi\in\mathcal{D}} \max\{ & \\ & 1 - \frac{1 - v(\boldsymbol{\phi}_{\boldsymbol{x}}^{\pi}(k))}{\prod_{i=1}^{k} e^{g_{i-1}(\boldsymbol{x},\pi)}}, \sup_{i\in[0,k-1]\cap\mathbb{N}} \{1 - e^{-\overline{V}}\} \\ \}, & \end{aligned} \tag{11}$$

where $\overline{V} = \sum_{j_1=1}^{i} g_{j_1-1}(\boldsymbol{x}, \pi) - \min_{j\in\{1,\ldots,n_X\}} h_{j,i}(\boldsymbol{x}, \pi)$.

Proof. (a). Let

$$\begin{aligned} W(\boldsymbol{x}_0, k) = \sup_{\pi\in\mathcal{D}} \max\{ & \\ & \sum_{i=1}^{k} g_{i-1}(\boldsymbol{x}_0, \pi) + V(\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi}(k)), \\ & \sup_{i\in[0,k-1]\cap\mathbb{N}} \{\sum_{j=1}^{i} g_{j-1}(\boldsymbol{x}_0, \pi) - \min_{j\in\{1,\ldots,n_X\}} h_{j,i}(\boldsymbol{x}_0, \pi)\} \\ \}. & \end{aligned} \tag{12}$$

We will prove that for $\forall \epsilon > 0$, $|W(\boldsymbol{x}_0, k) - V(\boldsymbol{x}_0)| \leq \epsilon$.

From (7), for any $\epsilon_1 > 0$, there exists $\pi \in \mathcal{D}$ such that

$$V(\boldsymbol{x}_0) \leq \epsilon_1 + \sup_{k\in\mathbb{N}} \{\sum_{i=1}^{k} g_{i-1}(\boldsymbol{x}_0, \pi) - \min_{j\in\{1,\ldots,n_X\}} h_{j,k}(\boldsymbol{x}_0, \pi)\}\}.$$

We respectively define $\pi_1 \in \mathcal{D}$ and $\pi_2 \in \mathcal{D}$ as follows: $\pi_1(i) = \pi(i)$ for $i = 0, \ldots, k$, and $\pi_2(i) = \pi(i + k)$ for $i \in \mathbb{N}$, and $\boldsymbol{y} = \boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi_1}(k)$, then obtain that

$$
\begin{aligned}
&W(\boldsymbol{x}_0,k)\geq \max\Big\{\\
&\sum_{i=1}^{k} g_{i-1}(\boldsymbol{x}_0,\pi)+V(\boldsymbol{y}),\\
&\quad\sup_{i\in[0,k-1]\cap\mathbb{N}}\{\sum_{j_1=1}^{i} g_{j_1-1}(\boldsymbol{x}_0,\pi)-\min_{j\in\{1,\dots,n_X\}} h_{j,i}(\boldsymbol{x}_0,\pi)\}\\
&\Big\}\\
&\geq \max\Big\{\\
&\sum_{i=1}^{k} g_{i-1}(\boldsymbol{x}_0,\pi_1)+\\
&\quad\sup_{l\in[k,\infty)\cap\mathbb{N}}\{\sum_{i'=1}^{l-k} g_{i'-1}(\boldsymbol{y},\pi_2)-\min_{j\in\{1,\dots,n_X\}} h_{j,l-k}(\boldsymbol{y},\pi_2)\},\\
&\quad\sup_{i\in[0,k-1]\cap\mathbb{N}}\{\sum_{j_1=1}^{i} g_{j_1-1}(\boldsymbol{x}_0,\pi_1)-\min_{j\in\{1,\dots,n_X\}} h_{j,i}(\boldsymbol{x}_0,\pi_1)\}\\
&\Big\}\\
&\geq \max\Big\{\\
&\quad\sup_{l\in[k,\infty)\cap\mathbb{N}}\{\sum_{j_1=1}^{l} g_{j_1-1}(\boldsymbol{x}_0,\pi)-\min_{j\in\{1,\dots,n_X\}} h_{j,l}(\boldsymbol{x}_0,\pi)\},\\
&\quad\sup_{i\in[0,k-1]\cap\mathbb{N}}\{\sum_{j_1=1}^{i} g_{j_1-1}(\boldsymbol{x}_0,\pi)-\min_{j\in\{1,\dots,n_X\}} h_{j,i}(\boldsymbol{x}_0,\pi)\}\\
&\Big\}\\
&\geq V(\boldsymbol{x}_0)-\epsilon_1.
\end{aligned}
$$

Therefore, $V(\boldsymbol{x}_0)\leq W(\boldsymbol{x}_0,k)+\epsilon_1$.

According to (12), for any $\epsilon_1>0$, there exists a perturbation input policy $\pi_1\in\mathcal{D}$ such that

$$
\begin{aligned}
&W(\boldsymbol{x}_0,k)\leq \epsilon_1+\max\Big\{\\
&\sum_{i=1}^{k} g_{i-1}(\boldsymbol{x}_0,\pi_1)+V(\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi_1}(k)),\\
&\quad\sup_{i\in[0,k-1]\cap\mathbb{N}}\{\sum_{j_1=1}^{i} g_{j_1-1}(\boldsymbol{x}_0,\pi_1)-\min_{j\in\{1,\dots,n_X\}} h_{j,i}(\boldsymbol{x}_0,\pi_1)\}\\
&\Big\}.
\end{aligned}
$$

Also, by the definition of V, i.e. (7), for any ϵ_1, there exists an input policy $\pi_2\in\mathcal{D}$ such that

$$
\begin{aligned}
V(\boldsymbol{y})\leq{}&\epsilon_1+\\
&\sup_{l\in\mathbb{N}}\Big\{\sum_{i=1}^{l} g_{i-1}(\boldsymbol{y},\pi_2)-\min_{j\in\{1,\dots,n_X\}} h_{j,l}(\boldsymbol{y},\pi_2)\Big\},
\end{aligned}
$$

where $\boldsymbol{y}=\boldsymbol{\phi}_{\boldsymbol{x}_0}^{\pi_1}(k)$. We define π:

$$
\pi(i)=\begin{cases}\pi_1(i), & i\in[0,k)\cap\mathbb{N}\\ \pi_2(i-k), & i\in[k,\infty)\cap\mathbb{N}\end{cases}.
$$

Therefore, we infer that

$$
\begin{aligned}
&W(\boldsymbol{x}_0,k)\leq \epsilon_1+\max\Big\{\\
&\sum_{i=1}^{k} g_{i-1}(\boldsymbol{x}_0,\pi_1)+V(\boldsymbol{y}),\\
&\quad\sup_{i\in[0,k-1]\cap\mathbb{N}}\{\sum_{j_1=1}^{i} g_{j_1-1}(\boldsymbol{x}_0,\pi_1)-\min_{j\in\{1,\dots,n_X\}} h_{j,i}(\boldsymbol{x}_0,\pi_1)\}\\
&\Big\}\\
&\leq 2\epsilon_1+\max\Big\{\\
&\sum_{i=1}^{k} g_{i-1}(\boldsymbol{x}_0,\pi_1)+\\
&\quad\sup_{l\in[k,\infty)\cap\mathbb{N}}\{\sum_{i'=1}^{l-k} g_{i'-1}(\boldsymbol{y},\pi_2)-\min_{j\in\{1,\dots,n_X\}} h_{j,l-k}(\boldsymbol{y},\pi_2),\\
&\quad\sup_{i\in[0,k-1]\cap\mathbb{N}}\{\sum_{j_1=1}^{i} g_{j_1-1}(\boldsymbol{x}_0,\pi_1)-\min_{j\in\{1,\dots,n_X\}} h_{j,i}(\boldsymbol{x}_0,\pi_1)\}\\
&\Big\}\\
&\leq 2\epsilon_1+\max\Big\{\\
&\quad\sup_{l\in[k,\infty)\cap\mathbb{N}}\{\sum_{j_1=1}^{l} g_{j_1-1}(\boldsymbol{x}_0,\pi)-\min_{j\in\{1,\dots,n_X\}} h_{j,i}(\boldsymbol{x}_0,\pi)\},\\
&\quad\sup_{i\in[0,k-1]\cap\mathbb{N}}\{\sum_{j_1=1}^{i} g_{j_1-1}(\boldsymbol{x}_0,\pi)-\min_{j\in\{1,\dots,n_X\}} h_{j,i}(\boldsymbol{x}_0,\pi)\}\\
&\Big\}\\
&\leq V(\boldsymbol{x}_0)+2\epsilon_1.
\end{aligned}
$$

Therefore, we finally have $|W-V|\leq\epsilon=2\epsilon_1$, implying that $V=W$ since ϵ_1 is arbitrary.

(b). (11) can be obtained using $v(\boldsymbol{x}_0)=1-e^{-V(\boldsymbol{x})}$. $\square$

Based on Lemma 2 we can infer that the value functions $V(\boldsymbol{x}_0)$ and $v(\boldsymbol{x}_0)$ are solutions to the two generalized Bellman equations (13) and (14), respectively.

Theorem 2. Under Assumption 1, the value function V is the unique continuous solution to the generalized Bellman equation

$$
\begin{aligned}
&\min\Big\{\inf_{\boldsymbol{d}\in D}\{V-V(\boldsymbol{f})-\ln(g+1)\},\\
&\qquad V+\min_{j\in\{1,\dots,n_X\}}\ln(l(1-h_j^X))\Big\}=0,\forall\boldsymbol{x}\in\mathcal{R}_0, \qquad (13)\\
&V(\boldsymbol{0})=0.
\end{aligned}
$$

The value function v is the unique bounded and continuous solution to the Bellman equation

$$
\begin{aligned}
&\min\Big\{\inf_{\boldsymbol{d}\in D}\{v-v(\boldsymbol{f})-g\cdot(1-v)\},\\
&\qquad v-1+\min_{j\in\{1,\dots,n_X\}} e^{\ln(l(1-h_j^X))}\Big\}=0,\forall\boldsymbol{x}\in\mathbb{R}^n, \qquad (14)\\
&v(\boldsymbol{0})=0.
\end{aligned}
$$

Proof. The fact that the value functions $V(\boldsymbol{x})$ in (7) and $v(\boldsymbol{x})$ in (8) are solutions to (13) and (14) respectively can be verified when $k=1$ in (10) and (11).

Here, we just prove the uniqueness of solutions to (14). The uniqueness of solution to (13) can be guaranteed by the relationship $v(\boldsymbol{x})=1-e^{-V(\boldsymbol{x})}$ for $\boldsymbol{x}\in\mathbb{R}^n$.

Assume that $\tilde{v}$ is a bounded and continuous solution to (14) as well, we need to prove that $v = \tilde{v}$ over $\boldsymbol{x} \in \mathbb{R}^n$, where $v < 1$ over $\mathcal{R}_0$ and $v = 1$ over $\mathbb{R}^n \setminus \mathcal{R}_0$. Assume that there exists $\boldsymbol{y}_0$ such that $\tilde{v}(\boldsymbol{y}_0) \neq v(\boldsymbol{y}_0)$. First let's assume $v(\boldsymbol{y}_0) > \tilde{v}(\boldsymbol{y}_0)$ and $v(\boldsymbol{y}_0) \geq 1$. Obviously, $\boldsymbol{y}_0 \neq \boldsymbol{0}$ and consequently $g(\boldsymbol{y}_0) > 0$. Since both v and $\tilde{v}$ satisfy (14), we have that

$$\inf_{\boldsymbol{d}\in D}\{v(\boldsymbol{y}_0) - v(\boldsymbol{f}(\boldsymbol{y}_0, \boldsymbol{d})) - g(\boldsymbol{y}_0)(1 - v(\boldsymbol{y}_0))\} = 0.$$

Since v is continuous over $\mathbb{R}^n$ and $\boldsymbol{f}$ is continuous over $\mathbb{R}^n \times D$, there exists $\boldsymbol{d}'_1 \in D$ such that $v(\boldsymbol{y}_0) - v(\boldsymbol{f}(\boldsymbol{y}_0, \boldsymbol{d}'_1)) - g(\boldsymbol{y}_0)(1-v(\boldsymbol{y}_0)) = 0$. Since $\tilde{v}(\boldsymbol{y}_0) - \tilde{v}(f(\boldsymbol{y}_0, \boldsymbol{d}'_1)) - g(\boldsymbol{y}_0)(1-\tilde{v}(\boldsymbol{y}_0)) \geq 0$, we obtain that

$$\begin{aligned} v(\boldsymbol{f}(\boldsymbol{y}_0, \boldsymbol{d}'_1)) - &\tilde{v}(\boldsymbol{f}(\boldsymbol{y}_0, \boldsymbol{d}'_1)) \\ &\geq (v(\boldsymbol{y}_0) - \tilde{v}(\boldsymbol{y}_0))(1 + g(\boldsymbol{y}_0)). \end{aligned}$$

Let $\boldsymbol{y}_1 = \boldsymbol{\phi}^{\pi_1}_{\boldsymbol{y}_0}(1)$, where $\pi_1(0) = \boldsymbol{d}'_1$, then $v(\boldsymbol{y}_1) > \tilde{v}(\boldsymbol{y}_1)$. Also, we have $v(\boldsymbol{y}_0) \leq v(\boldsymbol{y}_1)$. Moreover, $\boldsymbol{y}_1 \neq \boldsymbol{0}$, $g(\boldsymbol{y}_1) > 0$. We continue the above deduction for $\boldsymbol{y}_0$ to $\boldsymbol{y}_1$ and obtain that there exists $\boldsymbol{d}'_2 \in D$ such that

$$\begin{aligned} v(\boldsymbol{f}(\boldsymbol{y}_1, \boldsymbol{d}'_2)) - &\tilde{v}(\boldsymbol{f}(\boldsymbol{y}_1, \boldsymbol{d}'_2)) \\ &\geq (v(\boldsymbol{y}_1) - \tilde{v}(\boldsymbol{y}_1))(1 + g(\boldsymbol{y}_1)). \end{aligned}$$

Thus, we have

$$\begin{aligned} &v(\boldsymbol{f}(\boldsymbol{y}_1, \boldsymbol{d}'_2)) - \tilde{v}(\boldsymbol{f}(\boldsymbol{y}_1, \boldsymbol{d}'_2)) \geq \\ &(v(\boldsymbol{y}_0) - \tilde{v}(\boldsymbol{y}_0))(1 + g(\boldsymbol{y}_1))(1 + g(\boldsymbol{y}_0)). \end{aligned}$$

Let $\boldsymbol{y}_2 = \boldsymbol{\phi}^{\pi_2}_{\boldsymbol{y}_1}(1)$, where $\pi_2(0) = \boldsymbol{d}'_2$, then $v(\boldsymbol{y}_2) > \tilde{v}(\boldsymbol{y}_2)$. Also, $v(\boldsymbol{y}_1) \leq v(\boldsymbol{y}_2)$.

Analogously, we deduce that for $k \in \mathbb{N}$,

$$\begin{aligned} &v(\boldsymbol{f}(\boldsymbol{y}_k, \boldsymbol{d}'_{k+1})) - \tilde{v}(\boldsymbol{f}(\boldsymbol{y}_k, \boldsymbol{d}'_{k+1})) \geq \\ &\quad (v(\boldsymbol{y}_0) - \tilde{v}(\boldsymbol{y}_0))(1 + g(\boldsymbol{y}_k)) \cdots (1 + g(\boldsymbol{y}_0)). \end{aligned}$$

Moreover, let $\boldsymbol{y}_{k+1} = \boldsymbol{\phi}^{\pi_{k+1}}_{\boldsymbol{y}_k}(1)$, then $v(\boldsymbol{y}_k) \leq v(\boldsymbol{y}_{k+1})$, where $\pi_{k+1}(0) = \boldsymbol{d}'_{k+1}$. This implies that $\lim_{k\to\infty} \boldsymbol{y}_k \neq \boldsymbol{0}$ and thus $\boldsymbol{y}_k \notin B(\boldsymbol{0}, \bar{\epsilon})$ for $k \in \mathbb{N}$, where $B(\boldsymbol{0}, \bar{\epsilon})$ is defined in (2). Assume that $c_0 = \inf\{g(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathbb{R}^n \setminus B(\boldsymbol{0}, \bar{\epsilon})\}$. Obviously, $c_0 > 0$. Therefore,

$$\begin{aligned} v(\boldsymbol{f}(\boldsymbol{y}_k, \boldsymbol{d}'_{k+1})) - &\tilde{v}(\boldsymbol{f}(\boldsymbol{y}_k, \boldsymbol{d}'_{k+1})) \\ &\geq (v(\boldsymbol{y}_0) - \tilde{v}(\boldsymbol{y}_0))(1 + c_0)^{k+1}, \end{aligned}$$

implying that $\lim_{k\to\infty} v(\boldsymbol{y}_k) = \infty$, which contradicts the fact that v is bounded over $\mathbb{R}^n$.

Next, assume $v(\boldsymbol{y}_0) > \tilde{v}(\boldsymbol{y}_0)$ and $v(\boldsymbol{y}_0) < 1$. According to Theorem 1, every possible trajectory starting from $\boldsymbol{y}_0$ will eventually approach $\boldsymbol{0}$. Also, we have

$$\inf_{\boldsymbol{d}\in D}\{v(\boldsymbol{y}_0, \boldsymbol{d}) - v(\boldsymbol{f}(\boldsymbol{y}_0, \boldsymbol{d})) - g(\boldsymbol{y}_0)(1 - v(\boldsymbol{y}_0))\} = 0.$$

Following the deduction mentioned above, we have

$$v(\boldsymbol{y}_k) - \tilde{v}(\boldsymbol{y}_k) \geq v(\boldsymbol{y}_0) - \tilde{v}(\boldsymbol{y}_0), \forall k \in \mathbb{N}.$$

Since $\lim_{k\to\infty} \tilde{v}(\boldsymbol{y}_k) = 0$, $\lim_{k\to\infty} v(\boldsymbol{y}_k) \geq v(\boldsymbol{y}_0) - \tilde{v}(\boldsymbol{y}_0)$ holds, contradicting $\lim_{k\to\infty} v(\boldsymbol{y}_k) = 0$.

For the case that $\tilde{v}(\boldsymbol{y}_0) > v(\boldsymbol{y}_0)$, we can obtain similar contradiction by following the proof procedure mentioned above with v and $\tilde{v}$ reversed. □

4. ILLUSTRATIVE EXAMPLES

In this section we apply the equation (14) to the computation of robust regions of attraction on one example.

Example 1. In this example we consider a computer-based model of the following perturbed continuous-time system,

$$\begin{cases} \dot{x}_1 = \dfrac{x_2}{2} + x_1x_2 + (\dfrac{1}{2} + d)x_1^2 + \dfrac{1}{2}x_1^2x_2, \\ \dot{x}_2 = -2x_1 - x_2 - 2x_1x_2 - x_1^2 - x_1^2x_2, \end{cases} \tag{15}$$

where $d \in [-0.1, 0.1]$. Its unperturbed version is used to describe a chemical oscillator, and can be obtained by transforming the equilibrium (1,0.5) and making $x_1 = x, x_2 = 2y$ of the following system from (Papachristodoulou and Prajna, 2002):

$$\begin{cases} \dot{x} = a - x + x^2 + y, \\ \dot{x}_2 = b - x^2y, \end{cases} \tag{16}$$

where $a = b = 0.5$.

When performing computer simulations, the Euler's method is often used to analyze an ordinary differential equation, which uses the idea of local linearity or linear approximation. When the simulation time step is 0.2, the resulting discrete-time system is of the following form:

$$\begin{cases} x_1(k+1) = x_1(k) + 0.2\big(\dfrac{x_2(k)}{2} + x_1(k)x_2(k) \\ \qquad + (\dfrac{1}{2} + d(k))x_1^2(k) + \dfrac{1}{2}x_1^2(k)x_2(k)\big), \\ x_2(k) = -x_2(k) + 0.2\big(-2x_1(k) - x_2(k) \\ \qquad - 2x_1(k)x_2(k) - x_1^2(k) - x_1^2(k)x_2(k)\big), \end{cases} \tag{17}$$

where $d \in D = [-0.1, 0.1]$.

The equilibrium (0,0) for system (17) is uniformly locally exponentially stable. In this example we take the state constraint set $X = \{(x_1, x_2) \mid x_1^2 + x_2^2 < 1\}$. In our experiment, $g(x_1, x_2) = 0.01(x_1^2 + x_2^2)$ is used for solving (14) based on the well-known value iteration method together with the regularization technique in (Grüne and Zidani, 2015). The estimation is illustrated in Fig. 1, which also showcases the computed $v(\boldsymbol{x})$. Four trajectories, where two trajectories respect the state constraint and two trajectories violate the state constraint, are illustrated in Fig. 2. The trajectories are generated by extracting the perturbation input $d(j)$ from D randomly for $j \in \mathbb{N}$.

5. CONCLUSION

In this paper we presented a Bellman equation for computing robust regions of attraction for state-constrained perturbed discrete-time systems. The interior of the maximal robust region of attraction is characterized as the strict one sublevel set of the unique bounded and continuous solution to the derived Bellman equation. One example demonstrated the robust regions of attraction generation based on solving the derived Bellman equation.

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REFERENCES

Bardi, M. and Capuzzo-Dolcettae, I. (1997). *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations.* Springer Science & Business Media.

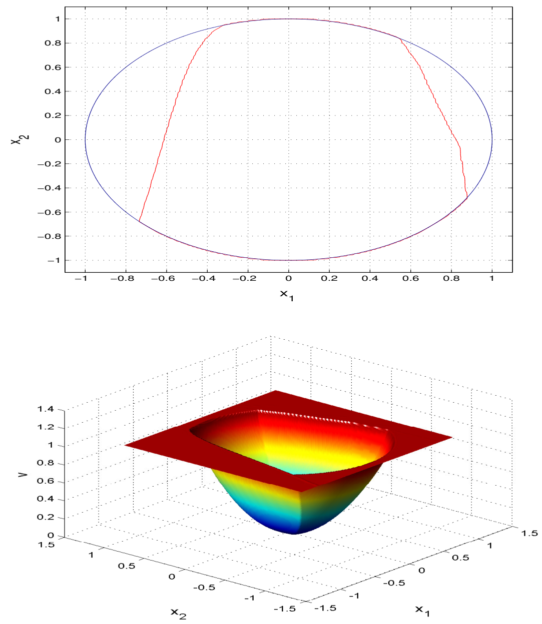

Fig. 1: An illustration of the interior of the maximal region of attraction computed via solving (14) for Example 1. Above: Blue and red curves denote the boundaries of the state constraint set X and the estimation of the interior of the maximal region of attraction, respectively. Below: $v(\boldsymbol{x})$ computed via solving (14).

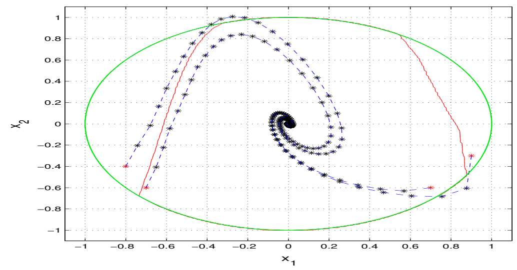

Fig. 2: An illustration of trajectories for Example 1. Green and red curves denote the boundaries of the state constraint set X and the estimation of the interior of the maximal robust region of attraction, respectively. Red stars and black stars denote the initial states and subsequent states, respectively. The dash blue line denotes the transition between states.

Bokanowski, O., Désilles, A., and Zidani, H. (2011). Rochj solver numerical parallel library for solving hamilton–jacobi equations. Technical report, Technical Report.

Bokanowski, O., Forcadel, N., and Zidani, H. (2010). Reachability and minimal times for state constrained nonlinear problems without any controllability assumption. *SIAM Journal on Control and Optimization*, 48(7), 4292–4316.

Camilli, F., Grüne, L., and Wirth, F. (2001). A generalization of zubov's method to perturbed systems. *SIAM Journal on Control and Optimization*, 40(2), 496–515.

Chesi, G. (2004). Estimating the domain of attraction for uncertain polynomial systems. *Automatica*, 40(11), 1981–1986.

Coutinho, D. and de Souza, C.E. (2013). Local stability analysis and domain of attraction estimation for a class of uncertain nonlinear discrete-time systems. *International Journal of Robust and Nonlinear Control*, 23(13), 1456–1471.

Genesio, R., Tartaglia, M., and Vicino, A. (1985). On the estimation of asymptotic stability regions: State of the art and new proposals. *IEEE Transactions on automatic control*, 30(8), 747–755.

Giesl, P. (2007). On the determination of the basin of attraction of discrete dynamical systems. *Journal of Difference Equations and Applications*, 13(6), 523–546.

Giesl, P. and Hafstein, S. (2014). Computation of Lyapunov functions for nonlinear discrete time systems by linear programming. *Journal of Difference Equations and Applications*, 20(4), 610–640.

Grüne, L. and Zidani, H. (2015). Zubov's equation for state-constrained perturbed nonlinear systems. *Mathematical Control and Related Fields*, 5(1), 55–71.

Korda, M., Henrion, D., and Jones, C.N. (2013). Inner approximations of the region of attraction for polynomial dynamical systems. *IFAC Proceedings Volumes*, 46(23), 534–539.

Ludwig, D., Walker, B., and Holling, C.S. (1997). Sustainability, stability, and resilience. *Conservation ecology*, 1(1).

Margellos, K. and Lygeros, J. (2011). Hamilton–Jacobi formulation for reach–avoid differential games. *IEEE Transactions on automatic control*, 56(8), 1849–1861.

Merola, A., Cosentino, C., and Amato, F. (2008). An insight into tumor dormancy equilibrium via the analysis of its domain of attraction. *Biomedical Signal Processing and Control*, 3(3), 212–219.

Mitchell, I.M. (2007). A toolbox of level set methods. *UBC Department of Computer Science Technical Report TR-2007-11*.

Mitchell, I.M., Bayen, A.M., and Tomlin, C.J. (2005). A time-dependent Hamilton-Jacobi formulation of reachable sets for continuous dynamic games. *IEEE Transactions on automatic control*, 50(7), 947–957.

Papachristodoulou, A. and Prajna, S. (2002). On the construction of Lyapunov functions using the sum of squares decomposition. In *CDC'02.*, volume 3, 3482–3487. IEEE.

Salle, J. and Lefschetz, S. (1961). Stability by Liapunov's direct method: with applications.

Valmorbida, G. and Anderson, J. (2014). Region of attraction analysis via invariant sets. In *ACC'14*, 3591–3596. IEEE.

Xue, B., Fränzle, M., and Naijun, Z. (2020). Inner-approximating reachable sets for polynomial systems with time-varying uncertainties. *IEEE Transactions on Automatic Control,to be published, doi: 10.1109/TAC.2019.2923049*.

Xue, B., Wang, Q., Zhan, N., and Fränzle, M. (2019). Robust invariant sets generation for state-constrained perturbed polynomial systems. In *HSCC'19*, 128–137.

Xue, B. and Zhan, N. (2018). Robust invariant sets computation for switched discrete-time polynomial systems. *arXiv preprint arXiv:1811.11454*.

Zubov, V.I. (1964). *Methods of AM Lyapunov and their Application*. P. Noordhoff.