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Safety Guarantee for Time-Delay Systems with Disturbances by Control Barrier Functionals

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Abstract Time delays occur in a variety of engineering problems because they may be inherent in the nature of plants or networks-induced. We investigate the safety verification of time-delay systems modeled by nonlinear delay differential equations subject to disturbances in their dynamics based on control barrier functionals in this paper. To the end, we propose the notion of input-to-state safety and input-to-state safe control barrier functional first, and then present an algorithm to synthesize input-to-state safe control barrier functional to guarantee the safety of the considered time-delay systems with disturbances. Three examples are provided to demonstrate the proposed approach.

Keywords Time-delay systems, delay differential equations, control barrier functionals, controllers, safety

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1 Introduction

Cyber-physical systems (CPSs) such as smart grids [1], intelligent traffic [2], and smart factories [3] have emerged and offered wide applications in our daily lives. Many of these systems are safety-critical, any fault in them may thus cause significant property damages, injuries, or even result in loss of life. Therefore, it is crucial to ensure the safety of these systems before applying them to social production and daily life.

Time delay is ubiquitous in the physical world. For instance, due to the limitation of reproductive age and food regeneration, the growth rate of biological species depends on the number of current species and the number of past species [4]. Similarly, the latent period of some infectious diseases inevitably introduces a delay in the process of its transmission [5]. During the past several decades, increasing attention has been paid to time-delay systems because of broad applications in engineering practice such as chemical processes control [6], human balancing [7], communications [8], and neural network systems [9]. Regarding the importance of these systems in applications, delay differential equations, as a new model of past dependence in the differential equation framework, have been developed by, e.g., Mishkis [10] and Bellman and Cooke [11], to capture the dynamics of time-delay systems.

Based on delay differential equations, stability analysis of time-delay systems was extensively studied in the past several decades. There are various investigations in existing literature, including, H_∞ and L_2-stability theories [12], absolute stability theory [13], input-to-state stability theory [14], Lyapunov-Krasovskii functionals [15], and many others. In contrast, safety analysis techniques ensuring the safety of time-delay systems are still in infancy.

One popular method for safety verification of CPSs is based on barrier certificates (BCs) [16]. A BC can separate all reachable sets of a considered system from the given unsafe set. To deal with CPSs with control inputs, the notion of BCs was extended to the one of *control barrier certificates* (CBCs) [17, 18]. On the other hand, BC based approach was further generalized to time-delay dynamical and hybrid systems [19–21], and the method of CBCs was also extended to time-delay systems with inputs for stability analysis [22, 23] recently.

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In practice, it is witnessed that numerous applications include models of disturbances in engineered systems ranging from wind forces [24] to pedestrian motion [25], models of financial instruments such as options [26], and models of biological/ecological processes such as predator-prey models [27]. One reason is that sampling and measuring data from environments in the design of CPSs are doomed to be subject to many kinds of noises and disturbances. The existence of disturbances further complicates the design and verification of CPSs. However, the existing BC based approaches cannot be applied to CPSs subject to both time delay and disturbances. To fill this gap, we investigate the safety guarantee of continuous time-delay systems subject to disturbances in this paper. Thus, we first propose the notion of input-to-state safe control barrier functionals (ISSf-CBFs), which can guarantee the safety of time-delay systems in the presence of disturbances, and then present an algorithm for synthesizing ISSf-CBFs. Finally, we demonstrate the proposed algorithm on three examples.

The main contributions are:

- We lift the control barrier function method of ordinary differential equations (ODEs) to time-delay systems modeled by delayed differential equations (DDEs) subject to disturbances for safety guarantee.
- Based on the concept of input-to-state safety (ISSf) in [18], input-to-state safe control barrier functionals (ISSf-CBFs) are proposed to guarantee the safety of time-delay systems in the presence of input perturbations.
- We give an algorithm for constructing input-to-state safe controllers and show the effectiveness of our algorithm via three illustrative examples.

The remainder of this paper is organized as follows. Section 2 gives an introduction to related work. Section 3 introduces the preliminaries. Section 4 develops the notion of CBFs for time-delay systems. Section 5 presents the safety guarantee for time-delay systems with input disturbances. Section 6 proposes a general algorithm framework for constructing ISSf controllers and three experiments. Finally, we make a conclusion and discuss future research in Section 7.

2 Related Work

As surveyed in [28], a variety of approaches have been proposed to verify the safety properties of CPSs over the past decades driven by the demand of safety-critical CPSs design. Among which, BC based approaches [16] are very promising, which are more closely related to this work. Similar to the idea of Lyapunov functions for stability analysis of dynamical systems, a BC can separate the state space of a considered system into safe and unsafe parts according to the safety properties to be verified, and guarantee all reachable states always keeping inside the safe side. Automatic generating BCs plays a cornerstone of BC based approaches, which can be reduced to constraint solving problems in general, can be solved by convex optimization based techniques efficiently subject to kinds of convexity conditions. Consequently, since [16], increasing attentions have been paid to how to relax such convexity conditions in order to synthesize more expressive BCs yet still efficiently with convex optimization techniques, e.g., exponential condition [29], Darboux condition [30], general relaxed barrier certificate condition [31], vector barrier certificate condition [32] and so on. But it still keeps open so far whether the necessary and sufficient condition of a formula being an inductive invariant given in [33] can be used as the weakest BC condition, according to which synthesizing BCs can still be done with convex optimization.

In order to deal with control inputs, [17, 18] extended BCs to control barrier certificates (CBCs) and presented convex optimization based approaches to synthesizing CBCs. Meanwhile, BC based approaches were also extended to time-delay systems [19–21]. However, they only considered natural time-delay systems without considering control inputs and disturbances, therefore cannot be applied to time-delay systems with disturbances.

In recent years, automatic verification of CPSs with delays draws increasing attention, a few other approaches have also been proposed, we just name a few below. [22, 23] investigated how to construct symbolic abstraction of time-delay systems possibly with inputs by exploiting input-to-state stability, so that the verification of time-delay systems can be correspondingly reduced to the reachability problem of finite state machines. In [34], the authors considered the verification of networked dynamical systems with delays associated to discrete jumps between different modes by reduction to computing reachable sets of ODEs with uncertain inputs, which is achieved by simulation together with sensitivity analysis in a bounded time horizon. [35, 36] tried to exploit local homeomorphism properties of DDEs to compute inner- and outer-approximations of their reachable sets in bounded time horizons, similar to the set boundary analysis for ODEs in [37]. In addition, a Taylor model based approach was proposed to over- and under-approximate reachable sets of DDEs with uncertain parameters and initial states in [38]. In order to conduct unbounded verification of DDEs, [39] proposed an approach based on interval Taylor model and stability analysis for a class of DDEs independent of states, and [40] proposed a Lyapunov function based approach. More recently, by utilizing linearization and spectral analysis, a method was proposed for analyzing the safety of exponentially stable time-delay systems in [20].

Clearly, all the above approaches share a common limitation, i.e., they cannot be applied to time-delay systems with control inputs and disturbances simultaneously. In contrast, ISSf-CBFs proposed in this paper can be used to guarantee the safety properties of time-delay systems subject to disturbances. Moreover, we propose a method to generate ISSf controllers based on safekeeping controllers, which facilitates the generation of ISSf controllers.

3 Preliminaries

Let $\mathbb{R}$ denote the set of real numbers, and $\mathbb{R}_{\geqslant 0}$ denote the set of non-negative real numbers. Given a vector $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm. For $d(\cdot) : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^m$, its $\mathbb{L}_\infty^m$ norm is given by $\|d\|_\infty := \operatorname{ess\,sup}_t |d(t)|$, where ess sup denotes the essential supremum (see Definition 1). Given a set $\mathcal{I} \subseteq \mathbb{R}^n$, let $\partial\mathcal{I}$, $\mathrm{Int}(\mathcal{I})$ and $\overline{\mathcal{I}}$ denote the boundary, interior and closure of $\mathcal{I}$, respectively. For given constants $a, b \in \mathbb{R}$ with $a \leqslant b$, let $\mathcal{B}([a,b],\mathbb{R}^n)$ denote the space of functions from the interval $[a,b]$ to $\mathbb{R}^n$.

Definition 1 (Essential supremum)**.** Given a measurable function $d(\cdot) : T \to \mathbb{R}$, where T is a measure space with measure μ. The essential supremum "ess sup" of d is the smallest number c such that the set $\{t \in T \mid d(t) > c\}$ has measure zero.

Definition 2 (Dirac delta function [41])**.** *Dirac delta function* is linear functional from a space of test function f, The action of δ on f, commonly denoted $\delta[f]$ or $< \delta, f >$, then gives the value at 0 of f for any function f, and has the following property:

$$\int_{-\infty}^{+\infty} f(x)\delta(x-a)dx = f(a).$$

Definition 3 (*Class* $\mathcal{K}$ function)**.** Given a constant $a > 0$ and a function $\alpha : [0,a) \to [0,+\infty)$, if α is continuous, strictly increasing and holds $\alpha(0) = 0$, then we say α is a *class* $\mathcal{K}$ function ($\alpha \in \mathcal{K}$).

The following notions are standard, will be used in the rest of this paper.

- Given a function $\alpha : (-b,c) \to (-\infty,+\infty)$, if α is continuous, strictly increasing with $\alpha(0) = 0$ for some $b > 0$ and $c > 0$, then we say α is an *extended class* $\mathcal{K}$ function. Here b, c can be $+\infty$.
- Given a function $\alpha : [0,+\infty) \to [0,+\infty)$, if α is continuous, strictly increasing with $\alpha(0) = 0$, and $\alpha(r) \to +\infty$ as $r \to +\infty$, then we say α is a *class* $\mathcal{K}_\infty$ function.
- Given a continuous function $\gamma : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$, if given a fixed n, $\gamma(m,n)$ belongs to *class* $\mathcal{K}$ with respect to m, and given a fixed m, $\gamma(m,n)$ is decreasing with respect to n and $\lim_{n\to\infty} \gamma(m,n) = 0$, then we say γ is a *class* $\mathcal{KL}$ function.

3.1 Time-delay Systems

In this paper, we consider time-delay systems described by delay differential equations (DDEs) of the following form:

$$\dot{x}(t) = f(x(t), x(t-\tau)) + g(x(t), x(t-\tau))(u(t) + d(t)), \tag{1}$$

where $x \in \mathbb{R}^n$ is system state, $d(\cdot) : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^m$ is input disturbance, $u(\cdot) : \mathbb{R}_{\geqslant 0} \to U \subset \mathbb{R}^m$ is control input, $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies the local Lipschitz condition, $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is continuous and satisfies the local Lipschitz condition, and τ is the discrete time delay term. We define a function $x_t(\omega) = x(t+\omega)$, $\omega \in [-\tau, 0]$ in Banach space $\mathcal{B}^\tau = \mathcal{B}([-\tau,0],\mathbb{R}^n)$, which maps the time interval $[-\tau, 0]$ to $\mathbb{R}^n$. The initial condition of system (1) is a function $x_0(\omega) = x(\omega)$, $\omega \in [-\tau, 0]$, rather than a single state $x(0)$ as in ODEs.

Note that system (1) is similar to the form of dynamic systems considered in article [18], both of them consider the issues of control inputs and disturbances, but [18] does not consider the existence of time delays.

System (1) can be transformed into the following form of functional differential equation:

$$\dot{x}(t) = \mathcal{F}(x_t) + \mathcal{G}(x_t)(u(t) + d(t)), \tag{2}$$

where the functionals $\mathcal{F} : \mathcal{B}^\tau \to \mathbb{R}^n$ and $\mathcal{G} : \mathcal{B}^\tau \to \mathbb{R}^{n\times m}$ are defined as follows:

$$\mathcal{F}(\psi) = f(\mathcal{R}_0(\psi), \mathcal{R}_{-\tau}(\psi)) \;\text{ and }\; \mathcal{G}(\psi) = g(\mathcal{R}_0(\psi), \mathcal{R}_{-\tau}(\psi)), \tag{3}$$

where

$$\mathcal{R}_\sigma(\psi) = \int_{-\tau}^{0} \psi(\omega)\delta(\omega - \sigma)d\omega = \psi(\sigma),$$

and δ stands for the Dirac delta function.

Given an initial condition $x_0 := x_0(\omega) \in \mathcal{B}^\tau, \omega \in [-\tau, 0]$, due to the assumption of local Lipschitz condition, (1) has a unique solution x_t to system (2) for $t < t_{\max}$, where $t_{\max}$ is the maximal execution time of (1). In the case that (2) is forward complete, then $t_{\max} = +\infty$. If for all $x_0(0) \in \mathcal{S}$, $x_t(0) \in \mathcal{S}$ for all $t \leqslant t_{\max}$, then we call that $\mathcal{S} \subset \mathbb{R}^n$ is a *forward invariant*. In this paper, a set $\mathcal{S}$ is called a *safe set* if $\mathcal{S}$ is a forward invariant.

4 Control Barrier Functionals

The notions of barrier certificates (BCs) and control barrier certificates (CBCs) are extensively and successfully applied in the verification of dynamical and hybrid systems whose continuous dynamics are described by ODEs [16, 17]. In recent years, a similar idea to BC based approaches was adapted to verifying the safety of DDEs, i.e., barrier functional (BF) based approach [19, 21]. In [22, 23], the notion of CBC was also extended to DDEs, called control barrier functionals (CBFs), for stability analysis. In this section, we generalize CBFs for DDEs stability analysis to the verification of DDEs and propose the input-to-state CBFs (ISSf-CBFs).

4.1 Barrier Functionals

In this subsection, we mainly recall the notion of BFs. Barrier functionals (BFs), which are used to ensure the safety of nature time-delay systems, were first proposed in [16] and were further investigated later in [21]. The idea of barrier functionals is similar to Lyapunov-Krasovskii functionals used in the stability analysis of DDEs.

Considering the time-delay system (1) without the term $g(x(t), x(t-\tau))(u(t) + d(t))$[1)], i.e.,

$$\dot{x}(t) = f(x(t), x(t-\tau)).$$

As discussed in Section 3.1, it can be transformed into a functional differential equation of the following form:

$$\dot{x}(t) = \mathcal{F}(x_t). \tag{4}$$

For ease of exposition, let's fix a set $\mathcal{C}$ and its boundary and interior, given by:

$$\mathcal{C} = \{\psi \in \mathcal{B}^\tau : \mathcal{H}(\psi) \geqslant 0\}, \tag{5}$$

$$\partial\mathcal{C} = \{\psi \in \mathcal{B}^\tau : \mathcal{H}(\psi) = 0\}, \tag{6}$$

$$\mathrm{Int}(\mathcal{C}) = \{\psi \in \mathcal{B}^\tau : \mathcal{H}(\psi) > 0\}. \tag{7}$$

where $\mathcal{H} : \mathcal{B}^\tau \to \mathbb{R}$ is a continuously differentiable functional. Clearly, $\overline{\mathcal{C}} = \mathcal{C}$. We assume that $\mathrm{Int}(\mathcal{C})$ is non-empty in what follows.

If $\mathcal{C}$ is a forward invariant, then the system (4) is safe with respect to $\mathcal{C}$. The following lemma in [21] presents conditions making $\mathcal{C}$ be a forward invariant.

1) Note that we use a DDE with a single constant time delay as an instance for illustration. However, it can be easily generalized to DDEs with multiple delays.

Lemma 1 ([21])**.** For a time-delay system (4) and a set $\mathcal{C} \subset \mathcal{B}^\tau$ defined as (5)-(7), if there exists $\alpha \in \mathcal{K}_{(-b,c)}$ for some $b, c \in \mathbb{R}_{\geqslant 0}$ satisfying

$$\dot{\mathcal{H}}(x_t) \geqslant -\alpha(\mathcal{H}(x_t)), \tag{8}$$

where $\dot{\mathcal{H}}(x_t)$ is the derivative of functional $\mathcal{H}$ with respect to t, i.e., $\dot{\mathcal{H}}(x_t) = d\mathcal{H}(x_t)/dt$, then we say the set $\mathcal{C}$ is forward invariant.

Lemma 1 gives a condition, which implies that any trajectory originating from $\mathcal{C}$ cannot punch through its boundary, to ensure the safety $\mathcal{C}$ for time-delay system (4). A functional $\mathcal{H}$ satisfying condition (8) is called *barrier functional*, which can guarantee the safety of time-delay system (4) with respect to the set $\mathcal{C}$.

Additionally, as $\alpha \in \mathcal{K}_{(-b,c)}$, the choice of b and c should guarantee $\mathcal{H}(x_t) \in (-b, c)$. So, in practice, they can be set by

$$b := -\inf_{x_t \in \mathcal{B}^\tau} \mathcal{H}(x_t), \quad c := \sup_{x_t \in \mathcal{B}^\tau} \mathcal{H}(x_t). \tag{9}$$

4.2 Control Barrier Functionals

By taking control inputs into account, the time-delay control system of interest is given below:

$$\dot{x}(t) = f(x(t), x(t-\tau)) + g(x(t), x(t-\tau))u(t),$$

where $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are locally Lipschitz, $x \in \mathbb{R}^n$ denotes system state, $u : \mathbb{R}_{\geqslant 0} \to U \subset \mathbb{R}^m$ denotes control input, and the constant $\tau \in \mathbb{R}_{\geqslant 0}$ denotes time delay. Similarly, the system can be rewritten as a functional differential equation of the following form:

$$\dot{x}(t) = \mathcal{F}(x_t) + \mathcal{G}(x_t)u(t). \tag{10}$$

where $\mathcal{F} : \mathcal{B}^\tau \to \mathbb{R}^n$ and $\mathcal{G} : \mathcal{B}^\tau \to \mathbb{R}^{n\times m}$ are functionals defined in (3).

In order to verify a time-delay system with inputs of the form (10), we extend the notion of barrier functionals to the one of control barrier functionals.

Definition 4 (Control barrier functionals, CBFs)**.** Consider a time-delay system of (10), let $\mathcal{C} \subset \mathcal{B}^\tau$ be a set defined as (5)-(7), if there exists an $\alpha \in \mathcal{K}_{(-b,c)}$ for some $b, c \in \mathbb{R}_{\geqslant 0}$ such that $\forall x_t \in \mathcal{C}$,

$$\sup_{u(t)\in U} [\dot{\mathcal{H}}(x_t)] \geqslant -\alpha(\mathcal{H}(x_t)), \tag{11}$$

then $\mathcal{H}$ is called control barrier functional.

Note that the above definition is similar to the notion of δ-ISS Lyapunov–Krasovskii functional defined in [22, 23], which is used for stability analysis of DDEs. We will prove in the following theorem (i.e., Theorem 1) that the control values satisfying the constraint (11) can guarantee the safety of time-delay systems with relative degree one (i.e., $\mathcal{H}_{\mathcal{G}}(x_t) \neq 0$). Before presenting Theorem 1 we need the following auxiliary lemma.

Lemma 2 (Comparison theory [42])**.** For a scalar differential equation as follows:

$$\dot{y} = f(t, y), \ y(t_0) = y_0,$$

where for all $t \geqslant 0$ and all $y \in J \subset \mathbb{R}$, $f(t, y)$ is locally Lipschitz in y and continuous in t. Let $[t_0, t_{\max})$ ($t_{\max}$ could be ∞) be the time interval over which the solution $y(t)$ exists; moreover assume $y(t) \in J$. Let $n(t)$ be a continuous function and for all $t \in [t_0, t_{\max})$, the following inequality holds

$$D^+ n(t) \geqslant f(t, n(t)), \ n(t_0) \geqslant y_0$$

with $n(t) \in J$, where $D^+ n(\cdot)$ is the upper right-hand derivative of $n(t)$. Then, $n(t) \geqslant y(t)$ holds for all $t \in [t_0, t_{\max})$.

Theorem 1. For the time-delay system (10), a set $\mathcal{C} \subset \mathcal{B}^\tau$ is defined as (5)-(7), if $\mathcal{H}$ is a control barrier functional, then $\mathcal{C}$ is a forward invariant.

Proof. Let

$$\dot{z}(t) = -\alpha(z(t)),\ z(0) = \mathcal{H}(x_0). \tag{12}$$

According to the existence and uniqueness theorem for ODEs, there is a unique solution to (12), say

$$z(t) = \gamma(z(0), t) = \gamma(\mathcal{H}(x_0), t),$$

for all $t \geqslant 0$. Since α is a *class* $\mathcal{K}$ function, we can get that γ is decreasing with respect to t for a fixed $\mathcal{H}(x_0)$ and γ is a *class* $\mathcal{K}$ function for a fixed t. Therefore, γ satisfies the properties of class $\mathcal{KL}$ function [43]. Applying Lemma 2 for (11) and (12), we get

$$\mathcal{H}(x_t) \geqslant \gamma(\mathcal{H}(x_0), t),$$

for $0 \leqslant t < t_{\max}$. According to the property of *class* $\mathcal{KL}$ function, we have for $0 \leqslant t < t_{\max}$, $\mathcal{H}(x_t) \geqslant 0$. Hence, $\mathcal{C}$ is a forward invariant by the definition of forward invariant. ∎

5 Safety Guarantee under Disturbances

In this section we study the ISSf for DDEs with disturbances. Suppose that we already have a safekeeping controller $k(x(t))$ to system (10), which could be contaminated by some disturbances $d(t)$, $k(x(t)) + d(t)$ therefore actually impacts on system (10). That means that we need to consider a system of the following form:

$$\dot{x}(t) = \mathcal{F}(x_t) + \mathcal{G}(x_t)(k(x(t)) + d(t)), \tag{13}$$

where $x_t(\omega) = x(t+\omega), \omega \in [-\tau, 0]$ and $d(\cdot) : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^m$.

5.1 Input-to-State Safety

The concept of ISSf is utilized for analyzing the safety properties of systems. It was first proposed in [44], and then in [45]. They aim to keep the underlying system away from the unsafe state set $\mathcal{S}_u \subset \mathbb{R}^n$. While [18] redefined the concept of ISSf, requiring the system to evolve in the safe set $\mathcal{S}$. We try to construct a controller to ensure the system ISSf, which is similar to the approach of constructing *input-to-state stabilizing* controllers in [46]. To guarantee the safety property $\mathcal{C}$, the goal is to ensure that the system always evolves in the set $\mathcal{C}$, or at least close to the set $\mathcal{C}$. The closeness to set $\mathcal{C}$ is related to the size of disturbances.

If a set $\mathcal{C}$ is forward invariant, then $\mathcal{C}$ is called safe. Similarly, if there exists a set $\mathcal{C}_d \supseteq \mathcal{C}$ such that $\mathcal{C}_d$ is forward invariant, then we call $\mathcal{C}$ is input-to-state safe. The set $\mathcal{C}_d$ is defined as

$$\mathcal{C}_d = \{\psi \in \mathcal{B}^\tau : \mathcal{H}(\psi) + \beta(\|d\|_\infty) \geqslant 0\}, \tag{14}$$

$$\partial\mathcal{C}_d = \{\psi \in \mathcal{B}^\tau : \mathcal{H}(\psi) + \beta(\|d\|_\infty) = 0\}, \tag{15}$$

$$\mathrm{Int}(\mathcal{C}_d) = \{\psi \in \mathcal{B}^\tau : \mathcal{H}(\psi) + \beta(\|d\|_\infty) > 0\}, \tag{16}$$

where $\beta \in \mathcal{K}_{[0,a)}$ with a satisfying $\lim_{r\to a}\beta(r) = b$ and $\|d\|_\infty \leqslant \bar{d} \in [0, a)$. Clearly, $\overline{\mathcal{C}_d} = \mathcal{C}_d$. We also assume $\mathrm{Int}(\mathcal{C}_\mathrm{d})$ is non-empty. The relationship between the safe set $\mathcal{C}$ and the input-to-state safe set $\mathcal{C}_d$ is shown in Figure 1.

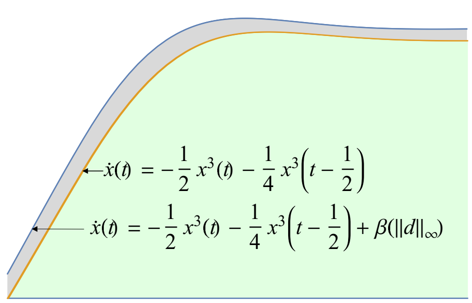

Figure 1 An example of the safe set $\mathcal{C}$ and the corresponding input-to-state safe set $\mathcal{C}_d$. The green region is set $\mathcal{C}$, Green + Grey region is set $\mathcal{C}_d$.

In [18], ISSf is defined with respect to the space $\mathbb{R}^n$. In this paper, we extend it to Banach space $\mathcal{B}^\tau$ and the corresponding definition of ISSf is given as follows.

Definition 5 (Input-to-state safety, ISSf)**.** Let $\mathcal{C} \subset \mathcal{B}^\tau$ be a set characterized by (5)-(7) for a continuous and differentiable functional $\mathcal{H} : \mathcal{B}^\tau \to \mathbb{R}$, if there exist a $\beta \in \mathcal{K}_{[0,a)}$ with $\lim_{r\to a} \beta(r) = b$ and a constant $\bar{d} \in [0, a)$, such that $\mathcal{C}_d$ characterized by (14)-(16) is forward invariant for all d with $\|d\|_\infty \leqslant \bar{d}$, then we say $\mathcal{C}$ is an *input-to-state safe set*.

According to Definition 5, we can find that $\mathcal{C}$ is input-to-state safe if and only if $\mathcal{C}_d$ is forward invariant. Therefore, let $\mathcal{C}_d$ be the safe set, we can get the corresponding set $\mathcal{C}$ according to the maximum disturbance value, which makes the ISSf set $\mathcal{C}$ conservative. We use the following example to explain the importance of ISSf.

Example 1. Consider the following time-delay system with nonlinear dynamics:

$$\dot{x}(t) = -\frac{1}{2}x^3(t) - \frac{1}{4}x^3(t-\tau) + x(t)u(t), \tag{17}$$

where $x \in \mathbb{R}$ and $u(\cdot) : \mathbb{R}_{\geqslant 0} \to U \subset \mathbb{R}$.

First, our goal here is to find a barrier functional by the controller $u(t) = k(x(t)) = 0$ and show that a safe set can be constructed when the time delay τ is small enough. According to [47], the system is asymptotically delay-independently stable and has one equilibrium $x_t(\omega) = x(t+\omega) = x^*$, $\omega \in [-\tau, 0]$, $t \geqslant 0$. Consequently, there exists an attraction region [42], a safe subset of the state space in which an equilibrium point is rendered to be asymptotically stable by a given controller. Closely resemble to the case without time delay, the equilibrium is $x^* = 0$, then a safe set around $x^* = 0$ can be constructed and we set the safe set to be $[-1, 1]$. Thus, the control barrier functional can be defined as

$$\mathcal{H}(\psi) = 1 - \psi^4(0).$$

Given the above barrier functional, we instantiate the parameter as $\tau = 1/2$ and choose $\alpha(\mathcal{H}) = \beta\mathcal{H}$ with $\beta \geqslant 0$ as the *class* $\mathcal{K}$ function in (11). Then the following equation can be obtained:

$$\begin{aligned}\dot{\mathcal{H}}(\psi) + \beta\mathcal{H}(\psi) &= 2\psi^6(0) + \psi^3(0)\psi^3(-\frac{1}{2}) + \beta\left(1 - \psi^4(0)\right) \\ &\geqslant \psi^4(0)(2\psi^2(0) - \beta) + \psi^3(0)\psi^3(-\frac{1}{2}) + \beta \\ &\geqslant \left(\psi^3(0) + \frac{1}{2}\psi^3(-\frac{1}{2})\right)^2 + \psi^6(0) - \frac{1}{4}\psi^6(-\frac{1}{2}) + \beta.\end{aligned}$$

If we choose a small enough β, the following equation holds:

$$\dot{\mathcal{H}}(\psi) + \beta\mathcal{H}(\psi) \geqslant 0.$$

That is, the condition (11) holds. According to Theorem 1, the system is safe. The state trajectories with different initial conditions $x(\omega) \equiv 1,\ 0.5\ ,0.3,\ 0,\ -0.3,\ -0.5,\ -1$ for $\omega \in [-1/2, 0]$ are shown in Figure 2.

According to the above analysis, we know that the controller $k(x(t)) = 0$ can guarantee the safety of the system (17). Now, under the same conditions, we consider the presence of disturbance d, i.e., the system becomes

$$\dot{x}(t) = -\frac{1}{2}x^3(t) - \frac{1}{4}x^3(t-\tau) + x(t)(u(t) + d). \tag{18}$$

We choose $k(x(t)) = 0$ and $\alpha(\mathcal{H}) = \beta\mathcal{H}$ with $\beta \geqslant 0$ as the *class* $\mathcal{K}$ function in (11). Hence, we get

$$\begin{aligned}\dot{\mathcal{H}}(\psi) + \beta\mathcal{H}(\psi) &= 2\psi^6(0) + \psi^3(0)\psi^3(-\frac{1}{2}) - 4\psi^4(0)d + \beta\left(1 - \psi^4(0)\right) \\ &\geqslant \psi^4(0)(2\psi^2(0) - \beta - 4d) + \psi^3(0)\psi^3(-\frac{1}{2}).\end{aligned} \tag{19}$$

According to (19), it is easy to find that the condition (11) may not hold when d becomes larger. We choose the initial condition $x(\omega) \equiv 0.5$ for $\omega \in [-1/2, 0]$, and use different values on $d(\cdot) \equiv 1, 2, 5, 10$, the resulting trajectories are presented in the right figure of Figure 2. Obviously, some trajectories leave the specified safe set $[-1, 1]$, implying that the system is not safe. Also, it is observed that when the disturbance reaches a certain limit, the resulting system trajectory even does not converge. That is, the controller $k(x(t)) = 0$ cannot guarantee the safety of the system subject to disturbances.

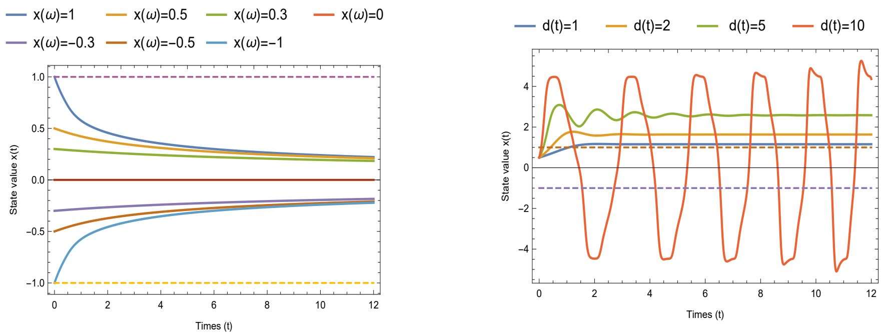

Figure 2 **Left**: state evolution of the system for different initial conditions $x(\omega)$. The dashed lines correspond to $x = -1$ and $x = 1$, the boundaries of the safe set.

Right: state evolution of the system for different disturbances d. The dashed lines correspond to $x = -1$ and $x = 1$, the boundaries of the safe set.

5.2 The Input-to-State Safe Control Barrier Functionals

Base on the definition of ISSf, we provide a way to guarantee the safety of DDEs with disturbances by ISSf-BFs and ISSf-CBFs in this subsection. Before defining the ISSf-BFs, we first state a constant e, which is useful for defining comparison function, as follows

$$e = -\lim_{r \to -b} \alpha(r).$$

Definition 6 (Input-to-state safe barrier functionals, ISSf-BFs)**.** For the time-delay system (13), let $\mathcal{C}$ be a set defined as (5)-(7) with a continuous and differentiable functional $\mathcal{H} : \mathcal{B}^\tau \to \mathbb{R}$. If there exist a function $\varepsilon \in \mathcal{K}_{[0,a)}$ with $\lim_{r\to a} \varepsilon(r) = e$, a function $\alpha \in \mathcal{K}_{(-b,c)}$, and $\bar{d} \in [0, a)$, such that $\forall x_t \in \mathcal{C}, \forall d(\cdot) : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^m$ satisfying $\|d\|_\infty \leqslant \bar{d}$,

$$\dot{H}(x_t) \geqslant -\alpha(\mathcal{H}(x_t)) - \varepsilon(\|d\|_\infty), \tag{20}$$

then we say $\mathcal{H}$ is an *input-to-state safe barrier functional*.

The following theorem can guarantee the safety of time-delay systems in the presence of disturbances.

Theorem 2. For the time-delay system (13), let $\mathcal{C} \subset \mathcal{B}^\tau$ be a set defined by (5)-(7) with a continuous and differentiable functionals $\mathcal{H} : \mathcal{B}^\tau \to \mathbb{R}$, $\mathcal{C}_d$ characterized by (14)-(16) for an $\beta \in \mathcal{K}_{[0,a)}$ with $\lim_{r\to a} \beta(a) = b$ and $\bar{d} \in [0, a)$. If $\mathcal{H}$ is an ISSf-BF, then $\mathcal{C}$ is input-to-state safe.

Proof. We just need to prove $\mathcal{C}_d$ is forward invariant. According to the definition of $\mathcal{C}_d$, we get a new functional:

$$\zeta(x_t, d) = \mathcal{H}(x_t) + \beta(\|d\|_\infty).$$

We can get the following equation from (20) for $\mathcal{H}$, which is an ISSf-BF:

$$\begin{aligned} \dot{\zeta}(x_t, d) = \dot{\mathcal{H}}(x_t) &\geqslant -\alpha(\mathcal{H}(x_t)) - \varepsilon(\|d\|_\infty) \\ &= -\alpha(\zeta(x_t, d) - \beta(\|d\|_\infty)) - \varepsilon(\|d\|_\infty). \end{aligned} \tag{21}$$

Consider the boundary $\partial\mathcal{C}_d$. Since $\zeta(x_t, d) = 0$ for $x_t \in \partial\mathcal{C}_d$, (21) can be reduced as

$$\dot{\zeta}(x_t, d) \geqslant -\alpha(-\beta(\|d\|_\infty)) - \varepsilon(\|d\|_\infty).$$

Setting $r = \beta(\|d\|_\infty)$ and defining $\rho(r) := -\alpha(-r)$, we have that for all $r_1, r_2 \in \mathbb{R}_{\geqslant 0}$ and $r_1 > r_2$ implies that $-\alpha(-r_1) > -\alpha(-r_2)$, implying that $\rho(r_1) > \rho(r_2)$. Thus, if $r < b$, ρ is a *class* $\mathcal{K}$ function. Consequently, by choosing $\beta = \rho^{-1} \circ \varepsilon$, it follows that for any $\bar{d} \in [0, a)$,

$$\rho^{-1} \circ \varepsilon(\bar{d}) < b,$$

which further implies $r < b$, thus

$$\dot{\zeta}(x_t, d) \geqslant \varepsilon(\|d\|_\infty) - \varepsilon(\|d\|_\infty) \geqslant 0. \tag{22}$$

The result follows immediately as $\mathcal{C}_d$ is forward invariant by (22). ∎

Theorem 2 shows that ISSf-BFs can guarantee the ISSf of time-delay systems under disturbances. In order to deal with control input, we give the definition of ISSf-CBFs in the following and demonstrate its role in safety guarantee.

Definition 7 (Input-to-state safe control barrier functionals, ISSf-CBFs)**.** For the time-delay system (13), let $\mathcal{C}$ be a set defined by (5)-(7) with a continuous and differentiable functional $\mathcal{H}:\mathcal{B}^\tau\to\mathbb{R}$. If there exist a $\varepsilon\in\mathcal{K}_{[0,a)}$ with $\lim_{r\to a}\varepsilon(r)=e$, an $\alpha\in\mathcal{K}_{(-b,c)}$, and a constant $\bar{d}\in[0,a)$ such that for $x_t\in\mathcal{C}$ and $d(\cdot):\mathbb{R}_{\geqslant 0}\to\mathbb{R}^m$ satisfying $\|d\|_\infty\leqslant\bar{d}$,

$$\sup_{u(t)\in U}[\dot{\mathcal{H}}(x_t)]\geqslant-\alpha(\mathcal{H}(x_t))-\varepsilon(\|d\|_\infty), \tag{23}$$

then we say $\mathcal{H}$ is an *input-to-state safe control barrier functional.*

Inspired by the controller constructed in [46], we construct our *ISSf controllers* in a similar way. For a safekeeping controller $k(x(t))$, we propose the ISSf controller as follows:

$$u(t)=k(x(t))+\mathcal{H}_\mathcal{G}(x_t), \tag{24}$$

where $\mathcal{H}_\mathcal{G}(x_t)$ denotes the derivative of $\mathcal{H}$ with respect to $\mathcal{G}$. We give a theorem below based on the ISSf controller (24). A new ISSf-CBF, which can guarantee the set $\mathcal{C}$ ISSf, is defined in the theorem.

Theorem 3. Given the time-delay system (13), a set $\mathcal{C}\subset\mathcal{B}^\tau$ defined by (5)-(7) with a continuous and differentiable functional $\mathcal{H}:\mathcal{B}^\tau\to\mathbb{R}$, if $\mathcal{H}$ satisfies

$$\sup_{u(t)\in U}[\mathcal{H}_\mathcal{F}(x_t)+\mathcal{H}_\mathcal{G}(x_t)u(t)-\mathcal{H}_\mathcal{G}(x_t)\mathcal{H}_\mathcal{G}(x_t)^T]\geqslant-\alpha(\mathcal{H}(x_t)), \tag{25}$$

for some $\alpha\in\mathcal{K}_{(-b,c)}$ and for $x_t\in\mathcal{C}$, then $\mathcal{H}$ is an input-to-state safe control barrier functional.

Proof. According to (25), we have

$$\dot{\mathcal{H}}(x_t)\geqslant-\alpha(\mathcal{H}(x_t))+\mathcal{H}_\mathcal{G}(x_t)\mathcal{H}_\mathcal{G}(x_t)^T+\mathcal{H}_\mathcal{G}(x_t)d(t)$$

Since $\mathcal{H}_\mathcal{G}(x_t)\mathcal{H}_\mathcal{G}(x_t)^T=|\mathcal{H}_\mathcal{G}(x_t)|^2$, we have

$$\dot{\mathcal{H}}(x_t)\geqslant-\alpha(\mathcal{H}(x_t))+|\mathcal{H}_\mathcal{G}(x_t)|^2-|\mathcal{H}_\mathcal{G}(x_t)|\|d\|_\infty. \tag{26}$$

implying that

$$\begin{aligned}\dot{\mathcal{H}}(x_t)&\geqslant-\alpha(\mathcal{H}(x_t))+\left(|\mathcal{H}_\mathcal{G}(x_t)|-\frac{\|d\|_\infty}{2}\right)^2-\frac{\|d\|_\infty^2}{4}\\&\geqslant-\alpha(\mathcal{H}(x_t))-\frac{\|d\|_\infty^2}{4},\end{aligned}$$

which has the same form as (23). Thus, $\mathcal{H}$ is a valid ISSf-CBF. ■

Theorem 3 shows that an ISSf controller can guarantee the safety of time-delay systems in the presence of disturbances. Similar to the CBF condition (11), when $\mathcal{H}_\mathcal{G}(x_t)=0$, Theorem 3 can only verify whether $k(x(t))=0$ is a safe controller. In the following, we use an example to enhance the understanding of ISSf controllers.

Example 2. Consider the time-delay system in Example 1. As discussed in Example 1, the safekeeping controller $k(x(t))=0$ can guarantee the safety of the system (17) in the absence of disturbances. However, if the system (17) is subject to disturbances, i.e., the system (18), the controller cannot ensure the safety of the system (18).

Now we apply the following ISSf controller to the system (18):

$$u(t)=k(x(x(t))+\mathcal{H}_\mathcal{G}(x_t)^T=-4x^4(t). \tag{27}$$

The system driven by this controller (27) will evolve in the safe set under small disturbance. We display some trajectories in Figure 3. These demonstrate that our ISSf controller can ensure the safety of the time-delay system subject to disturbances.

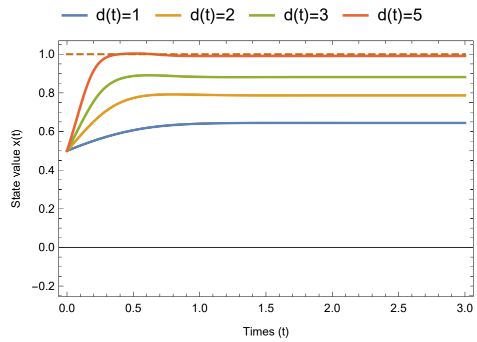

Figure 3 State evolution of the system for different disturbances when applying the ISSf controllers. The dashed lines correspond to $x = -1$ and $x = 1$, the boundaries of the safe set.

6 Implementation and Experiments

In this section, we first present our algorithm for synthesizing input-to-state safe controllers in Algorithm 1. If there are no disturbances in the system (line 2), the controller given by Theorem 1 (line 1) is safekeeping, so Algorithm 1 returns this controller directly. Otherwise, our algorithm gives an ISSf controller based on Theorem 3 (line 5). Algorithm 1 shows that safekeeping controllers can be synthesized by solving the constraint (11), and the ISSf controllers, which ensure the safety of the considered system in the presence of disturbances, can be easily obtained based on the safekeeping controller. However, now the generation of $k(x(t))$ is still a challenging problem, which is also our future research direction. The present work is based on the condition that $k(x(t))$ can be automatically generated or already exists.

Now we use three examples to show the applicability of our algorithm.

Algorithm 1 Input-to-State Safe Controller

Require:
Time-delay system S;
Barrier functional $\mathcal{H}$;
disturbance d;
Ensure:
Control input $u(t)$;
1: Synthesize $k(x(t))$ from the control barrier functional condition: $\sup_{k(x(t))\in U}[\dot{\mathcal{H}}(x_t)] \geqslant -\alpha(\mathcal{H}(x_t))$;
2: **if** $d = 0$ **then**
3: $u(t) = k(x(t))$;
4: **else**
5: $u(t) = k(x(t)) + \mathcal{H}_{\mathcal{G}}(x_t)^T$;
6: **end if**
7: **return** $u(t)$;

Example 3. Consider the following dynamic system, adapted from [47]:

$$\dot{x}(t) = -x(t) - \frac{1}{2}x(t-\tau) + x(t)(u(t) + d(t)), \tag{28}$$

where $x \in \mathbb{R}^n$, $u : \mathbb{R}_{\geqslant 0} \to U$, d is a constant disturbance and $d \in [0, 2]$. The safe set is $[-1, 1]$. In this case, we choose the barrier functional as

$$\mathcal{H}(\psi) = 1 - \psi^2(0). \tag{29}$$

Given the barrier functional (29), we discuss two cases with and without disturbances in the following:
(a) Without disturbances: set $d = 0$ as the input of Algorithm 1, and choose $\alpha(\mathcal{H}) = \beta\mathcal{H}$ with $\beta > 0$ as the *class* $\mathcal{K}$ function in (11). Then let $\tau = 1/2$ and $k(x(t)) = 0$, we get

$$\dot{\mathcal{H}}(\psi) + \beta\mathcal{H}(\psi) = 2\psi^2(0) + \psi(0)\psi(-\frac{1}{2}) + \beta\left(1 - \psi^2(0)\right)$$

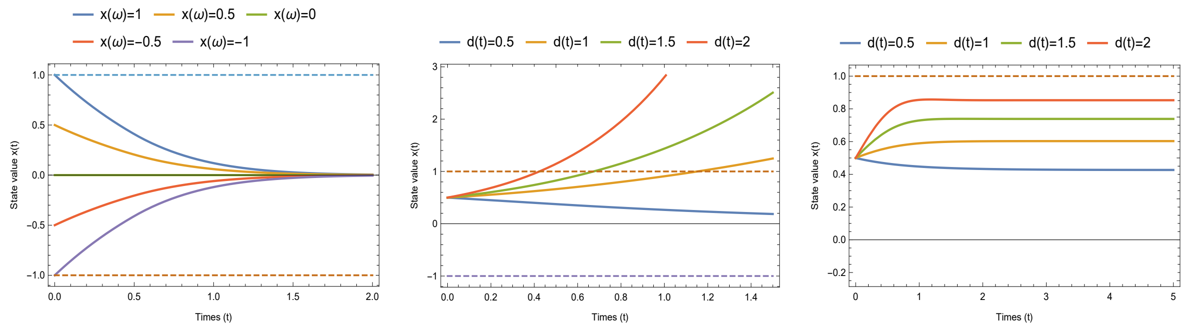

Figure 4 **Left**: state evolution of the system for different initial functions. The dashed lines correspond to $x = -1$ and $x = 1$, representing the boundaries of the safe set. **Middle**: state evolution of the system for different disturbances. The dashed lines correspond to $x = -1$ and $x = 1$, the boundaries of the safe set. **Right**: state evolution of the system for different disturbances with the ISSf controller. The dashed lines correspond to $x = -1$ and $x = 1$, the boundaries of the safe region.

$$\geqslant (2-\beta)\psi^2(0) + \psi(0)\psi(-\frac{1}{2}) + \beta,$$

when we choose a small β, then the following equation holds:

$$\dot{\mathcal{H}}(\psi) + \beta\mathcal{H}(\psi) \geqslant 0.$$

Thus $k(x(t)) = 0$ satisfies the control barrier functional condition (line 1), the set $[-1, 1]$ is safe. Algorithm 1 sets $u = 0$ (line 3) and returns it.

(b) With disturbances: Set different values on d : $d = 0.5,\ 1,\ 1.5,\ 2$, as the input of Algorithm 1, respectively. Choosing $\alpha(\mathcal{H}) = \beta\mathcal{H}$ with $\beta > 0$ as the *class* $\mathcal{K}$ function in (11) and assuming $k(x(t)) = 0$, we can get

$$\begin{aligned}\dot{\mathcal{H}}(\psi) + \beta\mathcal{H}(\psi) &= 2\psi^2(0) + \psi(0)\psi(-\frac{1}{2}) - 2\psi^2(0)d + \beta\left(1-\psi^2(0)\right)\\ &\geqslant \left(2-\beta-d\right)\psi^2(0) + \psi(0)\psi(-\frac{1}{2}) + \beta.\end{aligned}$$

We can find when the disturbance d reaches a certain limit, the condition (11) will not hold, so the system becomes unsafe. As we show in the middle of Figure 4, the system is safe when the disturbance d is very small, but when the disturbance d is large, the system even does not converge. Thus, the controller $k(x(t)) = 0$ cannot ensure the safety of the system.

Now we use the following ISSf controller as we proposed in (24) (line 5), we can get

$$u(t) = k(x(t)) + \mathcal{H}_{\mathcal{G}}(x_t) = -2x^2(t).$$

By applying the ISSf controller to system (28), the ISSf controller can ensure system (28) safe under small disturbances and the result is demonstrated in the right of Figure 4. Thus, the ISSf controller can ensure the system ISSf.

Example 4 (Population dynamics)**.** Let us consider a population dynamics example adapted from [48], given by:

$$\dot{q}(t) = \lambda[1 - q(t-\tau)/M]q(t),\ t \geqslant 0.$$

This equation is used to simulate the change of the number of individuals in a single population in nature. Due to restrictions such as the minimum breeding age or limited resources, the average growth rate $\dot{q}(t)/q(t) = \lambda[1 - q(t-\tau)/M]$ of the population relies on the size of the population in the past τ time units, because individual growth and resource recovery require a certain amount of time. If we set $x(t) = q(t)/M$ and adjust the time scale at the same time, we can get the following time-delay system proposed in [49]:

$$\dot{x}(t) = x(t)[1 - x(t-\tau)] - (u(t) + d(t)),\ t \geqslant 0,$$

where $x \in \mathbb{R}$ represents the individual number, $u : \mathbb{R}_{\geqslant 0} \to U$ denotes the capture behavior, $d : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ denotes the input disturbance. The goal is to keep the number of individuals within a reasonable range and set the safe set to be $x(t) \in [0.5, 1.6]$, so we can construct a barrier functional as

$$H(\psi) = 0.55^2 - (\psi(0) - 1.05)^2.$$

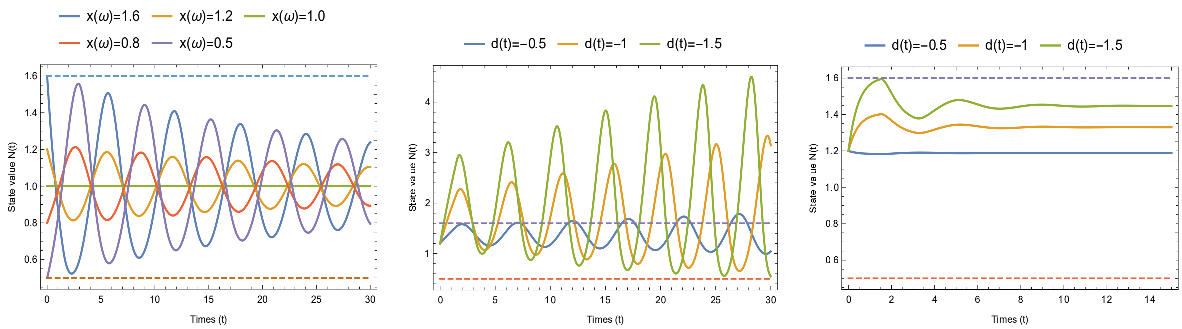

Figure 5 **Left**: state evolution of the system for different initial conditions. The dashed lines correspond to $x = 0.5$ and $x = 1.6$, the boundaries of the safe set. **Middle**: state evolution of the system for different disturbances. The dashed lines correspond to $x = 0.5$ and $x = 1.6$, the boundaries of the safe set. **Right**: state evolution of the system for different disturbances with the ISSf controller. The dashed lines correspond to $x = 0.5$ and $x = 1.6$, the boundaries of the safe region.

Given this barrier functional, we discuss two cases of the absence and presence of disturbance in the following:
(a) Without disturbances: we set $d = 0$ as the input of Algorithm 1 and choose $\alpha(\mathcal{H}) = \beta\mathcal{H}$ with $\beta > 0$ as the *class* $\mathcal{K}$ function in (11). Instantiating $\tau = 1.5$, we get

$$\begin{aligned}&\dot{\mathcal{H}}(\psi) + \beta H(\psi)\\&= -2(\psi(0) - 1.05)\psi(0)(1 - \psi(-1.5)) + 2(\psi(0) - 1.05)u(t) + \beta(0.55^2 - (\psi(0) - 1.05)^2) \geqslant 0.\end{aligned} \tag{30}$$

If we choose $u(t) = k(x(t)) = 0$, there exists a small β such that condition (30) holds, so $k(x(t)) = 0$ satisfies the control barrier functional condition (line 1), the set $[0.5, 1.6]$ is safe. Algorithm 1 sets $u = 0$ (line 3) and returns it. As we show in the left of Figure 5, the system always evolves within the safe set.
(b) With disturbances: we set different values on d : d =-0.5, -1, -1.5, as the input of Algorithm 1, respectively. Choosing $\alpha(\mathcal{H}) = \beta\mathcal{H}$ with $\beta > 0$ as the *class* $\mathcal{K}$ function in (11) and assuming $k(x(t)) = 0$, we have

$$\begin{aligned}&\dot{\mathcal{H}}(\psi) + \beta H(\psi)\\&= -2(\psi(0) - 1.05)\psi(0)(1 - \psi(-1.5)) + 2(\psi(0) - 1.05)d + \beta(0.55^2 - (\psi(0) - 1.05)^2) \geqslant 0.\end{aligned} \tag{31}$$

If the disturbance d is large, the condition (31) cannot hold. As we show in the middle of Figure 5, the system violates the safety constraints, and the system does not even converge under large disturbances.

Now by employing the following ISSf controller as proposed in (24) (line 5):

$$u(t) = k(x(t)) + \mathcal{H}_{\mathcal{G}}(x_t) = 2(x(t) - 1.05).$$

The right figure in Figure 5 reveals that the system is safe under small disturbances, that is, our ISSf controller can ensure the ISSf of the system subject to small disturbances.

Example 5 (PD-controller)**.** The following linear PD-controller is taken from [20, 38], defined as

$$\begin{cases}\dot{x}(t) = s(t),\\ \dot{s}(t) = -a_1(x(t-\tau) - x^*) - a_2 s(t-\tau) + s(t)(u(t) + d(t)),\end{cases} \tag{32}$$

where x denotes the position and s is velocity of an autonomous vehicle, u represents the input to control speed, d represents the unknown disturbance, and the constant τ is the time delay resulting from sensing, transmission, computation, actuation, etc. The vehicle needs adjust its acceleration in the light of the distance between the current location and the reference location x^*. The parameters are instantiated as $a_1 = 2$, $a_2 = 3$, $x^* = 1$, and $\tau = 0.35$. The system modeled by (32) is then transformed into the following form with $\hat{x} = x - 1$:

$$\begin{cases}\dot{\hat{x}}(t) = s(t),\\ \dot{s}(t) = -2\hat{x}(t-\tau) - 3s(t-\tau) + s(t)(u(t) + d(t)).\end{cases} \tag{33}$$

Let $x_1 = \hat{x}$, $x_2 = s$ and $\mathcal{X} = \left[\begin{smallmatrix}x_1\\ x_2\end{smallmatrix}\right]$, the system (33) can be rewritten as a dynamical system described by two-dimensional matrix as follows:

$$\dot{\mathcal{X}} = A_1\mathcal{X}(t) + A_2\mathcal{X}(t-\tau) + B\mathcal{X}(t)(u(t) + d(t)), \tag{34}$$

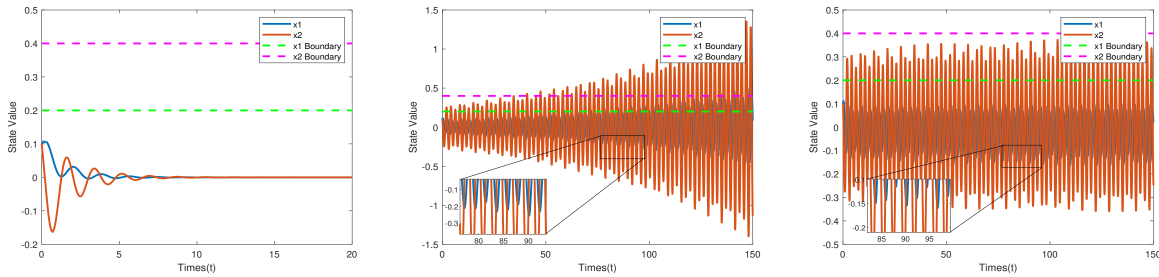

Figure 6 **Left**: state evolution of the system for initial conditions $x_1 = 0.1, x_2 = 0.1$. The dashed lines corresponding to $x = 0.2$ and $x = 0.4$ represent the safety boundaries of state x_1 and x_2, respectively. **Middle**: state evolution of the system for disturbance $d(t) = 1 + 0.2sin(t)$. The dashed lines corresponding to $x = 0.2$ and $x = 0.4$ represent the safety boundaries of state x_1 and x_2, respectively. **Right**: state evolution of the system for disturbance $d(t) = 1 + 0.2sin(t)$ with the ISSf controller. The dashed lines corresponding to $x = 0.2$ and $x = 0.4$ represent the safety boundaries of state x_1 and x_2, respectively.

where A_1, A_2, B are as follows:

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & 0 \\ -2 & -3 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The safe state set is $\mathcal{S} = \{(\hat{x}; s) \mid \hat{x} < 0.2, s < 0.4\}$ and the initial states set is $\mathcal{X} = \begin{bmatrix} -0.1, & 0.1 \end{bmatrix} \times \begin{bmatrix} 0, & 0.1 \end{bmatrix}$. We are concerned with studying the safety of system (34) under infinite-time horizon. According to the safe set, we choose the control barrier functional as

$$\mathcal{H}(\psi) = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix} - \psi(0).$$

Given the barrier functional, we discuss two cases, i.e., with and without disturbances, in the following:
(a) Without disturbances: consider the case of $d = 0$ and set $u = 0$. As discussed in [20], the system has one equilibrium at (0; 0) with $u = 0$ and would be safe under the conditions given above. By choosing $\alpha(\mathcal{H}) = \beta\mathcal{H}$ with $\beta > 0$ as the *class* $\mathcal{K}$ function in (11), we can also easily find that the following equation holds:

$$\dot{\mathcal{H}}(\psi) + \beta\mathcal{H}(\psi) \geqslant 0.$$

Therefore, $k(\mathcal{X}) = 0$ is a safekeeping controller. Some trajectories of the system are displayed in the left of Figure 6.
(b) With disturbances: consider the situation with disturbance, we set the following time-varying disturbance as the input of Algorithm 1.

$$d(t) = 1 + 0.2sin(t).$$

Also, we demonstrate the trajectory of the system in the middle of Figure 6. Obviously, the system violates the safety constraint. That is, $k(\mathcal{X}) = 0$ cannot guarantee the safety of the system.

By applying the following ISSf controller to the system:

$$u(t) = k(\mathcal{X}(t)) + \mathcal{H}_{\mathcal{G}}(\phi) = -x_2(t),$$

Figure 6 displays some trajectories of this system under the new ISSf controller, the ISSf controller can ensure the safety of the system under small disturbances.

7 Conclusion

In this work, we investigated the synthesis of safe controllers for time-delay systems. We proposed control barrier functionals and proved that it can guarantee the safety of time-delay systems. We redefined the concept of input-to-state safety and proposed the concept of input-to-state safe control barrier functionals to guarantee the safety of time-delay systems subject to disturbances. Finally, we gave an algorithm for synthesizing ISSf controllers and demonstrated it on three examples.

In the future, it deserves to investigate how to synthesize controllers for time-delay systems with control inputs and extend our method to high order control barrier functionals to guarantee the safety of time-delay systems with higher relative degree as well as time-delay systems with frequency domains and so on.

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