Robust Invariant Sets Generation for State-Constrained Perturbed Polynomial Systems

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ABSTRACT

In this paper we study the problem of computing robust invariant sets for state-constrained perturbed polynomial systems within the Hamilton-Jacobi reachability framework. A robust invariant set is a set of states such that every possible trajectory starting from it never violates the given state constraint, irrespective of the actual perturbation. The main contribution of this work is to describe the maximal robust invariant set as the zero level set of the unique Lipschitz-continuous viscosity solution to a Hamilton-Jacobi-Bellman (HJB) equation. The continuity and uniqueness property of the viscosity solution facilitates the use of existing numerical methods to solve the HJB equation for an appropriate number of state variables in order to obtain an approximation of the maximal robust invariant set. We furthermore propose a method based on semi-definite programming to synthesize robust invariant sets. Some illustrative examples demonstrate the performance of our methods.

KEYWORDS

Robust Invariant Sets; Polynomial Systems; Hamilton-Jacobi-Bellman Equations; Semi-Definite Programs

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1 INTRODUCTION

A fundamental problem in the theory of dynamical systems is the computation of robust invariant sets [7, 21, 30, 42], with applications ranging from system analysis over controller design to safety verification. A robust invariant is a set of states such that every possible trajectory starting from it always satisfies the specified state constraints, regardless of perturbations. It goes by numerous other names in the literature, e.g., infinite-time reachability tubes in [7] and invariance kernels in viability theory [2]. Synthesizing robust invariant sets has been the subject of extensive research over the past several decades, resulting in the emergence of a number of theories and corresponding computational approaches, e.g., Lyapunov function-based methods [20, 39], fixed-point methods [22, 31, 32, 35] and viability theory [2, 3, 12] and so on.

The present work studies the problem of computing robust invariant sets within the Hamilton-Jacobi reachability framework. Hamilton-Jacobi reachability analysis addresses reachability problems by exploiting the link to optimal control through viscosity solutions of HJB equations [4]. It extends the use of HJB equations, which are widely used in optimal control theory [5], to perform reachability analysis over both finite-time horizons [9, 15, 24, 26, 28] and the infinite time horizon [11, 18, 19, 37]. While computationally intensive, Hamilton-Jacobi reachability approaches are appealing nowadays due to the availability of modern numerical tools such as [8, 13, 27], which allow solving associated problems conveniently for appropriate numbers of state variables. Within the Hamilton-Jacobi framework, continuity of viscosity solutions is a desirable property from a theoretical point of view since discontinuities may invalidate uniqueness of the solution [5, 14]. Continuity is also desirable from a numerical computation point of view, since rigorous convergence results for numerical approximations to the derived HJB equation usually require continuity of the solution. Unfortunately, reachability analysis under state constraints may induce discontinuities in the viscosity solutions, see for instance [5, 6, 38], unless the dynamics satisfies special assumptions at the boundary of state constraints, e.g., the inward pointing qualification assumption [36] and outward pointing condition [16]. These conditions are, however, restrictive and viscosity solution can therefore be discontinuous in general. Recently, without requiring such assumptions, [9] infers a modified HJB equation and considers reachability problems over *finite* time horizons for state-constraind systems with control inputs. This modified HJB equation exhibits a unique continuous viscosity solution. Based on such Hamilton-Jacobi formulation in [9], [26] studies the *finite-time* reach-avoid differential game for state-constrained systems with competing inputs (control and perturbation). [15] further investigates differential games over *finite* time horizons where the target set, state constraint set and dynamics are allowed to be time-varying. Recently [19] considers the generation of the region of attraction over the infinite time horizon. The region of attraction here is the set of initial states that are controllable in that they can be driven, using an admissible control while respecting a set of state constraints, to *asymptotically* approach an equilibrium state. To the best of our knowledge, there is no previous work on the use of HJB equations having continuous viscosity solutions to characterize the maximal robust invariant set for state-constrained perturbed dynamical systems, where the notion of invariance refers to an *infinite* time horizon.

In this paper we therefore extend the HJB formulation from [9] to address computation of maximal robust invariant sets for state-constrained perturbed polynomal systems. In our framework, depending on whether a parameter introduced is greater than or equal to zero, we gain two types of HJB equations. One has a unique Lipschitz-continuous viscosity solution, whose zero level set equals the maximal robust invariant set. The formulation of this equation is the main contribution of this paper. Existing well-developed numerical methods [8, 13] can be employed to solve this equation for an appropriate number of state variables, thereby obtaining an approximation of the maximal robust invariant set. The other type does not feature this uniqueness property, but the zero sub-level set of its minimal lower semi-continuous viscosity solution again describes the maximal robust invariant set. As to this type, following [41] which considered the computation of reachable sets on finite time horizons for systems free of perturbation inputs and state constraints, we relax the HJB equation into a system of inequalities and then encode these inequalities in the form of sum-of-squares constraints, finally leading to a simple implementation method such that a robust invariant set can be generated via solving a single semi-definite program. Finally, three illustrative examples demonstrate the performance of our approaches.

The structure of this paper is as follows. In Section 2, basic notions used throughout this paper and the problem of interest are introduced. Then we present our approaches for generating robust invariant sets in Section 3. After demonstrating our approach on three examples in Section 4, we conclude this paper in Section 5.

2 PRELIMINARIES

In this section we describe the system of interest and the concept of robust invariant sets.

The following notations will be used throughout the rest of this paper: $\mathbb{R}^n$ denotes the set of n-dimensional real vectors; $\mathbb{R}[\cdot]$ denotes the ring of real polynomials in variables given by the argument, $\mathbb{R}_k[\cdot]$ denotes the set of real polynomials of degree at most k in variables given by the argument, $k \in \mathbb{N}$. $\mathbb{N}$ denotes the set of nonnegative integers. $\|\boldsymbol{x}\|$ denotes the 2-norm, i.e., $\|\boldsymbol{x}\| := \sqrt{\sum_{i=1}^{n} x_i^2}$, where $\boldsymbol{x} = (x_1, \ldots, x_n)$. Vectors are denoted by boldface letters.

The state-constrained perturbed dynamical system of interest in this paper is of the following form:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{d}(t)), \tag{1}$$

where $\boldsymbol{x}(\cdot) : [0, \infty) \to X$, $\boldsymbol{d}(\cdot) : [0, \infty) \to D$, $D = \{\boldsymbol{d} \in \mathbb{R}^m \mid \bigwedge_{i=1}^{n_D} h_i^D(\boldsymbol{d}) \leq 0\}$ is a compact set in $\mathbb{R}^m$ with $h_i^D \in \mathbb{R}[\boldsymbol{d}]$, $X = \{\boldsymbol{x} \in \mathbb{R}^n \mid \bigwedge_{i=1}^{n_X} h_i(\boldsymbol{x}) \leq 0\}$ is a compact set in $\mathbb{R}^n$ with $h_i \in \mathbb{R}[\boldsymbol{x}]$, and $\boldsymbol{f} \in \mathbb{R}[\boldsymbol{x}, \boldsymbol{d}]$.

In order to define our problem succinctly, we present the definition of an input policy d.

Definition 2.1. A perturbation policy, denoted by d, refers to a measurable function $\boldsymbol{d}(\cdot) : [0, \infty) \to D$.

The set of all perturbation policies is denoted by $\mathcal{D}$. Given a perturbation policy $d \in \mathcal{D}$, we denote the trajectory of system (1) initialized at $\boldsymbol{x}_0 \in X$ and subject to perturbation d by $\boldsymbol{\phi}_{\boldsymbol{x}_0}^d(\cdot) : [0, T] \to \mathbb{R}^n$, where $\boldsymbol{\phi}_{\boldsymbol{x}_0}^d(0) = \boldsymbol{x}_0$ and $T > 0$ is a time instant such that $\boldsymbol{\phi}_{\boldsymbol{x}_0}^d(t) \in X$ for all $t \in [0, T]$. Now, we define the (maximal) robust invariant set for system (1).

Definition 2.2 ((Maximal) Robust Invariant Set). The maximal robust invariant set $\mathcal{R}_0$ is the set of states such that every possible trajectory of system (1) starting from it never leaves X, i.e.

$$\mathcal{R}_0 = \{\boldsymbol{x} \mid \boldsymbol{\phi}_{\boldsymbol{x}}^d(t) \in X, \forall t \in [0, \infty), \forall d \in \mathcal{D}\}. \tag{2}$$

Correspondingly, a robust invariant set is a subset of the maximal robust invariant set $\mathcal{R}_0$.

3 INVARIANT SETS GENERATION

In this section we present our method to generate robust invariant sets for system (1). We first construct an auxiliary system in Subsection 3.1. Then based on the auxiliary system, in Subsection 3.2 we characterize the maximal robust invariant set $\mathcal{R}_0$ as the zero level set of the unique bounded and Lipschitz-continuous viscosity solution to a HJB equation, which can be solved by existing numerical methods. Furthermore, a computationally tractable semi-definite programming method is proposed to synthesize robust invariant sets in Subsection 3.3.

3.1 Reformulation of System (1)

As $\boldsymbol{f} \in \mathbb{R}[\boldsymbol{x}, \boldsymbol{d}]$ in system (1), $\boldsymbol{f}$ is only locally Lipschitz-continuous over $\boldsymbol{x}$. Therefore, existence of a global solution $\boldsymbol{\phi}_{\boldsymbol{x}_0}^d(t)$ over $t \in [0, \infty)$ to system (1) is not guaranteed for any initial state $\boldsymbol{x}_0 \in \mathbb{R}^n$. In this subsection we construct a system, to which a unique global solution over $t \in [0, \infty)$ starting from any initial state $\boldsymbol{x}_0 \in \mathbb{R}^n$ exists and coincides with the solution to system (1) over a compact set

$$\mathcal{X} = \{\boldsymbol{x} \mid h(\boldsymbol{x}) \geq 0\}, \tag{3}$$

where $h(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$. The compact set $\mathcal{X}$ satisfies $X \subset \mathcal{X}$ and $\partial X \cap \partial \mathcal{X} = \emptyset$. The primary reason for $\mathcal{X}$ to take the semi-algebraic set form as well as satisfy $\partial X \cap \partial \mathcal{X} = \emptyset$ is the need of constructing semi-definite programs to synthesize robust invariant sets. The constructed semi-definite program is shown in Subsection 3.3. The set $\mathcal{X}$ in (3) plays two important roles in our approach.

(1) The condition $X \subseteq \mathcal{X}$ guarantees that the maximal robust invariant set $\mathcal{R}_0$ for system (1) can be exactly characterized by trajectories to the auxiliary system (4), as formulated in Proposition 3.2.
(2) The condition $\partial\mathcal{X} \cap X = \emptyset$ assures that the zero sublevel set of the approximating polynomial returned by solving (42) in Subsection 3.3 is a robust invariant set. It is useful in justifying Theorem 3.9 in Subsection 3.3.

The auxiliary system is of the following form:

$$\dot{\boldsymbol{x}}(s) = \boldsymbol{F}(\boldsymbol{x}(s), \boldsymbol{d}(s)), \tag{4}$$

where $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{d}) : \mathbb{R}^n \times D \to \mathbb{R}^n$, which is globally Lipschitz over $\boldsymbol{x}$ uniformly over $\boldsymbol{d} \in D$, i.e. there exists a constant L_F such that $\|\boldsymbol{F}(\boldsymbol{x}_1, \boldsymbol{d}) - \boldsymbol{F}(\boldsymbol{x}_2, \boldsymbol{d})\| \leq L_F\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|$ for $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$ and $\boldsymbol{d} \in D$. Moreover, $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{d}) = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{d})$ over $\mathcal{X} \times D$, where $\mathcal{X}$ is defined in (3). This implies that trajectories to system (4) coincide with ones to system (1) over the state space $\mathcal{X}$.

The existence of system (4) is guaranteed via Kirszbraun's theorem [40], which is stated in Theorem 3.1.

Theorem 3.1 (Kirszbraun's Theorem). *Let $A \subset \mathbb{R}^n$ be a set and $\boldsymbol{f}' : A \to \mathbb{R}^m$ a function. Suppose there exists $\gamma \geq 0$ s.t. $\|\boldsymbol{f}'(\boldsymbol{x}) - \boldsymbol{f}'(\boldsymbol{y})\| \leq \gamma\|\boldsymbol{x} - \boldsymbol{y}\|$ for $\boldsymbol{x}, \boldsymbol{y} \in A$. Then there is a function $\boldsymbol{F}' : \mathbb{R}^n \to \mathbb{R}^m$ s.t. $\boldsymbol{F}'(\boldsymbol{x}) = \boldsymbol{f}'(\boldsymbol{x})$ for $\boldsymbol{x} \in A$ and $\|\boldsymbol{F}'(\boldsymbol{x}) - \boldsymbol{F}'(\boldsymbol{y})\| \leq \gamma\|\boldsymbol{x} - \boldsymbol{y}\|$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.*

For instance, $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{d}) = \inf_{\boldsymbol{y} \in \mathcal{X}}(\boldsymbol{f}(\boldsymbol{y}, \boldsymbol{d}) + \boldsymbol{z}L_{\boldsymbol{f}}\|\boldsymbol{x} - \boldsymbol{y}\|)$ satisfies (4), where $\boldsymbol{z}$ is an n-dimensional vector with each component equaling to one and $L_{\boldsymbol{f}}$ is the constant such that $\|\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{d}) - \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{d})\| \leq L_{\boldsymbol{f}}\|\boldsymbol{x} - \boldsymbol{y}\|$ over $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{X}$ and $\boldsymbol{d} \in D$. The existence of $L_{\boldsymbol{f}}$ can be guaranteed since $\boldsymbol{f} \in \mathbb{R}[\boldsymbol{x}, \boldsymbol{d}]$.

Therefore, for each pair $(d, \boldsymbol{x}_0) \in \mathcal{D} \times \mathbb{R}^n$, there exists a unique absolutely continuous trajectory $\boldsymbol{x}(t) = \boldsymbol{\psi}_{\boldsymbol{x}_0}^d(t)$ satisfying (4) for $t \geq 0$ and $\boldsymbol{x}(0) = \boldsymbol{x}_0$.

Proposition 3.2. *The maximal robust invariant set $\mathcal{R}_0$ for system (1) is equal to the set $\{\boldsymbol{x} \mid \boldsymbol{\psi}_{\boldsymbol{x}}^d(t) \in X, \forall t \in [0, \infty), \forall d \in \mathcal{D}\}$.*

Proof. Since $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{d}) = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{d})$ over $\boldsymbol{x} \in X$ and $\boldsymbol{d} \in D$, the trajectories for system (1) and (4) coincide in X. □

3.2 Characterization of $\mathcal{R}_0$

In this subsection, based on system (4) we characterize the maximal robust invariant set $\mathcal{R}_0$ by means of viscosity solutions to HJB equations.

Let $h_j'(\boldsymbol{x}) = \frac{h_j(\boldsymbol{x})}{1 + h_j^2(\boldsymbol{x})}$. Thus, $-1 < h_j'(\boldsymbol{x}) < 1$ over $\boldsymbol{x} \in \mathbb{R}^n$ for $j = 1, \ldots, n_X$. For $\boldsymbol{x} \in \mathbb{R}^n$, given a scalar value $\alpha \in [0, \infty)$, the value function $V : \mathbb{R}^n \to \mathbb{R}$ is defined by:

$$V(\boldsymbol{x}) := \sup_{d \in \mathcal{D}} \sup_{t \in [0, +\infty)} \max_{j \in \{1, \ldots, n_X\}} \left\{e^{-\alpha t} h_j'(\boldsymbol{\psi}_{\boldsymbol{x}}^d(t))\right\}. \tag{5}$$

Obviously,

$$-1 \leq V(\boldsymbol{x}) \leq 1, \forall \boldsymbol{x} \in \mathbb{R}^n, \forall \alpha \in [0, \infty). \tag{6}$$

The following theorem shows the relation between the value function V and the maximal robust invariant set $\mathcal{R}_0$.

Theorem 3.3. *$\mathcal{R}_0 = \{\boldsymbol{x} \in \mathbb{R}^n \mid V(\boldsymbol{x}) \leq 0\}$, where $\mathcal{R}_0$ is the maximal robust invariant set. Especially, when $\alpha > 0$, $V(\boldsymbol{x}) \geq 0$ for $\boldsymbol{x} \in \mathbb{R}^n$ and thus $\mathcal{R}_0 = \{\boldsymbol{x} \in \mathbb{R}^n \mid V(\boldsymbol{x}) = 0\}$.*

Proof. Assume $\boldsymbol{y}_0 \in \mathcal{R}_0$. According to Proposition 3.2, we have that for $j \in \{1, \ldots, n_X\}$,

$$h_j(\boldsymbol{\psi}_{\boldsymbol{y}_0}^d(t)) \leq 0, \forall t \in [0, \infty), \forall d \in \mathcal{D} \tag{7}$$

holds, implying that

$$h_j'(\boldsymbol{\psi}_{\boldsymbol{y}_0}^d(t)) \leq 0, \forall t \in [0, \infty), \forall d \in \mathcal{D}, \forall j \in \{1, \ldots, n_X\}.$$

Thus, $V(\boldsymbol{y}_0) \leq 0$. Therefore $\boldsymbol{y}_0 \in \{\boldsymbol{x} \mid V(\boldsymbol{x}) \leq 0\}$. On the other hand, if $\boldsymbol{y}_0 \in \{\boldsymbol{x} \in \mathbb{R}^n \mid V(\boldsymbol{x}) \leq 0\}$, then $V(\boldsymbol{y}_0) \leq 0$, implying that (7) holds. Therefore, $\boldsymbol{y}_0 \in \mathcal{R}_0$. This implies that $\mathcal{R}_0 = \{\boldsymbol{x} \in \mathbb{R}^n \mid V(\boldsymbol{x}) \leq 0\}$.

When $\alpha > 0$, $\lim_{t \to \infty} \max\{e^{-\alpha t} h_j'(\boldsymbol{\psi}_{\boldsymbol{x}}^d(t))\} = 0, \forall \boldsymbol{x} \in \mathbb{R}^n, \forall d \in \mathcal{D}$. Thus, $V(\boldsymbol{x}) \geq 0$ over $\boldsymbol{x} \in \mathbb{R}^n$. This implies that $\mathcal{R}_0 = \{\boldsymbol{x} \in \mathbb{R}^n \mid V(\boldsymbol{x}) = 0\}$. □

From Theorem 3.3, $\mathcal{R}_0$ can be constructed by computing $V(\boldsymbol{x})$, which is Lipschitz-continuous when $\alpha > 0$ and is lower semi-continuous when $\alpha = 0$. We just present the formal statement for $\alpha > 0$ in Lemma 3.4. The lower semicontinuity statement for $\alpha = 0$ is guaranteed by Lemma 4 in [14].

Lemma 3.4. *If $\alpha > 0$, $V(\boldsymbol{x})$ in (5) is locally Lipschitz-continuous over $\mathbb{R}^n$.*

Proof. First, for $\boldsymbol{x}_0 \in \mathbb{R}^n$ and $\boldsymbol{y}_0 \in B$, where $B = \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x} - \boldsymbol{x}_0\| \leq \delta\}$ with $\delta > 0$, we have

$$|V(\boldsymbol{x}_0) - V(\boldsymbol{y}_0)| \leq \sup_{d \in \mathcal{D}} \sup_{t \in [0, \infty)} \max_{j \in \{1, \ldots, n_X\}} e^{-\alpha t}|h_j'(\boldsymbol{\psi}_{\boldsymbol{x}_0}^d(t)) - h_j'(\boldsymbol{\psi}_{\boldsymbol{y}_0}^d(t))|. \tag{8}$$

Since $-1 < h_j'(\boldsymbol{x}) < 1$ over $\mathbb{R}^n$ for $j = 1, \ldots, n_X$ and $\lim_{t \to \infty} e^{-\alpha t} = 0$, this implies that the supremum

$$\sup_{d \in \mathcal{D}} \sup_{t \in [0, \infty)} \max_{j \in \{1, \ldots, n_X\}} e^{-\alpha t}|h_j'(\boldsymbol{\psi}_{\boldsymbol{x}_0}^d(t)) - h_j'(\boldsymbol{\psi}_{\boldsymbol{y}_0}^d(t))| \tag{9}$$

is attained on a finite time interval $[0, K]$. This conclusion can be obtained as follows: If the supremum (9) is equal to 0, then $\max_{j \in \{1, \ldots, n_X\}} |e^{-\alpha t} h_j'(\boldsymbol{\psi}_{\boldsymbol{x}_0}^d(t)) - e^{-\alpha t} h_j'(\boldsymbol{\psi}_{\boldsymbol{y}_0}^d(t))| \equiv 0$ for $t \in [0, \infty)$ and $d \in \mathcal{D}$, implying that the supremum can be attained on any finite time interval $[0, K]$. Otherwise, assume that the supremum equals a positive value ϵ_1. Since

$$\max_{j \in \{1, \ldots, n_X\}} e^{-\alpha t}|h_j'(\boldsymbol{\psi}_{\boldsymbol{x}_0}^d(t)) - h_j'(\boldsymbol{\psi}_{\boldsymbol{y}_0}^d(t))| \leq 2e^{-\alpha t}, \forall d \in \mathcal{D},$$

there exists a finite value $K > 0$ such that

$$\sup_{d \in \mathcal{D}} \sup_{t \in (K, \infty)} \max_{j \in \{1, \ldots, n_X\}} |e^{-\alpha t} h_j'(\boldsymbol{\psi}_{\boldsymbol{x}_0}^d(t)) - e^{-\alpha t} h_j'(\boldsymbol{\psi}_{\boldsymbol{y}_0}^d(t))| \leq 2e^{-\alpha K} \leq \frac{\epsilon_1}{2}.$$

Therefore, ϵ_1 is attained on the time interval $[0, K]$, i.e.

$$\sup_{d \in \mathcal{D}} \sup_{t \in [0, \infty)} \max_{j \in \{1, \ldots, n_X\}} |e^{-\alpha t} h_j'(\boldsymbol{\psi}_{\boldsymbol{x}_0}^d(t)) - e^{-\alpha t} h_j'(\boldsymbol{\psi}_{\boldsymbol{y}_0}^d(t))| = \sup_{d \in \mathcal{D}} \sup_{t \in [0, K]} \max_{j \in \{1, \ldots, n_X\}} e^{-\alpha t}|h_j'(\boldsymbol{\psi}_{\boldsymbol{x}_0}^d(t)) - h_j'(\boldsymbol{\psi}_{\boldsymbol{y}_0}^d(t))|.$$

$h'_j(\boldsymbol{x})$ is locally Lipschitz-continuous over $\boldsymbol{x} \in \mathbb{R}^n$ since $h_j(\boldsymbol{x})$ is locally Lipschitz-continuous over $\boldsymbol{x} \in \mathbb{R}^n$ for $j \in \{1, \ldots, n_X\}$. Therefore,

$$\begin{aligned}
&|V(\boldsymbol{x}_0) - V(\boldsymbol{y}_0)| \\
&= \sup_{d \in \mathcal{D}} \sup_{t \in [0,K]} \max_{j \in \{1,\ldots,n_X\}} e^{-\alpha t} |h'_j(\boldsymbol{\psi}^d_{\boldsymbol{x}_0}(t)) - h'_j(\boldsymbol{\psi}^d_{\boldsymbol{y}_0}(t))| \\
&\leq \sup_{d \in \mathcal{D}} \sup_{t \in [0,K]} \max_{j \in \{1,\ldots,n_X\}} |h'_j(\boldsymbol{\psi}^d_{\boldsymbol{x}_0}(t)) - h'_j(\boldsymbol{\psi}^d_{\boldsymbol{y}_0}(t))| \\
&\leq \sup_{d \in \mathcal{D}} \sup_{t \in [0,K]} L_{h'} \|\boldsymbol{\psi}^d_{\boldsymbol{x}_0}(t) - \boldsymbol{\psi}^d_{\boldsymbol{y}_0}(t)\| \\
&\leq L_{h'} \sup_{t \in [0,K]} e^{L_{\boldsymbol{F}} t} \|\boldsymbol{x}_0 - \boldsymbol{y}_0\| \\
&\leq L_{h'} e^{L_{\boldsymbol{F}} K} \|\boldsymbol{x}_0 - \boldsymbol{y}_0\|,
\end{aligned} \tag{10}$$

where $L_{h'}$ is the Lipschitz constant of $\max_{j \in \{1,\ldots,n_X\}} h'_j(\boldsymbol{x})$ over a compact set covering the set $\Omega(B, K)$, $\Omega(B, K) = \{\boldsymbol{x} \mid \boldsymbol{x} = \boldsymbol{\psi}^d_{\boldsymbol{x}'_0}(t), t \in [0, K], d \in \mathcal{D}, \boldsymbol{x}'_0 \in \mathcal{X}\}$ is the reachable set of B within the time interval $[0, K]$ and $L_{\boldsymbol{F}}$ is the Lipschitz constant of the vector field $\boldsymbol{F}$. The second-to-last inequality in (10) is obtained by applying Grönwall's inequality [17] to

$$\begin{aligned}
&\|\boldsymbol{\psi}^d_{\boldsymbol{x}_0}(t) - \boldsymbol{\psi}^d_{\boldsymbol{y}_0}(t)\| \\
&= \|\boldsymbol{x}_0 - \boldsymbol{y}_0 + \int_0^t \big(\boldsymbol{F}(\boldsymbol{\psi}^d_{\boldsymbol{x}_0}(s), \boldsymbol{d}(s)) - \boldsymbol{F}(\boldsymbol{\psi}^d_{\boldsymbol{y}_0}(s), \boldsymbol{d}(s))\big) ds\| \\
&\leq \|\boldsymbol{x}_0 - \boldsymbol{y}_0\| + L_{\boldsymbol{F}} \int_0^t \|\boldsymbol{\psi}^d_{\boldsymbol{x}_0}(s) - \boldsymbol{\psi}^d_{\boldsymbol{y}_0}(s)\| ds.
\end{aligned}$$

This shows the desired Lipschitz-continuity of $V(\boldsymbol{x})$. □

Next we show that $V(\boldsymbol{x})$ in (5) satisfies the dynamic programming principle.

Lemma 3.5. *For $\boldsymbol{x} \in \mathbb{R}^n$ and $t \in [0, \infty)$, we have*

$$\begin{aligned}
V(\boldsymbol{x}) = \sup_{d \in \mathcal{D}} \max \big\{ & e^{-\alpha t} V(\boldsymbol{\psi}^d_{\boldsymbol{x}}(t)), \\
& \sup_{\tau \in [0,t]} \max_{j \in \{1,\ldots,n_X\}} e^{-\alpha \tau} h'_j(\boldsymbol{\psi}^d_{\boldsymbol{x}}(\tau)) \big\}.
\end{aligned} \tag{11}$$

Proof. Let

$$\begin{aligned}
W(t, \boldsymbol{x}) := \sup_{d \in \mathcal{D}} \max \big\{ & e^{-\alpha t} V(\boldsymbol{\psi}^d_{\boldsymbol{x}}(t)), \\
& \sup_{\tau \in [0,t]} \max_{j \in \{1,\ldots,n_X\}} e^{-\alpha \tau} h'_j(\boldsymbol{\psi}^d_{\boldsymbol{x}}(\tau)) \big\}.
\end{aligned}$$

We will prove that for any $\epsilon > 0$, $|W(t, \boldsymbol{x}) - V(\boldsymbol{x})| < \epsilon$.

According to (5), for any ϵ_1, there exists a perturbation policy d' such that

$$V(\boldsymbol{x}) \leq \sup_{t \in [0,\infty)} \max_{j \in \{1,\ldots,n_X\}} \{e^{-\alpha t} h'_j(\boldsymbol{\psi}^{d'}_{\boldsymbol{x}}(t))\} + \epsilon_1.$$

We introduce two perturbation policies d_1 and d_2 such that $\boldsymbol{d}_1(\tau) = \boldsymbol{d}'(\tau)$ for $\tau \in [0, t]$ and $\boldsymbol{d}_2(\tau) = \boldsymbol{d}'(t + \tau)$ for $\tau \in [0, \infty)$, and let $\boldsymbol{y} = \boldsymbol{\psi}^{d_1}_{\boldsymbol{x}}(t)$. We have

$$\begin{aligned}
W(t, \boldsymbol{x}) &\geq \max \big\{ e^{-\alpha t} V(\boldsymbol{y}), \sup_{\tau \in [0,t]} \max_{j \in \{1,\ldots,n_X\}} e^{-\alpha \tau} h'_j(\boldsymbol{\psi}^{d_1}_{\boldsymbol{x}}(\tau)) \big\} \\
&\geq \max \big\{ \sup_{\tau \in [t,+\infty)} \max_{j \in \{1,\ldots,n_X\}} \{e^{-\alpha \tau} h'_j(\boldsymbol{\psi}^{d_2}_{\boldsymbol{y}}(\tau - t))\}, \\
&\qquad \sup_{\tau \in [0,t]} \max_{j \in \{1,\ldots,n_X\}} \{e^{-\alpha \tau} h'_j(\boldsymbol{\psi}^{d_1}_{\boldsymbol{x}}(\tau))\} \big\} \\
&= \max \big\{ \sup_{\tau \in [t,+\infty)} \max_{j \in \{1,\ldots,n_X\}} \{e^{-\alpha \tau} h'_j(\boldsymbol{\psi}^{d'}_{\boldsymbol{x}}(\tau))\}, \\
&\qquad \sup_{\tau \in [0,t]} \max_{j \in \{1,\ldots,n_X\}} \{e^{-\alpha \tau} h'_j(\boldsymbol{\psi}^{d'}_{\boldsymbol{x}_0}(\tau))\} \big\} \\
&= \sup_{\tau \in [0,\infty)} \max_{j \in \{1,\ldots,n_X\}} \{e^{-\alpha \tau} h'_j(\boldsymbol{\psi}^{d'}_{\boldsymbol{x}}(\tau))\} \geq V(\boldsymbol{x}) - \epsilon_1.
\end{aligned}$$

Therefore, $V(\boldsymbol{x}) \leq W(t, \boldsymbol{x}) + \epsilon_1$.

On the other hand, for any $\epsilon_1 > 0$, there exists $d_1 \in \mathcal{D}$ such that

$$\begin{aligned}
W(t,\boldsymbol{x}) \leq \max \big\{ & e^{-\alpha t} V(\boldsymbol{\psi}^{d_1}_{\boldsymbol{x}}(t)), \\
& \sup_{\tau \in [0,t]} \max_{j \in \{1,\ldots,n_X\}} \{e^{-\alpha \tau} h'_j(\boldsymbol{\psi}^{d_1}_{\boldsymbol{x}}(\tau))\} \big\} + \epsilon_1.
\end{aligned} \tag{12}$$

Also, for any $\epsilon_1 > 0$, there exists d_2 such that

$$V(\boldsymbol{y}) \leq \sup_{\tau \in [0,\infty)} \max_{j \in \{1,\ldots,n_X\}} \{e^{-\alpha \tau} h'_j(\boldsymbol{\psi}^{d_2}_{\boldsymbol{y}}(\tau))\} + \epsilon_1,$$

where $\boldsymbol{y} = \boldsymbol{\psi}^{d_1}_{\boldsymbol{x}}(t)$. We define $d \in \mathcal{D}$ such that $\boldsymbol{d}(\tau) = \boldsymbol{d}_1(\tau)$ for $\tau \in [0, t]$ and $\boldsymbol{d}(t + \tau) = \boldsymbol{d}_2(\tau)$ for $\tau \in (0, \infty)$. Then,

$$\begin{aligned}
W(t, \boldsymbol{x}) &\leq \max\{ \sup_{\tau \in [t,\infty)} \max_{j \in \{1,\ldots,n_X\}} \{e^{-\alpha \tau} h'_j(\boldsymbol{\psi}^{d_2}_{\boldsymbol{y}}(\tau - t))\}, \\
&\qquad \sup_{\tau \in [0,t]} \max_{j \in \{1,\ldots,n_X\}} \{e^{-\alpha \tau} h'_j(\boldsymbol{\psi}^{d_1}_{\boldsymbol{x}}(\tau))\}\} + 2\epsilon_1 \\
&\leq \sup_{\tau \in [0,+\infty)} \max_{j \in \{1,\ldots,n_X\}} \{e^{-\alpha \tau} h'_j(\boldsymbol{\psi}^{d}_{\boldsymbol{x}}(\tau))\} + 2\epsilon_1 \\
&\leq V(\boldsymbol{x}) + 2\epsilon_1.
\end{aligned} \tag{13}$$

Consequently, $|V(\boldsymbol{x}) - W(t, \boldsymbol{x})| \leq \epsilon = 2\epsilon_1$, implying $V(\boldsymbol{x}) = W(t, \boldsymbol{x})$ since ϵ_1 is arbitrary. □

In the following we show that $V(\boldsymbol{x})$ in (5) is a viscosity solution to the HJB partial differential equation (14):

$$\begin{aligned}
\min \big\{ & \inf_{\boldsymbol{d} \in D} (\alpha V(\boldsymbol{x}) - \frac{\partial V}{\partial \boldsymbol{x}} \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{d})), \\
& V(\boldsymbol{x}) - \max_{j \in \{1,\ldots,n_X\}} h'_j(\boldsymbol{x}) \big\} = 0.
\end{aligned} \tag{14}$$

The concept of a viscosity solution to (14) is given below.

Definition 3.6. [14] *A locally bounded function $V(\boldsymbol{x})$ on $\mathbb{R}^n$ is a viscosity solution to* (14)*, if 1) for any continuously differentiable function $v(\boldsymbol{x})$ such that $V_* - v$ attains a local minimum at $\boldsymbol{x}_0 \in \mathbb{R}^n$,*

$$\begin{aligned}
\min \big\{ & \inf_{\boldsymbol{d} \in D} (\alpha V_*(\boldsymbol{x}_0) - \frac{\partial v}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x} = \boldsymbol{x}_0} \boldsymbol{F}(\boldsymbol{x}_0, \boldsymbol{d})), \\
& V_*(\boldsymbol{x}_0) - \max_{j \in \{1,\ldots,n_X\}} h'_j(\boldsymbol{x}_0) \big\} \geq 0.
\end{aligned} \tag{15}$$

holds (i.e., V_* is a viscosity super-solution); 2) for any continuously differentiable function $v(\boldsymbol{x})$ such that $V^* - v$ attains *a local maximum at* $\boldsymbol{x}_0 \in \mathbb{R}^n$,

$$\min\Big\{\inf_{\boldsymbol{d}\in D}(\alpha V^*(\boldsymbol{x}_0) - \frac{\partial v}{\partial \boldsymbol{x}}\mid_{\boldsymbol{x}=\boldsymbol{x}_0} \boldsymbol{F}(\boldsymbol{x}_0,\boldsymbol{d})), \\ V^*(\boldsymbol{x}_0) - \max_{j\in\{1,\ldots,n_X\}} h'_j(\boldsymbol{x}_0)\Big\} \leq 0. \tag{16}$$

holds (i.e., V^* *is a viscosity sub-solution), where* V_* *(respectively,* V^**) denotes the lower (respectively, upper) semi-continuous envelope of* V*. Note that if* V *is continuous,* $V = V_* = V^*$.

When $\alpha = 0$, $V(\boldsymbol{x})$ is lower semi-continuous and consequently the uniqueness of viscosity solutions to (14) is not expected generally. According to Proposition 4 and 5 in [14], $V(\boldsymbol{x})$ is the minimal lower semi-continuous viscosity solution to (14). In contrast, when $\alpha > 0$, $V(\boldsymbol{x})$ provides a unique continuous viscosity solution to (14):

THEOREM 3.7. *If* $\alpha > 0$, $V(\boldsymbol{x})$ *in* (5) *is the unique bounded and Lipschitz-continuous viscosity solution to* (14).

PROOF. 1. The continuity and boundedness of $V(\boldsymbol{x})$ are assured by Lemma 3.4 and (6) respectively. From Definition 3.6, a continuous function is a viscosity solution to (14) if it is both a viscosity sub-solution and a viscosity super-solution.

First we prove that $V(\boldsymbol{x})$ is a sub-solution to (14). Let v be a continuously differentiable function such that $V - v$ attains a local maximum at $\boldsymbol{y}_0$. Without loss of generality, assume that this maximum is 0, i.e. $V(\boldsymbol{y}_0) = v(\boldsymbol{y}_0)$. Thus, there exists $\overline{\delta} > 0$ such that $V(\boldsymbol{x}) - v(\boldsymbol{x}) \leq 0$ for $\boldsymbol{x}$ satisfying $\|\boldsymbol{x} - \boldsymbol{y}_0\| \leq \overline{\delta}$. Suppose (16) in Definition 3.6 is false. Then there exists $\epsilon_1 > 0$ such that $\max_{i\in\{1,\ldots,n_X\}}\{h'_i(\boldsymbol{y}_0)\} \leq v(\boldsymbol{y}_0) - \epsilon_1$. Thus, there exists a positive value δ' satisfying $\delta' \leq \overline{\delta}$ such that $\max_{i\in\{1,\ldots,n_X\}} e^{-\alpha t}\{h'_i(\boldsymbol{x})\} \leq v(\boldsymbol{y}_0) - \frac{\epsilon_1}{2}$ for $\boldsymbol{x}$ and t satisfying $\|\boldsymbol{x} - \boldsymbol{y}_0\| \leq \delta'$ and $0 \leq t \leq \delta'$. Also, there exists $\epsilon_2 > 0$ such that

$$\frac{\partial v}{\partial \boldsymbol{x}}\mid_{\boldsymbol{x}=\boldsymbol{y}_0} \boldsymbol{F}(\boldsymbol{y}_0,\boldsymbol{d}) \leq \alpha v(\boldsymbol{y}_0) - \epsilon_2, \forall \boldsymbol{d} \in D. \tag{17}$$

Since v is continuously differentiable, there exists a positive value δ'' satisfying $\delta'' \leq \delta'$ such that

$$\frac{\partial v}{\partial \boldsymbol{x}}\boldsymbol{F}(\boldsymbol{x},\boldsymbol{d}) \leq \alpha v(\boldsymbol{x}) - \frac{\epsilon_2}{2} \tag{18}$$

for $\boldsymbol{x}$ satisfying $\|\boldsymbol{x} - \boldsymbol{y}_0\| \leq \delta''$ and $\boldsymbol{d} \in D$. Moreover, let Ω be a compact set which covers all states traversed by trajectories starting from $\boldsymbol{y}_0$ within the time interval $[0, \delta']$, and let M be the upper bound of $\boldsymbol{F}(\boldsymbol{x},\boldsymbol{d})$ over $\Omega \times D$. We have

$$\|\boldsymbol{\psi}^d_{\boldsymbol{y}_0}(t) - \boldsymbol{y}_0\| = \|\int_{\tau=0}^{t} \boldsymbol{F}(\boldsymbol{x}(\tau),\boldsymbol{d}(\tau))d\tau\| \leq M|t-0|, \forall d \in \mathcal{D},$$

where $t \in [0, \delta']$. Therefore there exists $\delta > 0$ such that $\|\boldsymbol{\psi}^d_{\boldsymbol{y}_0}(\tau) - \boldsymbol{y}_0\| \leq \delta''$ for all $\tau \in [0, \delta]$ and $d \in \mathcal{D}$. By applying Grönwall's inequality [17] to (18) with the time interval $[0, \delta]$, we have

$$v(\boldsymbol{\psi}^d_{\boldsymbol{y}_0}(\delta)) \leq e^{\delta\alpha} v(\boldsymbol{y}_0) + \frac{\epsilon_2}{2\alpha}(1 - e^{\delta\alpha}), \forall d \in \mathcal{D}. \tag{19}$$

Therefore, $e^{-\alpha\delta} v(\boldsymbol{\psi}^d_{\boldsymbol{y}_0}(\delta)) \leq v(\boldsymbol{y}_0) - \frac{\epsilon_2}{2\alpha}(1 - e^{-\delta\alpha})$. Further, since $V - v$ has a local maximum of 0 at $\boldsymbol{y}_0$,

$$e^{-\alpha\delta} V(\boldsymbol{\psi}^d_{\boldsymbol{y}_0}(\delta)) \leq V(\boldsymbol{y}_0) - \epsilon_3$$

holds, where $\epsilon_3 = \frac{\epsilon_2}{2\alpha}(1 - e^{-\delta\alpha}) > 0$. Therefore, according to (11) in Lemma 3.5, we finally have

$$\begin{aligned} V(\boldsymbol{y}_0) = &\sup_{d\in\mathcal{D}} \max\{e^{-\alpha\delta} V(\boldsymbol{\psi}^d_{\boldsymbol{y}_0}(\delta)), \\ &\max_{i\in\{1,\ldots,n_X\}}\{\sup_{s\in[0,\delta]} e^{-\alpha s} h'_i(\boldsymbol{\psi}^d_{\boldsymbol{y}_0}(s))\}\} \\ &\leq V(\boldsymbol{y}_0) - \min\{\frac{\epsilon_1}{2}, \epsilon_3\}. \end{aligned} \tag{20}$$

This is a contradiction, since ϵ_1 and ϵ_3 are positive. Thus V is a sub-solution to (14).

Next we prove V is also a viscosity super-solution to (14). Let v be a continuously differentiable function such that $V - v$ attains a local minimum at $\boldsymbol{y}_0$. We assume that this minimum is 0, i.e. $V(\boldsymbol{y}_0) = v(\boldsymbol{y}_0)$. Thus, there exists $\overline{\delta} > 0$ such that $V(\boldsymbol{x}) - v(\boldsymbol{x}) \geq 0$ for $\boldsymbol{x}$ satisfying $\|\boldsymbol{x} - \boldsymbol{y}_0\| \leq \overline{\delta}$.

If (15) is false, then either

$$\max_{i\in\{1,\ldots,n_X\}}\{h'_i(\boldsymbol{y}_0)\} \geq v(\boldsymbol{y}_0) + \epsilon_1 \text{ or} \tag{21}$$

$$\sup_{\boldsymbol{d}\in D} \frac{\partial v}{\partial \boldsymbol{x}}\mid_{\boldsymbol{x}=\boldsymbol{y}_0} \boldsymbol{F}(\boldsymbol{y}_0,\boldsymbol{d}) - \alpha v(\boldsymbol{y}_0) \geq \epsilon_2 \tag{22}$$

holds for some ϵ_1, $\epsilon_2 > 0$.

If (21) holds then $\max_{i\in\{1,\ldots,n_X\}} e^{-\alpha t}\{h'_i(\boldsymbol{y}_0)\} \geq v(\boldsymbol{y}_0) + \epsilon_1$ when $t = 0$. Therefore there is $\delta' > 0$ with $\delta' \leq \overline{\delta}$ such that $\max_{i\in\{1,\ldots,n_X\}} e^{-\alpha t}\{h'_i(\boldsymbol{x})\} \geq v(\boldsymbol{y}_0) + \frac{\epsilon_1}{2} = V(\boldsymbol{y}_0) + \frac{\epsilon_1}{2}$ for $\boldsymbol{x}$ and t satisfying $\|\boldsymbol{x} - \boldsymbol{y}_0\| \leq \delta'$ and $0 \leq t \leq \delta'$. Moreover, there exists a sufficiently small $\delta > 0$ such that $\|\boldsymbol{\psi}^d_{\boldsymbol{y}_0}(\tau) - \boldsymbol{y}_0\| \leq \delta'$ for $\tau \in [0, \delta]$ and $d \in \mathcal{D}$.

Then according to (11) in Lemma 3.5, we obtain

$$\begin{aligned} V(\boldsymbol{y}_0) = &\sup_{d\in\mathcal{D}} \max\{e^{-\alpha\delta} V(\boldsymbol{\psi}^d_{\boldsymbol{y}_0}(\delta)), \\ &\max_{i\in\{1,\ldots,n_X\}}\{\sup_{s\in[0,\delta]} e^{-\alpha s} h'_i(\boldsymbol{\psi}^d_{\boldsymbol{y}_0}(s))\}\} \\ &\geq V(\boldsymbol{y}_0) + \frac{\epsilon_1}{2}. \end{aligned} \tag{23}$$

This is a contradiction since $\epsilon_1 > 0$.

However, if (22) holds then there is $\delta_1 > 0$ with $\delta_1 \leq \overline{\delta}$ such that there exists a strategy $d \in \mathcal{D}$ such that

$$\frac{\epsilon_2}{2} \leq \frac{\partial v}{\partial \boldsymbol{x}}\boldsymbol{F}(\boldsymbol{x},\boldsymbol{d}) - \alpha v(\boldsymbol{x}) \tag{24}$$

for $\boldsymbol{x}$ satisfying $\|\boldsymbol{x} - \boldsymbol{y}_0\| \leq \delta_1$. By applying Grönwall's inequality [17] to (24) with the time interval $[0, \delta]$, where δ is a positive value such that $\|\boldsymbol{\psi}^d_{\boldsymbol{y}_0}(\tau) - \boldsymbol{y}_0\| \leq \delta_1$ for $\tau \in [0, \delta]$ and $d \in \mathcal{D}$, we have

$$e^{-\alpha\delta} v(\boldsymbol{\psi}^d_{\boldsymbol{y}_0}(\delta)) - v(\boldsymbol{y}_0) \geq \frac{\epsilon_2}{2\alpha}(1 - e^{-\delta\alpha}). \tag{25}$$

Further, since $V - v$ attains a local minimum at $\boldsymbol{y}_0$ and $V(\boldsymbol{y}_0) = v(\boldsymbol{y}_0)$, we have

$$e^{-\alpha\delta} V(\boldsymbol{\psi}^d_{\boldsymbol{y}_0}(\delta)) - V(\boldsymbol{y}_0) \geq \frac{\epsilon_2}{2\alpha}(1 - e^{-\delta\alpha}). \tag{26}$$

Therefore, the following contradiction is obtained:

$$\begin{aligned} V(\boldsymbol{y}_0) = &\sup_{d\in\mathcal{D}}\{\max(e^{-\alpha\delta} V(\boldsymbol{\psi}^d_{\boldsymbol{y}_0}(\delta)), \\ &\max_{i\in\{1,\ldots,n_X\}}\{\max_{s\in[0,\delta]} e^{-\alpha s} h'_i(\boldsymbol{\psi}^d_{\boldsymbol{y}_0}(s)))\}\} \\ &\geq V(\boldsymbol{y}_0) + \frac{\epsilon_2}{2\alpha}(1 - e^{-\delta\alpha}). \end{aligned} \tag{27}$$

We thus conclude that (15) holds and V is a super-solution to (14). Therefore, V is a bounded and Lipschitz-continuous viscosity solution to (14).

2. We show the uniqueness of $V(\boldsymbol{x})$. We first prove a comparison principle: If V_1 and V_2 are bounded Lipschitz-continuous functions over $\boldsymbol{x} \in \mathbb{R}^n$, and they are respectively a viscosity sub- and super-solution to (14), then $V_1 \leq V_2$ in $\mathbb{R}^n$. For ease of exposition, we define $H(\boldsymbol{x}, \boldsymbol{p}) = \sup_{\boldsymbol{d}\in D} \boldsymbol{p}\cdot\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{d})$, $H(\overline{\boldsymbol{x}}) = H(\overline{\boldsymbol{x}}, \frac{\partial \phi(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\overline{\boldsymbol{x}}})$ and $H(\overline{\boldsymbol{y}}) = H(\overline{\boldsymbol{y}}, \frac{\partial \psi(\boldsymbol{y})}{\partial \boldsymbol{y}} \mid_{\boldsymbol{y}=\overline{\boldsymbol{y}}})$., where $\cdot$ denotes the inner product between $\boldsymbol{p}$ and $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{d})$.

Let

$$\Phi(\boldsymbol{x}, \boldsymbol{y}) = V_1(\boldsymbol{x}) - V_2(\boldsymbol{y}) - \frac{\|\boldsymbol{x} - \boldsymbol{y}\|^2}{2\epsilon} - \delta(\langle \boldsymbol{x} \rangle^m + \langle \boldsymbol{y} \rangle^m),$$

where $\langle \boldsymbol{x} \rangle = (1 + \|\boldsymbol{x}\|^2)^{\frac{1}{2}}$, and ϵ, δ, m are positive parameters to be chosen conveniently. Let us prove by contradiction that there is $\beta > 0$ and $\boldsymbol{z}$ such that $V_1(\boldsymbol{z}) - V_2(\boldsymbol{z}) = \beta$. Choosing $\delta > 0$ such that $2\delta\langle \boldsymbol{z} \rangle \leq \frac{\beta}{2}$, we have for $0 < m \leq 1$,

$$\frac{\beta}{2} < \beta - 2\delta\langle \boldsymbol{x} \rangle^m = \Phi(\boldsymbol{z}, \boldsymbol{z}) \leq \sup_{\boldsymbol{x},\boldsymbol{y}} \Phi(\boldsymbol{x}, \boldsymbol{y}). \quad (28)$$

Since Φ is continuous and $\lim_{\|\boldsymbol{x}\|+\|\boldsymbol{y}\|\to\infty} \Phi(\boldsymbol{x}, \boldsymbol{y}) = -\infty$, there exist $\overline{x}$, $\overline{y}$ such that

$$\Phi(\overline{x}, \overline{y}) = \sup_{\boldsymbol{x},\boldsymbol{y}} \Phi(\boldsymbol{x}, \boldsymbol{y}). \quad (29)$$

From $\Phi(\overline{x}, \overline{\boldsymbol{x}}) + \Phi(\overline{\boldsymbol{y}}, \overline{\boldsymbol{y}}) \leq 2\Phi(\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}})$ we easily get

$$\frac{\|\overline{\boldsymbol{x}} - \overline{\boldsymbol{y}}\|^2}{\epsilon} \leq V_1(\overline{\boldsymbol{x}}) - V_1(\overline{\boldsymbol{y}}) + V_2(\overline{\boldsymbol{x}}) - V_2(\overline{\boldsymbol{y}}). \quad (30)$$

Then the boundedness of V_1 and V_2 implies that

$$\|\overline{\boldsymbol{x}} - \overline{\boldsymbol{y}}\| \leq c\sqrt{\epsilon} \quad (31)$$

for a suitable constant c. By plugging (31) into (30) and using the Lipschitz-continuity of V_1 and V_2 we get

$$\frac{\|\overline{\boldsymbol{x}} - \overline{\boldsymbol{y}}\|}{\epsilon} \leq w\sqrt{\epsilon} \quad (32)$$

for some constant w.

Next, define the continuously differentiable functions

$$\begin{aligned} \phi(\boldsymbol{x}) &:= V_2(\overline{\boldsymbol{y}}) + \frac{\|\boldsymbol{x} - \overline{\boldsymbol{y}}\|^2}{2\epsilon} + \delta(\langle \boldsymbol{x} \rangle^m + \langle \overline{\boldsymbol{y}} \rangle^m), \\ \psi(\boldsymbol{y}) &:= V_1(\overline{\boldsymbol{x}}) - \frac{\|\overline{\boldsymbol{x}} - \boldsymbol{y}\|^2}{2\epsilon} - \delta(\langle \overline{\boldsymbol{x}} \rangle^m + \langle \boldsymbol{y} \rangle^m), \end{aligned} \quad (33)$$

and observe that $V_1 - \phi$ attains its maximum at $\overline{\boldsymbol{x}}$ and $V_2 - \psi$ attains its minimum at $\overline{\boldsymbol{y}}$. It is easy to compute

$$\begin{aligned} \frac{\partial \phi(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\overline{\boldsymbol{x}}} &= \frac{\overline{\boldsymbol{x}} - \overline{\boldsymbol{y}}}{\epsilon} + \gamma\overline{\boldsymbol{x}}, \gamma = \delta m \langle \overline{\boldsymbol{x}} \rangle^{m-2} \\ \frac{\partial \psi(\boldsymbol{y})}{\partial \boldsymbol{y}} \mid_{\boldsymbol{y}=\overline{\boldsymbol{y}}} &= \frac{\overline{\boldsymbol{x}} - \overline{\boldsymbol{y}}}{\epsilon} - \tau\overline{\boldsymbol{y}}, \tau = \delta m \langle \overline{\boldsymbol{y}} \rangle^{m-2}. \end{aligned} \quad (34)$$

According to Definition 3.6, we have

$$\begin{aligned} &\min\big\{\alpha V_1(\overline{\boldsymbol{x}}) - H(\overline{\boldsymbol{x}}), V_1(\overline{\boldsymbol{x}}) - \max_{j\in\{1,\dots,n_X\}} h'_j(\overline{\boldsymbol{x}})\big\} \\ &\leq \min\big\{\alpha V_2(\overline{\boldsymbol{y}}) - H(\overline{\boldsymbol{y}}), V_2(\overline{\boldsymbol{y}}) - \max_{j\in\{1,\dots,n_X\}} h'_j(\overline{\boldsymbol{y}})\big\}, \end{aligned} \quad (35)$$

implying that

$$\begin{aligned} &\min\big\{\alpha V_1(\overline{\boldsymbol{x}}) - H(\overline{\boldsymbol{x}}) - (\alpha V_2(\overline{\boldsymbol{y}}) - H(\overline{\boldsymbol{y}})), V_1(\overline{\boldsymbol{x}}) - \\ &\max_{j\in\{1,\dots,n_X\}} h'_j(\overline{\boldsymbol{x}}) - (V_2(\overline{\boldsymbol{y}}) - \max_{j\in\{1,\dots,n_X\}} h'_j(\overline{\boldsymbol{y}}))\big\} \leq 0. \end{aligned} \quad (36)$$

Obviously, either

$$\alpha V_1(\overline{\boldsymbol{x}}) - H(\overline{\boldsymbol{x}}) - (\alpha V_2(\overline{\boldsymbol{y}}) - H(\overline{\boldsymbol{y}})) \leq 0 \text{ or} \quad (37)$$

$$V_1(\overline{\boldsymbol{x}}) - \max_{j\in\{1,\dots,n_X\}} h'_j(\overline{\boldsymbol{x}}) - V_2(\overline{\boldsymbol{y}}) + \max_{j\in\{1,\dots,n_X\}} h'_j(\overline{\boldsymbol{y}}) \leq 0. \quad (38)$$

If (37) holds, $V_1(\overline{\boldsymbol{x}}) - V_2(\overline{\boldsymbol{y}}) \leq \frac{1}{\alpha}(H(\overline{\boldsymbol{x}}) - H(\overline{\boldsymbol{y}})) \leq \frac{1}{\alpha}(\epsilon + L_{\boldsymbol{F}} w\sqrt{\epsilon} + \delta m K(\langle \overline{\boldsymbol{y}} \rangle^m + \langle \overline{\boldsymbol{x}} \rangle^m)$ with $K = L_{\boldsymbol{F}} + \sup_{\boldsymbol{d}\in D}\{\|\boldsymbol{F}(\boldsymbol{0}, \boldsymbol{d})\|\}$ and the last inequality can be obtained as follows:

$$\begin{aligned} &H^-(\overline{\boldsymbol{x}}) - H^-(\overline{\boldsymbol{y}}) \\ &\leq \sup_{\boldsymbol{d}\in D}\big(\frac{\partial \phi(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\overline{\boldsymbol{x}}} \cdot\boldsymbol{F}(\overline{\boldsymbol{x}}, \boldsymbol{d}) - \frac{\partial \psi(\boldsymbol{y})}{\partial \boldsymbol{y}} \mid_{\boldsymbol{y}=\overline{\boldsymbol{y}}} \cdot\boldsymbol{F}(\overline{\boldsymbol{y}}, \boldsymbol{d})\big) \\ &\leq \frac{\partial \phi(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\overline{\boldsymbol{x}}} \cdot\boldsymbol{F}(\overline{\boldsymbol{x}}, \boldsymbol{d}_1) - \frac{\partial \psi(\boldsymbol{y})}{\partial \boldsymbol{y}} \mid_{\boldsymbol{y}=\overline{\boldsymbol{y}}} \cdot\boldsymbol{F}(\overline{\boldsymbol{y}}, \boldsymbol{d}_1) + \epsilon \\ &\leq \frac{\|\overline{\boldsymbol{x}} - \overline{\boldsymbol{y}}\|^2}{\epsilon} L_{\boldsymbol{F}} + \gamma\overline{\boldsymbol{x}} \cdot \boldsymbol{F}(\overline{\boldsymbol{x}}, \boldsymbol{d}_1) + \tau\overline{\boldsymbol{y}} \cdot \boldsymbol{F}(\overline{\boldsymbol{y}}, \boldsymbol{d}_1) + \epsilon \\ &\leq \frac{\|\overline{\boldsymbol{x}} - \overline{\boldsymbol{y}}\|^2}{\epsilon} L_{\boldsymbol{F}} + \gamma\overline{\boldsymbol{x}} \cdot (\boldsymbol{F}(\overline{\boldsymbol{x}}, \boldsymbol{d}_1) - \boldsymbol{F}(\boldsymbol{0}, \boldsymbol{d}_1) + \boldsymbol{F}(\boldsymbol{0}, \boldsymbol{d}_1)) \\ &\qquad + \tau\overline{\boldsymbol{y}} \cdot (\boldsymbol{F}(\overline{\boldsymbol{y}}, \boldsymbol{d}_1) - \boldsymbol{F}(\boldsymbol{0}, \boldsymbol{d}_1) + \boldsymbol{F}(\boldsymbol{0}, \boldsymbol{d}_1)) + \epsilon \\ &\leq \frac{\|\overline{\boldsymbol{x}} - \overline{\boldsymbol{y}}\|^2}{\epsilon} L_{\boldsymbol{F}} + \gamma L_{\boldsymbol{F}}\|\overline{\boldsymbol{x}}\|^2 + \gamma\|\overline{\boldsymbol{x}}\|\|\boldsymbol{F}(\boldsymbol{0}, \boldsymbol{d}_1)\| \\ &\qquad + \tau L_{\boldsymbol{F}}\|\overline{\boldsymbol{y}}\|^2 + \tau\|\overline{\boldsymbol{y}}\|\|\boldsymbol{F}(\boldsymbol{0}, \boldsymbol{d}_1)\| + \epsilon \\ &\leq \frac{\|\overline{\boldsymbol{x}} - \overline{\boldsymbol{y}}\|^2}{\epsilon} L_{\boldsymbol{F}} + \gamma K(1 + \|\overline{\boldsymbol{x}}\|^2) + \tau K(1 + \|\overline{\boldsymbol{y}}\|^2) + \epsilon \\ &\leq L_{\boldsymbol{F}} w\sqrt{\epsilon} + \delta m K(\langle \overline{\boldsymbol{y}} \rangle^m + \langle \overline{\boldsymbol{x}} \rangle^m) + \epsilon, \end{aligned} \quad (39)$$

where $\boldsymbol{d}_1$ satisfies

$$\begin{aligned} &\sup_{\boldsymbol{d}\in D}\big(\frac{\partial \phi(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\overline{\boldsymbol{x}}} \cdot\boldsymbol{F}(\overline{\boldsymbol{x}}, \boldsymbol{d}) - \frac{\partial \psi(\boldsymbol{y})}{\partial \boldsymbol{y}} \mid_{\boldsymbol{y}=\overline{\boldsymbol{y}}} \cdot\boldsymbol{F}(\overline{\boldsymbol{y}}, \boldsymbol{d})\big) \\ &\leq \frac{\partial \phi(\boldsymbol{x})}{\partial \boldsymbol{x}} \mid_{\boldsymbol{x}=\overline{\boldsymbol{x}}} \cdot\boldsymbol{F}(\overline{\boldsymbol{x}}, \boldsymbol{d}_1) - \frac{\partial \psi(\boldsymbol{y})}{\partial \boldsymbol{y}} \mid_{\boldsymbol{y}=\overline{\boldsymbol{y}}} \cdot\boldsymbol{F}(\overline{\boldsymbol{y}}, \boldsymbol{d}_1) + \epsilon. \end{aligned}$$

Therefore, choosing $0 < m \leq \frac{\alpha}{K}$, we obtain $\Phi(\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}}) \leq V_1(\overline{\boldsymbol{x}}) - V_2(\overline{\boldsymbol{y}}) - \delta(\langle \overline{\boldsymbol{x}} \rangle^m + \langle \overline{\boldsymbol{y}} \rangle^m) \leq \frac{1}{\alpha}(L_{\boldsymbol{F}} w\sqrt{\epsilon} + \epsilon)$. $\Phi(\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}})$ can be smaller than $\frac{\beta}{2}$ for ϵ small enough, contradicting (28).

If (38) holds, $V_1(\overline{\boldsymbol{x}}) - V_2(\overline{\boldsymbol{y}}) \leq \max_{j\in\{1,\dots,n_X\}} h'_j(\overline{\boldsymbol{x}}) - \max_{j\in\{1,\dots,n_X\}} h'_j(\overline{\boldsymbol{y}}) \leq L_{h'} c\sqrt{\epsilon}$, where $L_{h'}$ is the Lipschitz constant over a set covering $\overline{\boldsymbol{x}}$ and $\overline{\boldsymbol{y}}$. Thus, $\Phi(\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}})$ can be made smaller than $\frac{\beta}{2}$ for ϵ small enough, contradicting (28).

Therefore, $V_1 \leq V_2$ over $\boldsymbol{x} \in \mathbb{R}^n$. It is obvious that if $U(\boldsymbol{x})$ is a bounded Lipschitz-continuous viscosity solution to (14), $U(\boldsymbol{x})$ and $V(\boldsymbol{x})$ are both sub- and super-viscosity solutions and consequently $U(\boldsymbol{x}) = V(\boldsymbol{x})$. Therefore, the uniqueness of bounded Lipschitz solutions to (14) is guaranteed. □

If $\alpha > 0$ then $V(\boldsymbol{x})$ is the unique bounded and Lipschitz-continuous solution to (14) according to Theorem 3.7. This is the main contribution of this paper. Such continuity facilitates the use of existing numerical methods (e.g., [34]) and tools (e.g., [9]) to solve equation (14) for an appropriate number of state and perturbation variables and consequently provides a practical method for computing an approximation of the maximal robust invariant set. For $\alpha = 0$, it is however nontrivial to compute its minimal lower semi-continuous solution. In the sequel we propose a novel semi-definite programming formulation based on (14) with $\alpha = 0$ to synthesize robust invariant sets.

3.3 Semi-definite Programming Implementation

In this subsection we present a method based on semi-definite programming to compute robust invariant sets.

From (14) we observe that if a continuously differentiable function $u(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$ satisfies (14), then $u(\boldsymbol{x})$ satisfies for $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{d} \in D$ the constraints:

$$\begin{cases} \alpha u(\boldsymbol{x}) - \frac{\partial u}{\partial \boldsymbol{x}}\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{d}) \geq 0, \forall \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{d} \in D, \\ u(\boldsymbol{x}) - h'_j(\boldsymbol{x}) \geq 0, \forall \boldsymbol{x} \in \mathbb{R}^n, \forall j \in \{1, \ldots, n_X\}. \end{cases} \tag{40}$$

Corollary 3.8. *If a continuously differentiable function $u(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$ is a solution to (40), $\{\boldsymbol{x} \mid u(\boldsymbol{x}) \leq 0\}$ is a robust invariant set for $\alpha = 0$ and $\{\boldsymbol{x} \mid u(\boldsymbol{x}) = 0\}$ a robust invariant set for $\alpha > 0$.*

Proof. Assume $\boldsymbol{x}_0 \in \{\boldsymbol{x} \in \mathbb{R}^n \mid u(\boldsymbol{x}) \leq 0\}$. According to (40), we have that $u(\boldsymbol{\psi}^d_{\boldsymbol{x}_0}(t)) \leq e^{\alpha t} u(\boldsymbol{x}_0) \leq 0$ for $d \in \mathcal{D}$ and $t \in [0, \infty)$. This implies that $\boldsymbol{\psi}^d_{\boldsymbol{x}_0}(t) \in \{\boldsymbol{x} \in \mathbb{R}^n \mid u(\boldsymbol{x}) \leq 0\}$ for $t \in [0, \infty)$ and $d \in \mathcal{D}$.

Since $u(\boldsymbol{x}) - h'_j(\boldsymbol{x}) \geq 0$ for $\boldsymbol{x} \in \mathbb{R}^n$ and $j \in \{1, \ldots, n_X\}$, $u(\boldsymbol{x}) \leq 0$ implies $h_j(\boldsymbol{x}) \leq 0$ for $\boldsymbol{x} \in \mathbb{R}^n$ and $j \in \{1, \ldots, n_X\}$. Consequently, $\{\boldsymbol{x} \in \mathbb{R}^n \mid u(\boldsymbol{x}) \leq 0\} \subset X$. Thus, $\{\boldsymbol{x} \in \mathbb{R}^n \mid u(\boldsymbol{x}) \leq 0\} \subset \mathcal{R}_0$ is a robust invariant set.

In the following we prove that $u(\boldsymbol{x})$ is positive semi-definite when $\alpha > 0$, i.e. that $u(\boldsymbol{x}) \geq 0$ for $\boldsymbol{x} \in \mathbb{R}^n$. Suppose that there exists a state $\boldsymbol{x}_0 \in \mathbb{R}^n$ such that $u(\boldsymbol{x}_0) < 0$. According to the first constraint in (40), we obtain that $u(\boldsymbol{\psi}^d_{\boldsymbol{x}_0}(t)) \leq e^{\alpha t} u(\boldsymbol{x}_0)$ holds for $t \geq 0$. Thus, $\lim_{t \to \infty} u(\boldsymbol{\psi}^d_{\boldsymbol{x}_0}(t)) = -\infty$, which contradicts the fact that $u(\boldsymbol{x}) \geq \max_j\{h'_j(\boldsymbol{x})\} > -1$ for $\boldsymbol{x} \in \mathbb{R}^n$. Therefore, $u(\boldsymbol{x}) \geq 0$ over $\mathbb{R}^n$ and consequently $\{\boldsymbol{x} \mid u(\boldsymbol{x}) = 0\}$ is a robust invariant set for $\alpha > 0$. □

From Corollary 3.8 we observe that a robust invariant set could be found by solving (40) rather than (14). Aspired by $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{d}) = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{d})$ for $(\boldsymbol{x}, \boldsymbol{d}) \in \mathcal{X} \times D$, where $\mathcal{X}$ is defined in (3), we construct a semi-definite program to compute robust invariant sets when $u(\boldsymbol{x})$ in (40) is a polynomial function and is restricted in $\mathcal{X}$, i.e.

$$\begin{cases} \alpha u(\boldsymbol{x}) - \frac{\partial u}{\partial \boldsymbol{x}}\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{d}) \geq 0, \forall \boldsymbol{x} \in \mathcal{X}, \forall \boldsymbol{d} \in D, \\ u(\boldsymbol{x}) - h'_j(\boldsymbol{x}) \geq 0, \forall \boldsymbol{x} \in \mathcal{X}, \forall j \in \{1, \ldots, n_X\}. \end{cases} \tag{41}$$

Denote the set of sum-of-squares polynomials over $\boldsymbol{y}$ by $\texttt{SOS}(\boldsymbol{y})$, i.e.

$$\texttt{SOS}(\boldsymbol{y}) := \{p \in \mathbb{R}[\boldsymbol{y}] \mid p = \sum_{i=1}^{r} q_i^2, q_i \in \mathbb{R}[\boldsymbol{y}], i = 1, \ldots, r\}.$$

The constructed semi-definite program is formulated below:

Theorem 3.9. *If $u(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$ is a solution to (42) then the sets $\{\boldsymbol{x} \in \mathcal{X} \mid u(\boldsymbol{x}) \leq 0\}$ and $\{\boldsymbol{x} \in \mathcal{X} \mid u(\boldsymbol{x}) = 0\}$ are robust invariant sets for $\alpha = 0$ and $\alpha > 0$, respectively.*

$$\begin{aligned} &\min_{u, s_i, s_i^X} \boldsymbol{c} \cdot \boldsymbol{w} \\ &\alpha u(\boldsymbol{x}) - \frac{\partial u}{\partial \boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{d}) + \sum_{i=1}^{n_D} s_i h_i^D(\boldsymbol{d}) - s_0 h(\boldsymbol{x}) \in \texttt{SOS}(\boldsymbol{x}, \boldsymbol{d}), \\ &(1 + h_j^2)u(\boldsymbol{x}) - h_j(\boldsymbol{x}) - s_j^X h(\boldsymbol{x}) \in \texttt{SOS}(\boldsymbol{x}), \\ &j = 1, \ldots, n_X, \end{aligned} \tag{42}$$

where $\boldsymbol{c} \cdot \boldsymbol{w} = \int_{\mathcal{X}} u d\boldsymbol{x}$, $\boldsymbol{c}$ is the vector composed of unknown coefficients in $u(\boldsymbol{x}) \in \mathbb{R}_k[\boldsymbol{x}]$, $\boldsymbol{w}$ is the constant vector computed by integrating the monomials in $u(\boldsymbol{x})$ over $\mathcal{X}$, $s_i \in \texttt{SOS}[\boldsymbol{x}, \boldsymbol{d}]$, $i = 0, \ldots, n_D$, and $s_j^X \in \texttt{SOS}[\boldsymbol{x}]$, $j = 1, \ldots, n_X$. The constraints that polynomials are sum-of-squares can be written explicitly as linear matrix inequalities, and the objective is linear in the coefficients of $u(\boldsymbol{x})$. Therefore problem (42) is a semi-definite program.

Proof. Since $u(\boldsymbol{x})$ satisfies the constraint in (42), we obtain that $u(\boldsymbol{x})$ satisfies according to the $\mathcal{S}-$ procedure presented in [10] the inequations

$$\alpha u(\boldsymbol{x}) - \frac{\partial u}{\partial \boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{d}) \geq 0, \forall \boldsymbol{d} \in D, \forall \boldsymbol{x} \in \mathcal{X} \text{ and} \tag{43}$$

$$(1 + h_j^2(\boldsymbol{x}))u(\boldsymbol{x}) - h_j(\boldsymbol{x}) \geq 0, \forall \boldsymbol{x} \in \mathcal{X}, \forall j \in \{1, \ldots, n_X\}. \tag{44}$$

Assume that there exist a state $\boldsymbol{y}_0 \in \{\boldsymbol{x} \in \mathcal{X} \mid u(\boldsymbol{x}) \leq 0\}$, a policy d', and a time instant $\tau > 0$ such that $\boldsymbol{\phi}^{d'}_{\boldsymbol{y}_0}(\tau) \notin X$. Inequation (44) implies $\{\boldsymbol{x} \in \mathcal{X} \mid u(\boldsymbol{x}) \leq 0\} \subset X$ and thus $\boldsymbol{y}_0 \in X$. Let t_0 be the time instant such that $\boldsymbol{\phi}^{d'}_{\boldsymbol{y}_0}(t_0)$ belongs to the boundary of X and $\boldsymbol{\phi}^{d'}_{\boldsymbol{y}_0}(s) \in \mathcal{X} \setminus X$ for $s \in (t_0, \tau]$. t_0 exists since $X \subset \mathcal{X}$ and $\partial X \cap \partial \mathcal{X} = \emptyset$. Also, since $X \subset \mathcal{X}$ with $\partial X \cap \partial \mathcal{X} = \emptyset$ as well as (44), we obtain that $\boldsymbol{\phi}^{d'}_{\boldsymbol{y}_0}(t_0) \in \{\boldsymbol{x} \in \mathcal{X} \mid u(\boldsymbol{x}) \leq 0\}$ and $u(\boldsymbol{\phi}^{d'}_{\boldsymbol{y}_0}(s) > 0$ for $s \in (t_0, \tau]$. However, according to (43) we get $u(\boldsymbol{\phi}^{d'}_{\boldsymbol{y}_0}(s) \leq 0$ for $s \in (t_0, \tau]$. This is a contradiction. Thus, all trajectories of system (1) initialized in $\{\boldsymbol{x} \in \mathcal{X} \mid u(\boldsymbol{x}) \leq 0\}$ live in $\{\boldsymbol{x} \in \mathcal{X} \mid u(\boldsymbol{x}) \leq 0\}$ and thus stay inside X always.

A similar argument as in the proof of Corollary 3.8 can be used to prove that $u(\boldsymbol{x}) \geq 0$ over $\{\boldsymbol{x} \in \mathcal{X} \mid u(\boldsymbol{x}) \leq 0\}$ when $\alpha > 0$. Therefore, Theorem 3.9 is justified. □

Remark 1. *From Theorem 3.9, we observe that a robust invariant set is described by the zero set of a polynomial function when applying (42) with $\alpha > 0$. Extremely conservative robust invariant sets could thus be returned by solving (42), thus potentially rendering the application of the semi-definite program (42) useless in practice. This effect is shown on some examples in Section 4. Therefore we generally assign 0 to α when employing the semi-definite program (42) for synthesizing robust invariant sets.*

4 EXPERIMENTS

In this section we evaluate the performance of grid-based numerical methods for solving (14) with $\alpha > 0$ and of the semi-definite programming method (42) on three illustrative examples. All computations were performed on an i7-8550U 1.80GHz CPU with 4GB RAM running Fedora 29. We employ the ROC-HJ[1] solver [8] to solve (14) with $\alpha > 0$, and use YALMIP [2] [23] and Mosek[3] [29] to implement (42). The parameters that control the performance of these two methods are presented in Table 1. Note that in solving (14), uniform grids of 500^2 on the state space $[-1.1, 1.1] \times [-1.1, 1.1]$, 400^2 on $[-0.25, 0.25] \times [-0.25, 0.25]$ and 20^7 on $[-1, 1]^7$ are respectively adopted for Examples 4.1, 4.2 and 4.3.

	SDP					HJB	
Ex.	d_u	d_{s_i}	d_j^X	α	$T_{\mathtt{SDP}}$	α	$T_{\mathtt{HJB}}$
1	10	10	12	0	1.48	1	628.33
	14	14	16	0	6.84		
	16	16	18	0	16.05		
2	10	12	12	0	3.61	1	110.93
	18	20	20	0	32.46		
3	2	2	4	0	4.25	1	–
	4	4	6	0	156.03		

Table 1: *Parameters and the performance of our implementations of solving* (42) *and* (14) *on the examples presented in this section.* $d_u, d_{s_i}, d_{s_j^X}$*: the degree of the polynomials* u, s_i, s_j^X *in* (42)*, respectively,* $i = 1, \ldots, n_D$*,* $j = 1, \ldots, n_X$*;* α*: the scalar value* α *in* (42) *and* (14)*;* $T_{\mathtt{SDP}}$ *and* $T_{\mathtt{HJB}}$*: computation times (seconds) for solving* (42) *and* (14) *respectively.*

Example 4.1. Consider a two-dimensional system

$$\dot{x} = -0.5x, \dot{y} = 10x^2 - (0.5 + d)y,$$

where $X = \{(x, y) \mid x^2 + y^2 - 1 \leq 0\}$, $D = \{d \mid d^2 - 0.01 \leq 0\}$, and $\mathcal{X} = \{(x, y) \mid 1.1 - x^2 - y^2 \geq 0\}$.

The estimated maximal robust invariant set, which is computed by solving (14) with $\alpha = 1$, is displayed in Fig. 1. The level sets of the corresponding computed viscosity solution are shown in Fig. 2. Plots of robust invariant sets computed by solving (42) when $d_u = 10, 12, 16$ are presented in Fig. 1.

Example 4.2. Consider a two-dimensional system, corresponding to a Moore-Greitzer model of a jet engine with the controller $u = 0.8076x - 0.9424y$ from [33],

$$\dot{x} = -y - \frac{3}{2}x^2 - \frac{1}{2}x^3 + d, \dot{y} = u,$$

where $X = \{(x, y) \mid x^2 + y^2 - 0.04 \leq 0\}$, $D = \{d \mid d^2 - 0.02^2 \leq 0\}$, and $\mathcal{X} = \{(x, y) \mid 0.041 - x^2 - y^2 \geq 0\}$.

The estimated maximal robust invariant set, which is computed by solving (14) with $\alpha = 1$, is displayed in Fig. 3. The level sets of the corresponding computed viscosity solution are shown in Fig. 4. Plots of robust invariant sets computed by solving (42) when $d_u = 10, 18$ are presented in Fig. 3.

[1]Download from https://uma.ensta-paristech.fr/soft/ROC-HJ/.
[2]Download from https://yalmip.github.io/.
[3]Mosek for academic use can be obtained free of charge from https://www.mosek.com/.

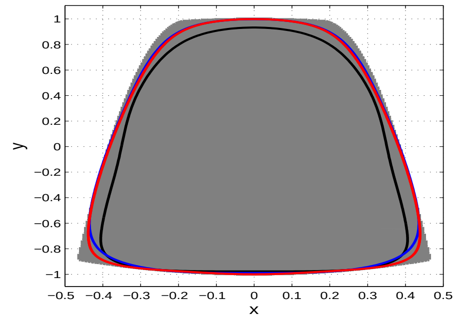

Figure 1: Computed robust invariant sets for Ex. 4.1. Black, blue and red curves represent boundaries of robust invariant sets computed by solving (42) when $d_u = 10$, 14, and 16, respectively. Gray region is an estimate of the maximal robust invariant set obtained by numerically solving (14).

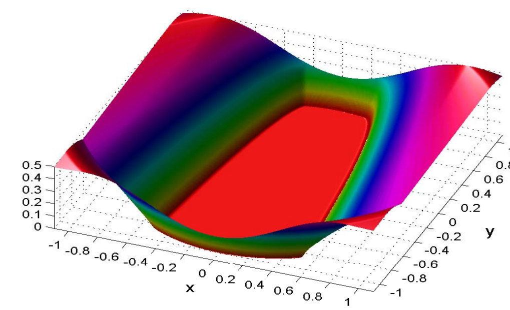

Figure 2: Level sets of the computed viscosity solution to (14) for Ex. 4.1 with $\alpha = 1$.

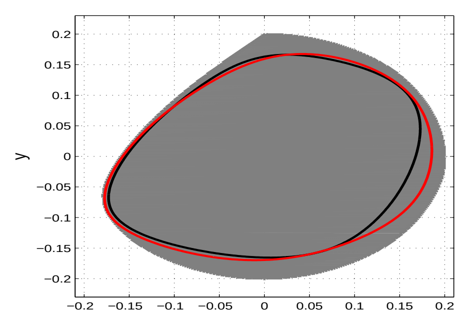

Figure 3: Computed robust invariant sets for Ex. 4.2. Black and red curves represent boundaries of robust invariant sets computed by solving (42) when $d_u = 10$ and 18, respectively. Gray region is an estimate of the maximal robust invariant set obtained by numerically solving (14).

The level sets displayed in Fig. 2 and Fig. 4 further confirm that the viscosity solution $V(\boldsymbol{x})$ to HJB (14) with $\alpha > 0$ is non-negative, as stated in Theorem 3.3. We apply the semi-definite program (42) with $\alpha = 1$ to Examples 4.1 and 4.2 as

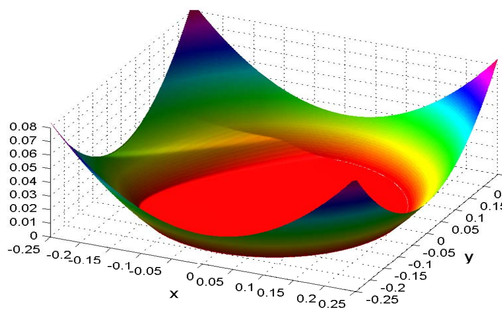

Figure 4: Level sets of the computed viscosity solution to (14) for Ex. 4.2 with $\alpha = 1$.

well. However, we did not obtain non-empty robust invariant sets for both examples based on the parameters in Table 1. This justifies Remark 1.

Although grid-based numerical methods can be employed to solve HJB equation (14) with $\alpha > 0$ and thus produce an estimate of the maximal robust invariant set, they generally require gridding state and perturbation spaces, thereby exhibiting exponential growth in computational complexity with the number of state and perturbation variables and preventing their application to higher dimensional problems. As opposed to grid-based numerical methods, the semi-definite programming based method (42) trades off accuracy for computing speed. It falls within the convex programming framework and can be applied to systems with moderately high dimensionality. We illustrate this issue through an example with seven state variables.

Example 4.3. Consider a seven-dimensional system

$$\begin{aligned}&\dot{x}_1 = -x_1 + 0.5x_2, \dot{x}_2 = -x_2 + 0.4x_3,\\&\dot{x}_3 = -x_3 + 0.5x_4, \dot{x}_4 = -x_4 + 0.7x_5,\\&\dot{x}_5 = -x_5 + 0.5x_6, \dot{x}_6 = -x_6 + 0.8x_7,\\&\dot{x}_7 = -x_7 + 10x_1^2 + x_2^2 - x_3^2 - x_4^2 + x_5^2 + x_6 d,\end{aligned} \tag{45}$$

where $X = \{\boldsymbol{x} \mid \|\boldsymbol{x}\|^2 - 1 \leq 0\}$, $D = \{d \mid d^2 - 0.25 \leq 0\}$, and $\mathcal{X} = \{\boldsymbol{x} \mid -\|\boldsymbol{x}\|^2 + 1.01 \leq 0\}$.

Unlike for the low-dimensional Examples 4.1 and 4.2, the grid-based numerical method for solving (14) here runs out of memory and thus does not return an estimate. The semi-definite programming based method (42), however, is still able to compute robust invariant sets, which are illustrated in Fig. 5. In order to shed light on the accuracy of the computed robust invariant sets, we employ the first-order Euler method to synthesize coarse estimates of the maximal robust invariant sets on planes $x_1 - x_2$ with $x_3 = x_4 = x_5 = x_6 = x_7 = 0$ and $x_5 - x_6$ with $x_1 = x_2 = x_3 = x_4 = x_7 = 0$ respectively. These estimates are also depicted in Fig. 5.

Although the size of the program (42) grows extremely fast with the number of state and perturbation variables and the degree of the polynomials in (42), engineering insight can further enhance the computational efficiency and scalability advantages of the semi-definite programming based method

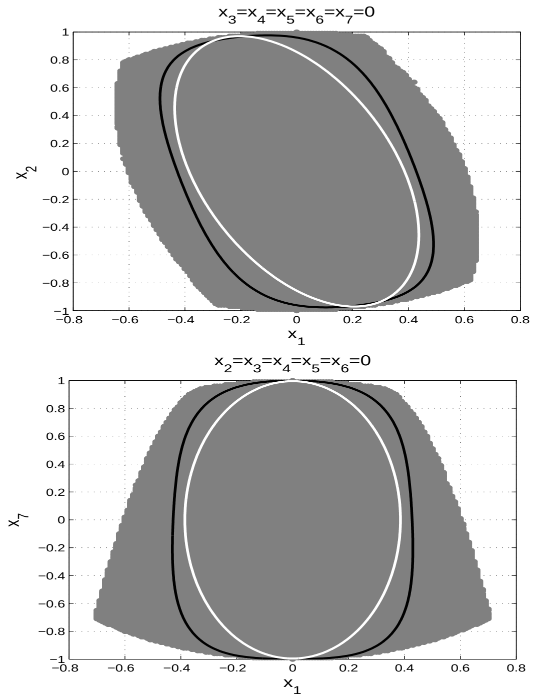

Figure 5: Projections of computed robust invariant sets for Ex. 4.3. White and black curves represent boundaries of robust invariant sets computed by solving (42) when $d_u = 2$ and 4, respectively. Gray region is an estimate of the maximal robust invariant set obtained via simulation techniques.

(42) over the grid-based numerical method. Some options are the use of template polynomials such as diagonally dominant sum-of-squares (DSOS) and scaled diagonally dominant-sum-of-squares (SDSOS) polynomials [1, 25]. DSOS and SDSOS result in converting the semi-definite programming based relaxations into linear programs and second-order cone programs with lower complexity than the semi-definite programs.

Finally, it is worth remarking that the semi-definite programming based method (42) is limited to polynomial systems, while the HJB method (14) is capable of dealing with more general nonlinear systems, which are however not the focus of this paper.

5 CONCLUSION AND FUTURE WORK

In this paper we studied the computation of robust invariant sets for state-constrained perturbed polynomial systems within the Hamilton-Jacobi reachability framework. We formulated the maximal robust invariant set as the zero level set of the unique Lipschitz viscosity solution to a HJB equation. Existing numerical methods were employed to solve this HJB equation for an appropriate number of variables and thus produce an estimate of the maximal robust invariant set. Moreover, based on the derived HJB equation a novel semi-definite programming method was proposed such that a robust invariant set could be computed by solving a single semi-definite program, which provides for better scalability than the numeric method. Three illustrative examples demonstrated the performance of the approaches.

In future work we will compare the methods in this paper with other existing methods [31, 39]. Additionally, we will address two open problems for the program (42), namely conditions for existence of a solution and how well the computed robust invariant sets approximate the maximal robust invariant set with the degree of polynomials tending to infinity.

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