Branching Time Temporal Logics

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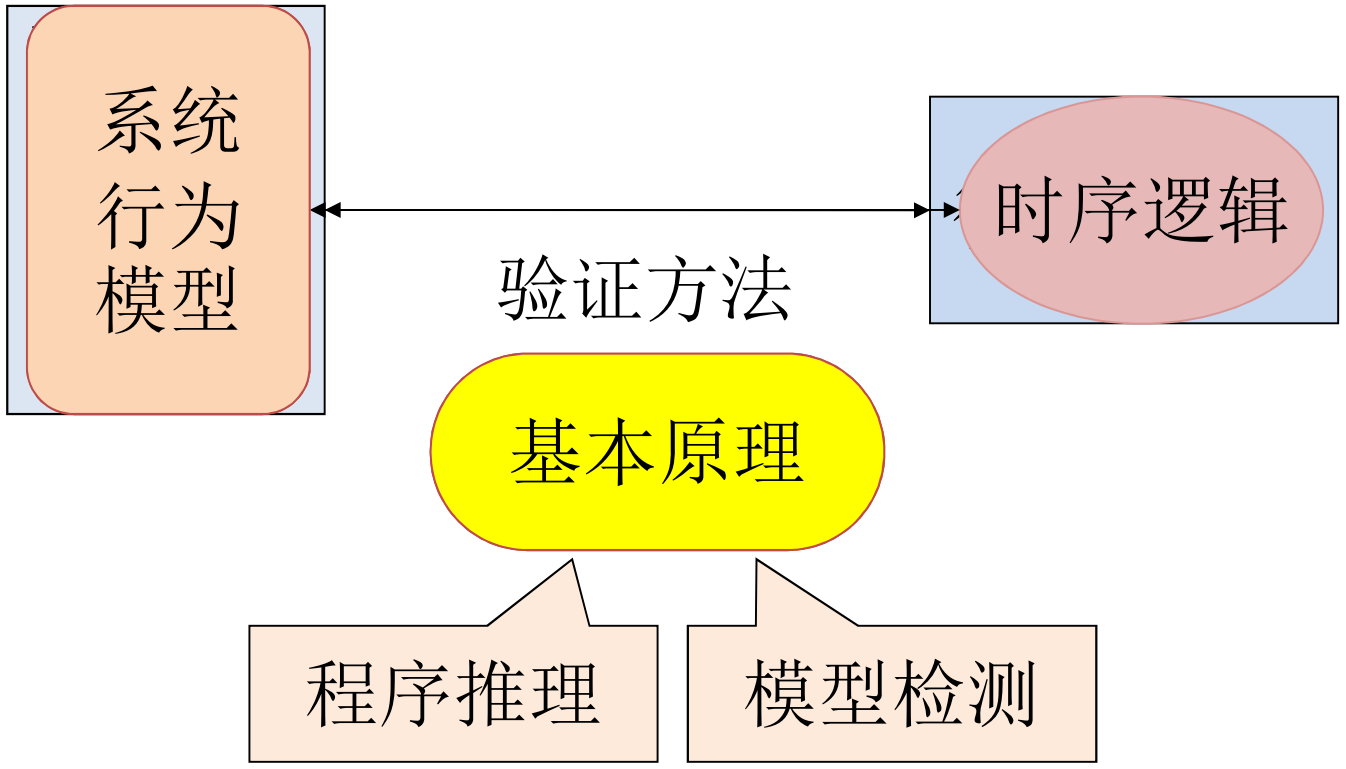
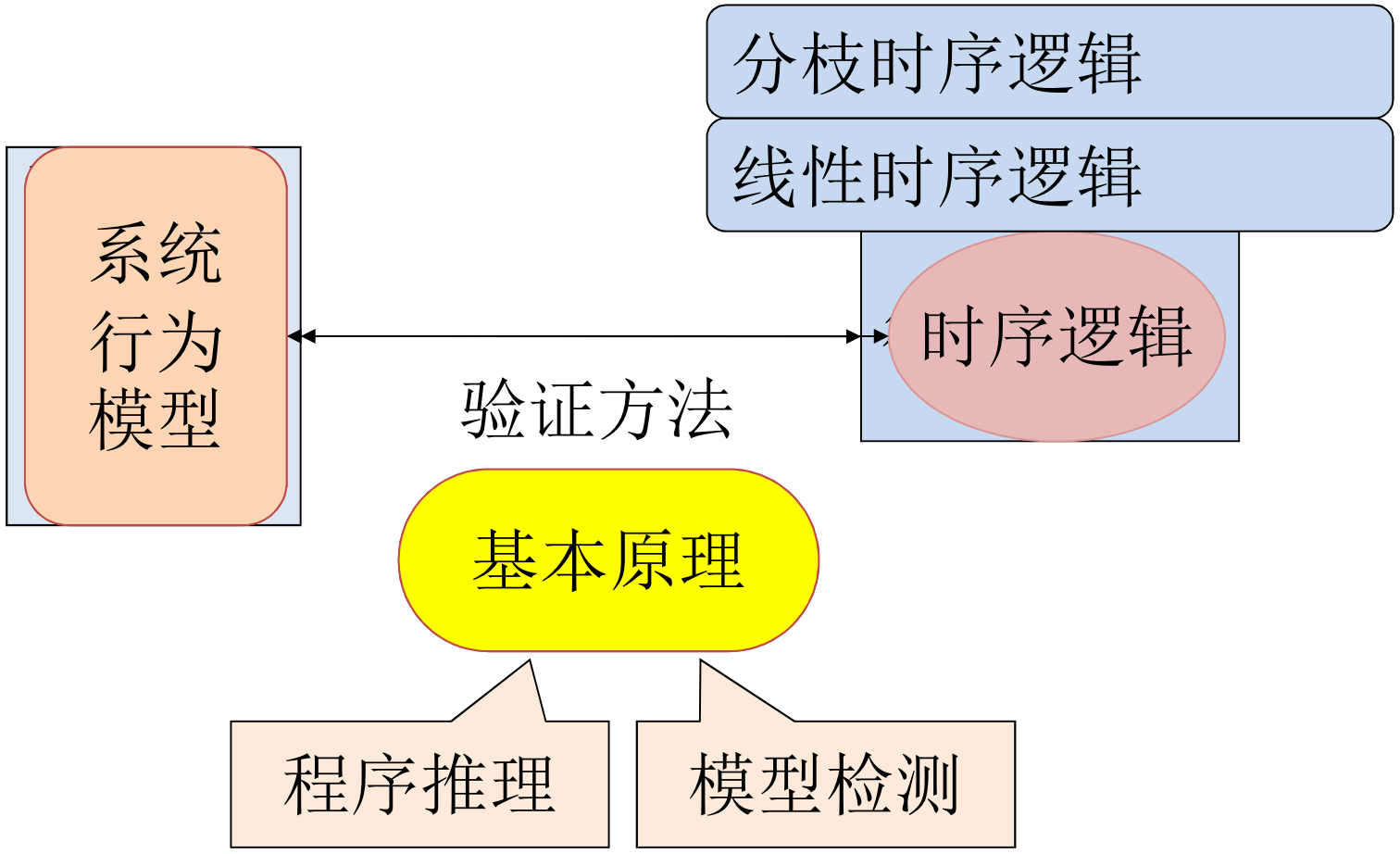
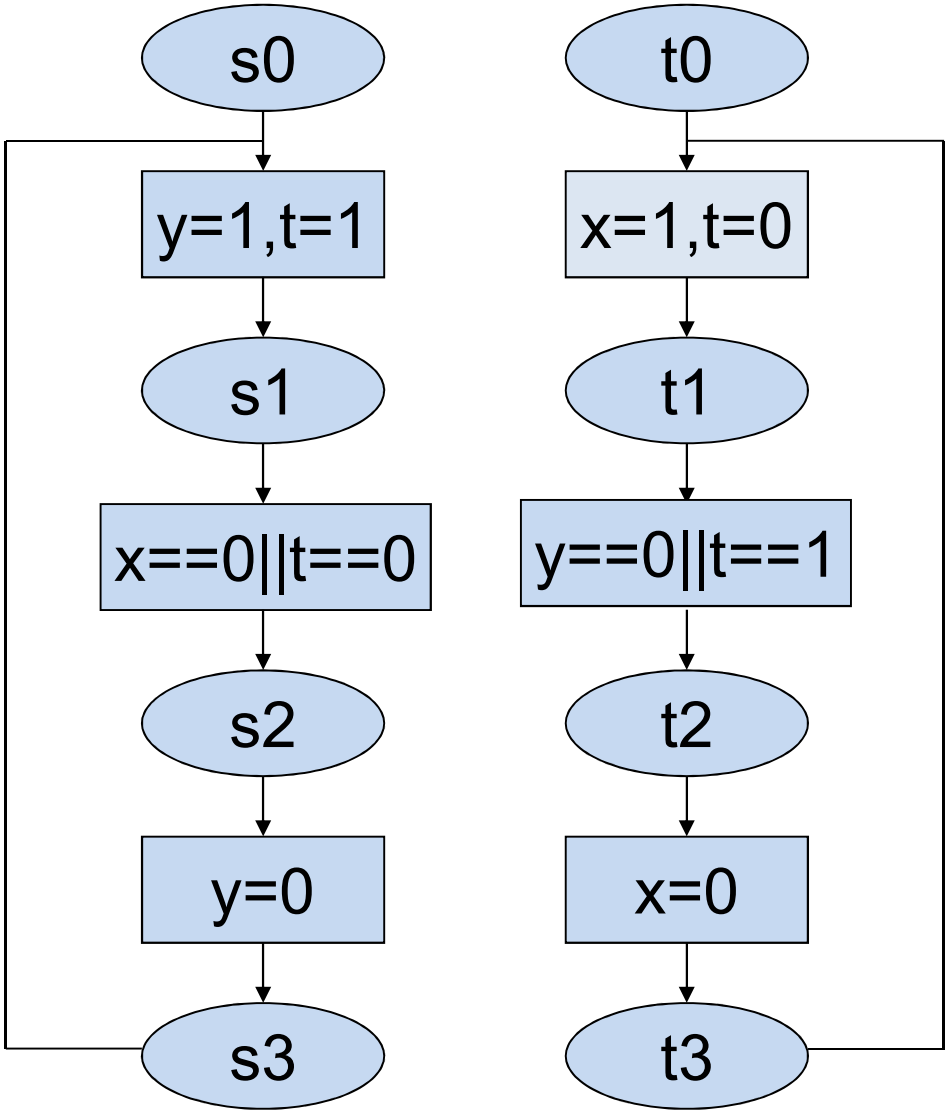
课程内容



课程内容

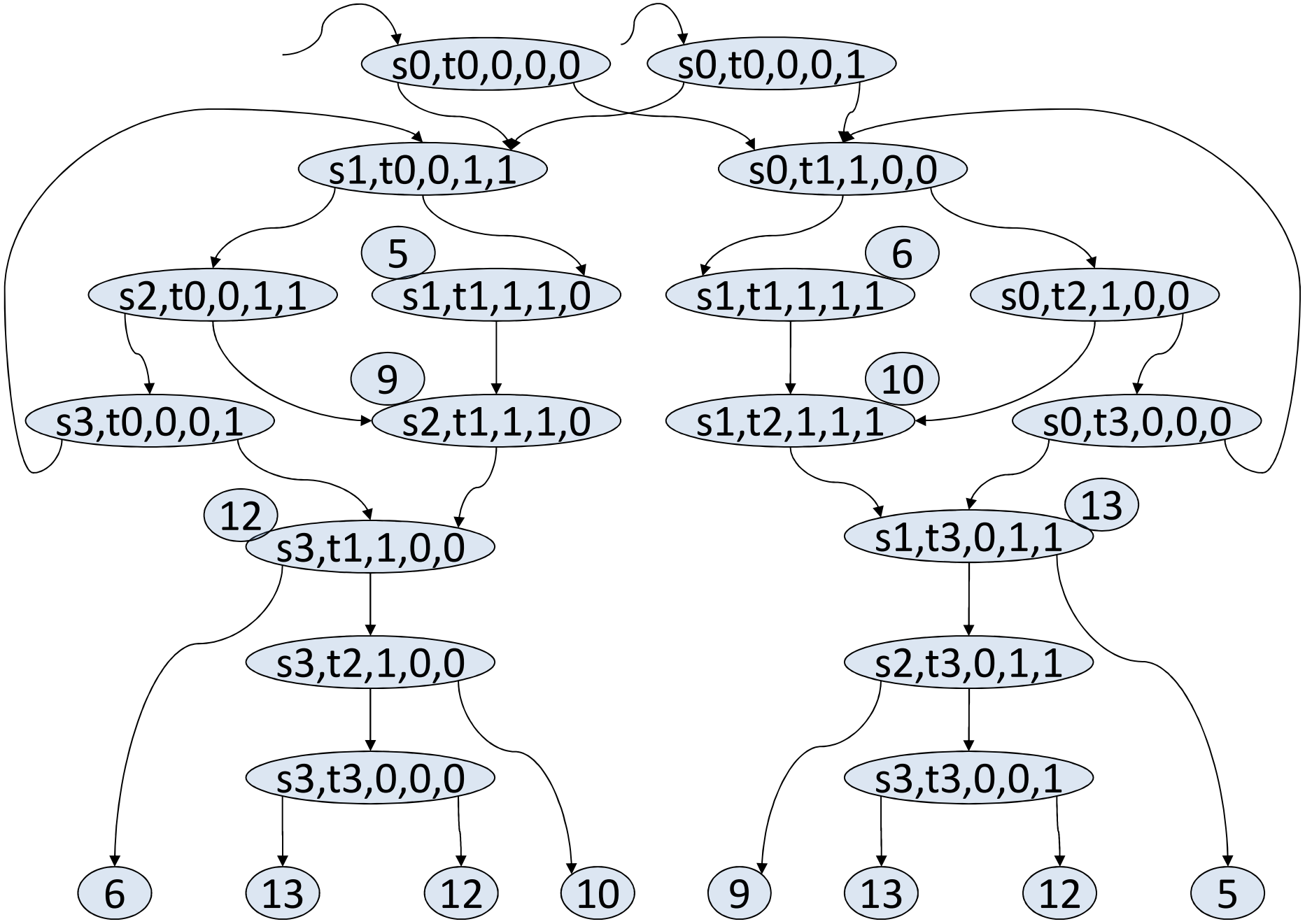


Example



Initial States

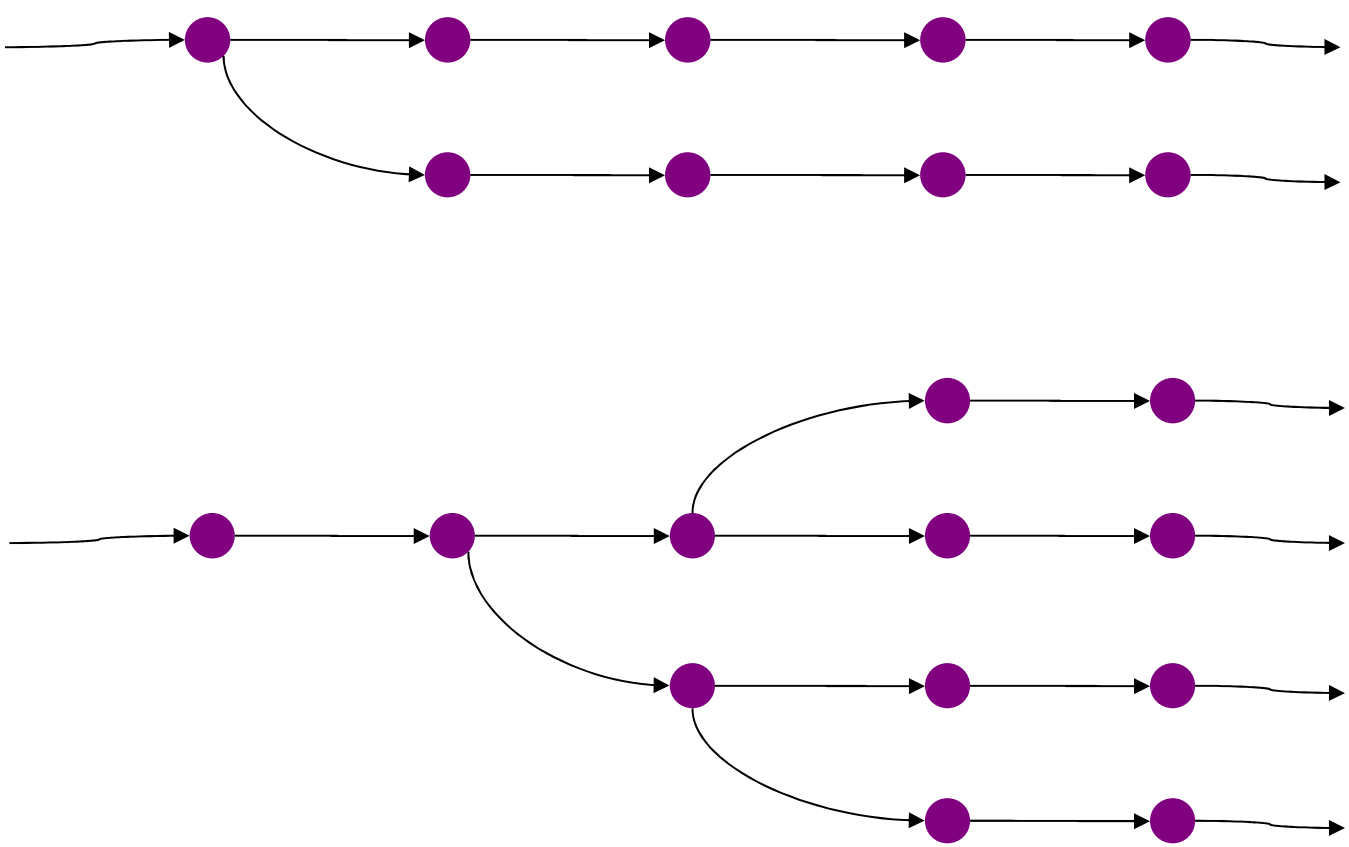
s0
t0
x=0
y=0


s0,t0,0,0,0
s0,t0,0,0,1
s1,t0,0,1,1
s0,t1,1,0,0
5
6
s2,t0,0,1,1
s1,t1,1,1,0
s1,t1,1,1,1
s0,t2,1,0,0
9
10
s3,t0,0,0,1
s2,t1,1,1,0
s1,t2,1,1,1
s0,t3,0,0,0
12
13
s3,t1,1,0,0
s1,t3,0,1,1
s3,t2,1,0,0
s2,t3,0,1,1
s3,t3,0,0,0
s3,t3,0,0,1
6
13
12
10
9
13
12
5

Examples of Properties

- Safety
- Inevitability
- Response
- Immediate Response
- Priority
- First Come – First Served
- Globally Exists a Branch such that F(a=s2)

Tree Structures

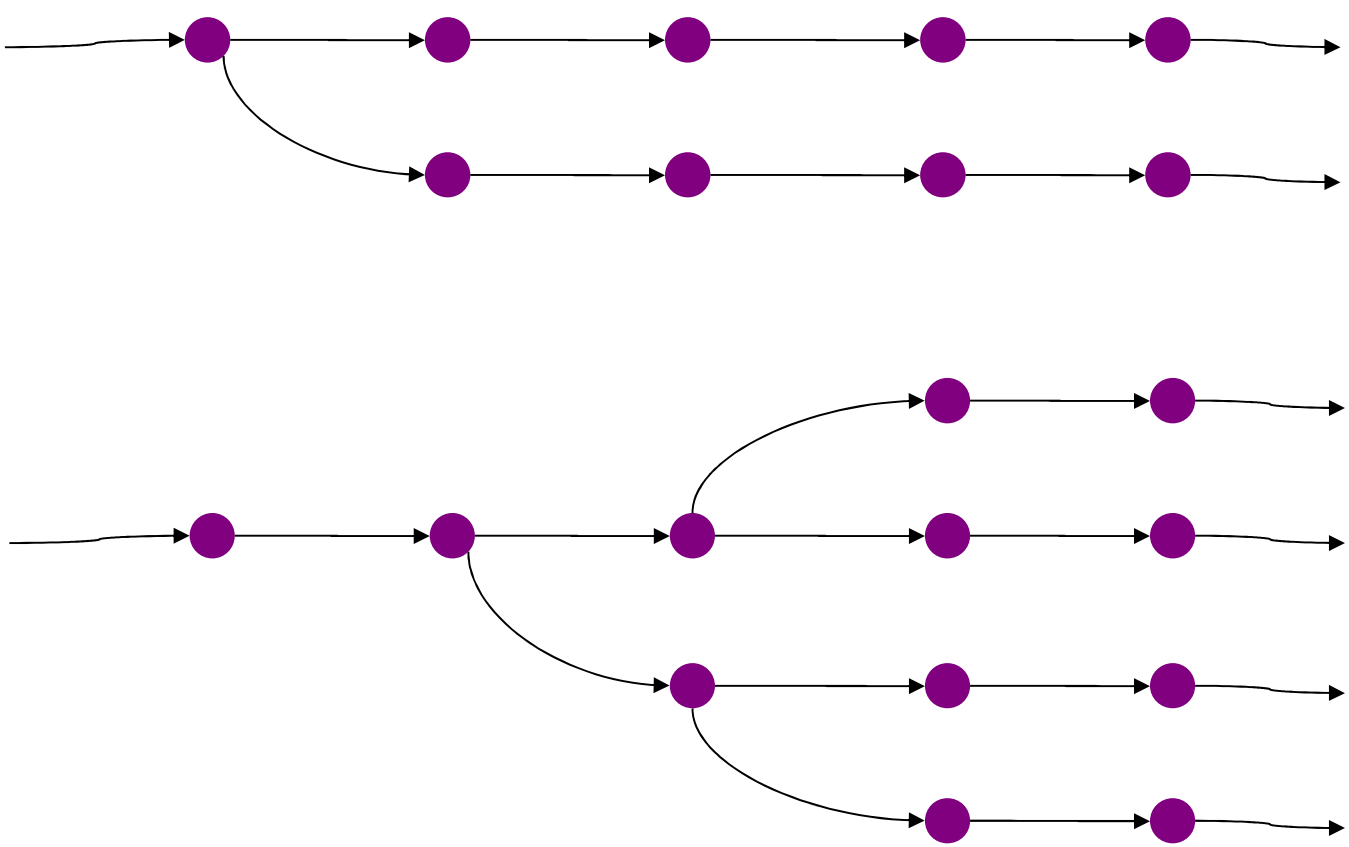


Contents

- Computation Tree Logic (CTL)
- Bounded Semantics of CTL
- Fixpoint Representation of CTL Formulas
- μ-Calculus

(I) Computation Tree Logic (CTL)

Tree Structures



Path Quantifiers: A, E

Examples of Properties

- Safety: AG (!(a=s2$\wedge$b=t2))
- Inevitability AF (a=s2$\vee$b=t2)
- Response AG (a=s1 → AF (a=s2))
- Imm. Response AG (a=s1 → AX (a=s2))
- Priority AG (a=s1$\wedge$b!=t1$\wedge$b!=t2 → A(a=s2 R b!=t2))
- FCFS AG (a=s1$\wedge$b!=t1$\wedge$b!=t2 → A(b!=t2 U a=s2))
- GE.. AG (EF(a=s2))

Syntax of CTL

Let AP be a set of proposition symbols.

Definition

Let p range over AP.

The set Φ of CTL formulas is defined as follows.

$$\Phi == p \mid \Phi \wedge \Phi \mid \Phi \vee \Phi \mid \neg \Phi \mid$$
$$AX\,\Phi \mid AG\,\Phi \mid AF\,\Phi \mid A(\Phi\ R\ \Phi) \mid A(\Phi\ U\ \Phi) \mid$$
$$EX\,\Phi \mid EG\,\Phi \mid EF\,\Phi \mid E(\Phi\ R\ \Phi) \mid E(\Phi\ U\ \Phi)$$

Examples of Properties

- Safety: AG (!(a=s2$\wedge$b=t2))
- Inevitability AF (a=s2$\vee$b=t2)
- Response AG (a=s1 → AF (a=s2))
- Imm. Response AG (a=s1 → AX (a=s2))
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- FCFS AG (a=s1$\wedge$b!=t1$\wedge$b!=t2 → A(b!=t2 U a=s2))
- GE.. AG (EF(a=s2))

Proposition Symbols

Let AP be the set of proposition symbols {p0,p1,…,p13} with the following meaning:

$p_0 \equiv (x = 0)$	$p_1 \equiv (x = 1)$
$p_2 \equiv (y = 0)$	$p_3 \equiv (y = 1)$
$p_4 \equiv (t = 0)$	$p_5 \equiv (t = 1)$
$p_{6+i} \equiv (a = s_i)$	$p_{10+i} \equiv (b = t_i)$

$$i \in \{0, 1, 2, 3\}$$

Examples of Properties

- Safety: AG ($\neg(p8 \wedge p12)$)
- Inevitability AF ($p8 \vee p12$)
- Response AG ($p7 \rightarrow AF\ p8$)
- Imm. Response AG ($p7 \rightarrow AX\ p8$)
- Priority AG ($p7 \wedge \neg p11 \wedge \neg p12 \rightarrow A(p8\ R\ \neg p12)$)
- FCFS AG ($p7 \wedge \neg p11 \wedge \neg p12 \rightarrow A(\neg p12\ U\ p8)$)
- GE… AG (EF (p8))

Semantics

Kripke Structures

Let K=<S,R,I,L> be a Kripke structure.

- S : A finite set of states
- $R \subseteq S \times S$: A total transition relation
- $I \subseteq S$: A set of initial states
- L: S → 2^{AP} is a labeling function

Semantics: M |= ϕ

A CTL formula may be interpreted on a Kripke structure.

(1) Define M,s |= ϕ for a state s

(2) Define M |= ϕ as follows:

M |= ϕ if M,s |= ϕ for every initial state s.

Semantics: $M,u \models \phi$

Definition

$M,u \models p$, if $p \in AP$ and $p \in L(u)$

$M,u \models \neg\phi$, if $M,u \not\models \phi$

$M,u \models \phi \vee \psi$, if $M,u \models \phi$ or $M,u \models \psi$

$M,u \models \phi \wedge \psi$, if $M,u \models \phi$ and $M,u \models \psi$

$M,u \models A\ \psi$, if for every path π of u, $(M,\pi \models \psi)$

$M,u \models E\ \psi$, if there is a path π of u, $(M,\pi \models \psi)$

Semantics: $M,\pi \models \psi$

Definition

$M,\pi \models X\,\phi$, if $M,\pi_1 \models \phi$

$M,\pi \models F\,\phi$, if $\exists i \geq 0, (M,\pi_i \models \phi)$

$M,\pi \models G\,\phi$, if $\forall i \geq 0, (M,\pi_i \models \phi)$

$M,\pi \models \phi U \psi$, if $\exists i \geq 0, M,\pi_i \models \psi$ and $\forall 0 \leq j < i, M,\pi_j \models \phi$

$M,\pi \models \phi R \psi$, if $\forall i \geq 0, (\forall 0 \leq j < i, M,\pi_j \not\models \phi) \rightarrow M,\pi_i \models \psi$

Satisfiability and Validity

Satisfiability and Validity

Definition

A formula ϕ is satisfiable, if there is a model M such that $M \models \phi$.

Definition

A formula ϕ is valid, if for every model M, $M \models \phi$ holds.

Satisfiability and Validity Checking

The complexities of CTL satisfiability and validity checking are EXPTIME-complete.

Equivalences

Equivalences

Definition

A formula ϕ is equivalent to a formula ψ, if for every model M, (M $\models \phi$ iff M $\models \psi$).

Examples of Equivalences

$$EX\,\phi \equiv \neg AX \neg \phi$$

$$E(\phi\ R\ \psi) \equiv E(\psi\ U\ (\phi \wedge \psi)) \vee EG\ \psi$$

These equivalences can be proved by applying the semantics.

Dual Operators and the Negation Normal Form (NNF)

Dual Operators

$$EX\,\phi \equiv \neg AX\neg\,\phi$$

$$EG\,\phi \equiv \neg AF\neg\,\phi$$

$$E(\phi\ R\ \psi) \equiv \neg A(\neg\phi\ U\ \neg\psi)$$

$$EF\,\phi \equiv \neg AG\neg\,\phi$$

$$E(\phi\ U\ \psi) \equiv \neg A(\neg\phi\ R\ \neg\psi)$$

NNF

Definition

A formula is in NNF, if the negation symbol is only applied to atomic formulas.

Every formula is equivalent to a formula in NNF.

Recursive Equations

Operators R and U

$$E(\phi \; R \; \psi) \equiv \psi \wedge (\phi \vee EX \; E(\phi \; R \; \psi))$$

$$E(\phi \; U \; \psi) \equiv \psi \vee (\phi \wedge EX \; E(\phi \; U \; \psi))$$

$$A(\phi \; R \; \psi) \equiv \psi \wedge (\phi \vee AX \; A(\phi \; R \; \psi))$$

$$A(\phi \; U \; \psi) \equiv \psi \vee (\phi \wedge AX \; A(\phi \; U \; \psi))$$

Operators G and F

Let $p_0 \in$ AP be given.

Let $\perp$ denote $(p_0 \wedge \neg p_0)$

Then

$EG\ \psi \equiv E(\perp R\ \psi)$

$EF\ \psi \equiv E(\neg\perp U\ \psi)$

$AG\ \psi \equiv A(\perp R\ \psi)$

$AF\ \psi \equiv A(\neg\perp U\ \psi)$

Minimal Complete Set of Temporal Operators

Minimal Complete Set

EX,EF,EG,ER,EU

ER is expressible by EU and EG

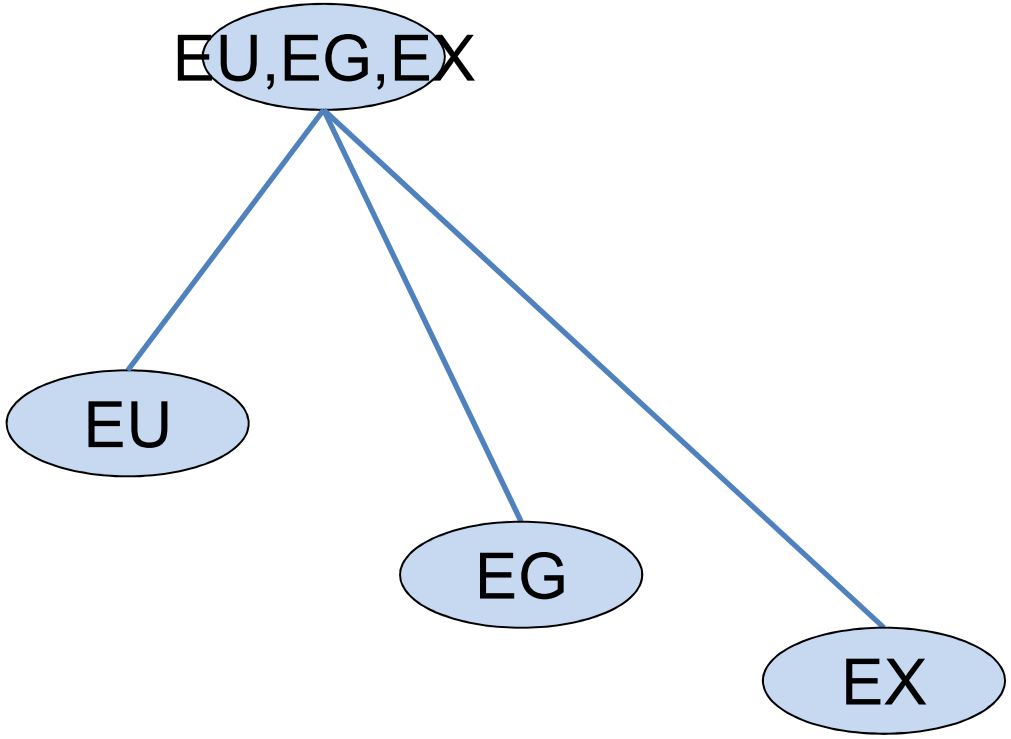
EF is expressible by EU

Then {EX,EU,EG} is a complete set.

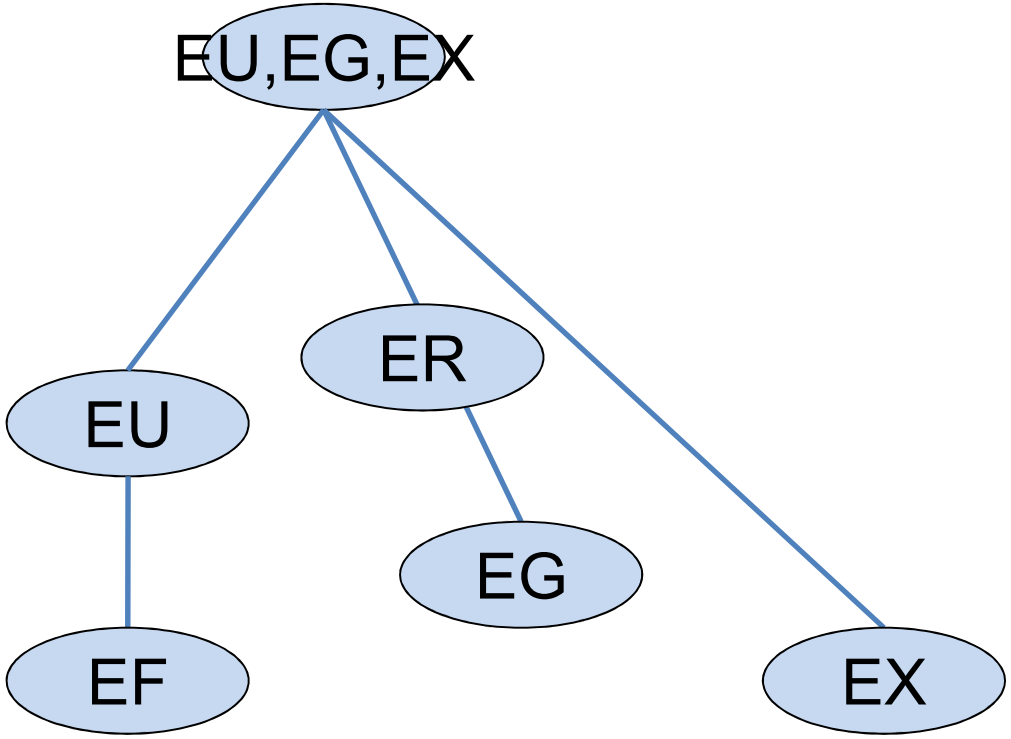
{EX,EU,EG} is also a minimal complete set.

Expressiveness

Expressiveness of Subsets of CTL



Expressivity of Subsets of CTL



CTL Proof System

Proof System for {EX,EU,EG,EF,AX,AU,AG,AF}

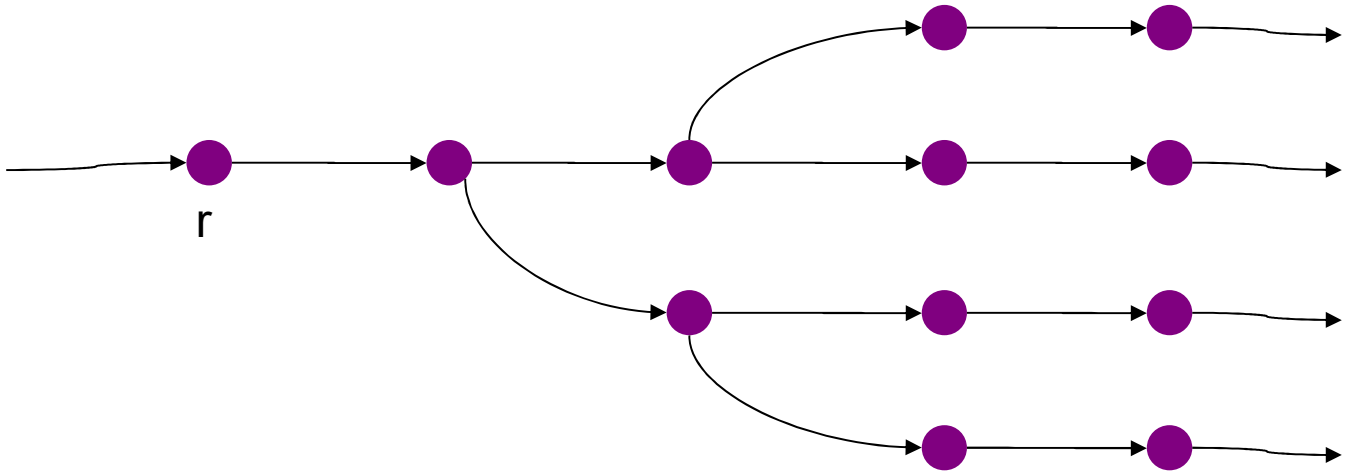
- Axioms for temporal logics formulas
- Proof rules for temporal formulas
- Propositional proof system

Axioms for Temporal Logics Formulas

- $EFp \leftrightarrow E(\neg\bot\ U\ p)$
- $AGp \leftrightarrow \neg EF\neg p$
- $AFp \leftrightarrow A(\neg\bot\ U\ p)$
- $EGp \leftrightarrow \neg AF\ \neg p$
- $EX(p \vee q) \leftrightarrow EXp \vee EXq$
- $AXp \leftrightarrow \neg EX\ \neg p$
- $E(p\ U\ q) \leftrightarrow (q \vee(p \wedge EXE(p\ U\ q))$
- $A(p\ U\ q) \leftrightarrow (q \vee (p \wedge AXA(p\ U\ q))$
- $EX\neg\bot \wedge AX\neg\bot$

Axioms for Temporal Logics Formulas

- $AG(r \rightarrow (\neg q \wedge EXr)) \rightarrow (r \rightarrow \neg A(p\ U\ q))$
- $AG(r \rightarrow (\neg q \wedge EXr)) \rightarrow (r \rightarrow \neg AFq)$
- $AG(r \rightarrow (\neg q \wedge (p \rightarrow AXr))) \rightarrow (r \rightarrow \neg E(p\ U\ q))$
- $AG(r \rightarrow (\neg q \wedge AXr)) \rightarrow (r \rightarrow \neg EFq)$
- $AG(p \rightarrow q) \rightarrow (EXp \rightarrow EXq)$



Proof Rule (Generalization)

$$\frac{\vdash p}{\vdash AGp}$$

Propositional proof system

- Axioms: all tautologies are axioms
- Proof Rule (MP):

$$\frac{\vdash p \rightarrow q \qquad \vdash p}{\vdash q}$$

Proof System

The proof system is sound and complete.

Applications of CTL

CTL as a Specification Language

System Models: Kripke Structures (K)

System Specifications: Formulas of CTL (φ)

Verification of correctness

Solving the model checking problem: $K \models \varphi$

Model Checking

Model Checking

Definition

Given a model K and a formula ϕ.

The model checking problem is the problem of checking whether K |= ϕ holds.

Model Checking

The complexity of model checking is P-complete.

On CTL*

The set Φ of CTL* formulas is defined as follows.

$$\Phi \;::=\; p \;\mid\; \Phi \wedge \Phi \mid \Phi \vee \Phi \mid \neg \Phi \mid$$
$$X\,\Phi \mid G\,\Phi \mid F\,\Phi \mid (\Phi\ R\ \Phi) \mid (\Phi\ U\ \Phi) \mid$$
$$E\,\Phi \mid A\,\Phi$$

CTL: every path quantifier of {E,A}

must be followed by one of X,G,F,R,U

CTL* is more expressive than CTL

(II) CTL限界语义

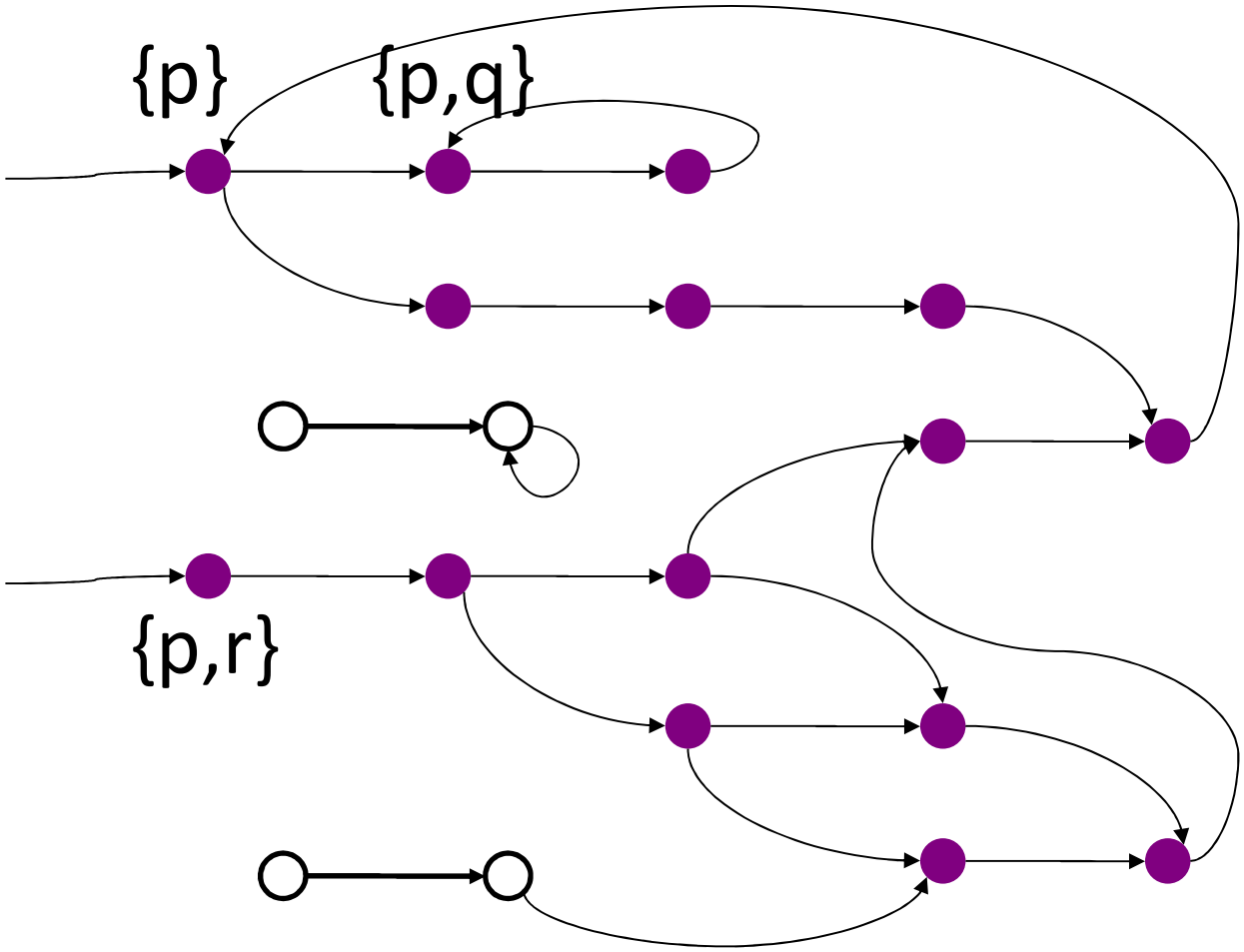
CTL语义

$M \models \varphi$

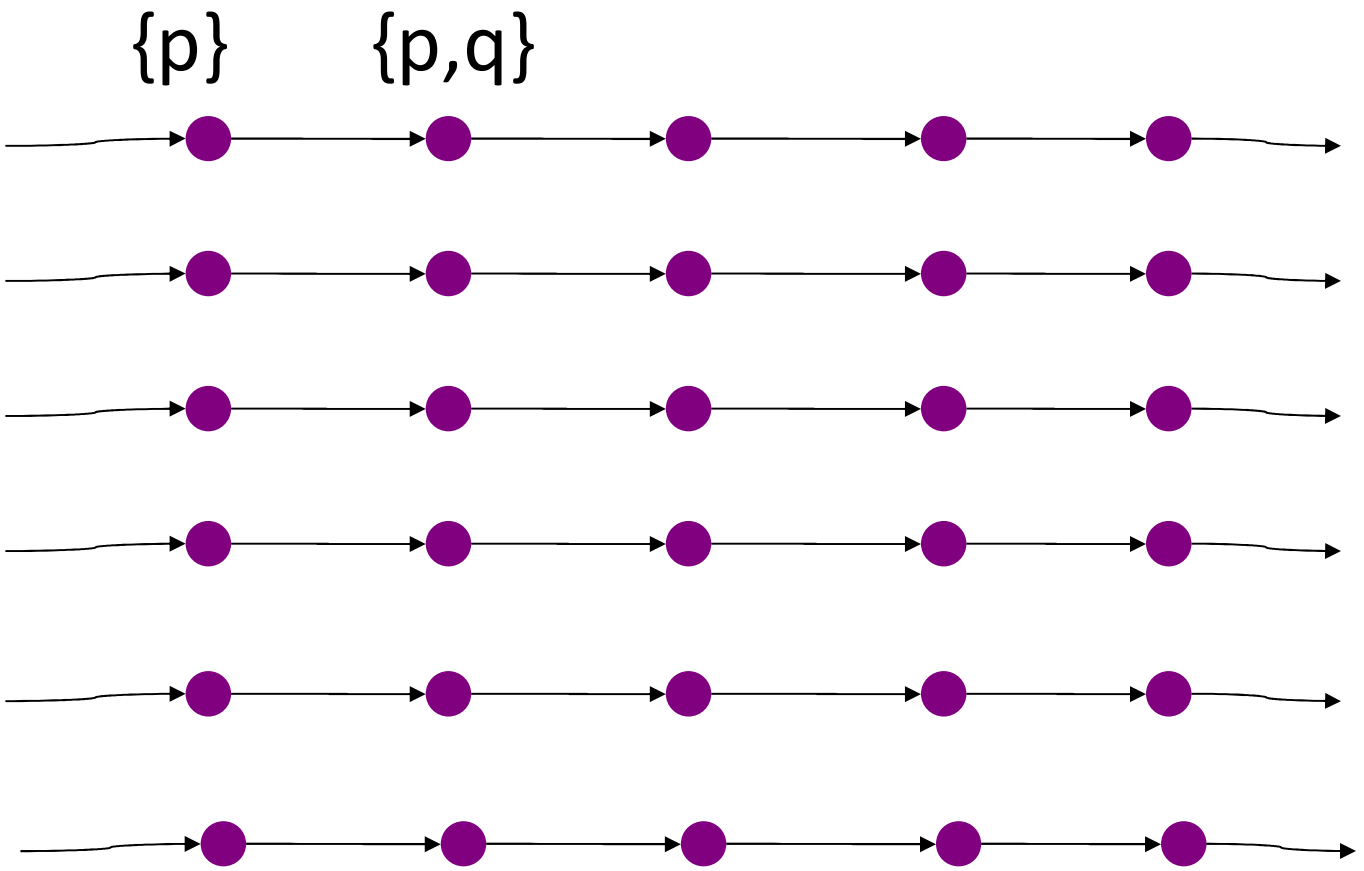
语义上牵涉无穷路径

考虑有穷路径

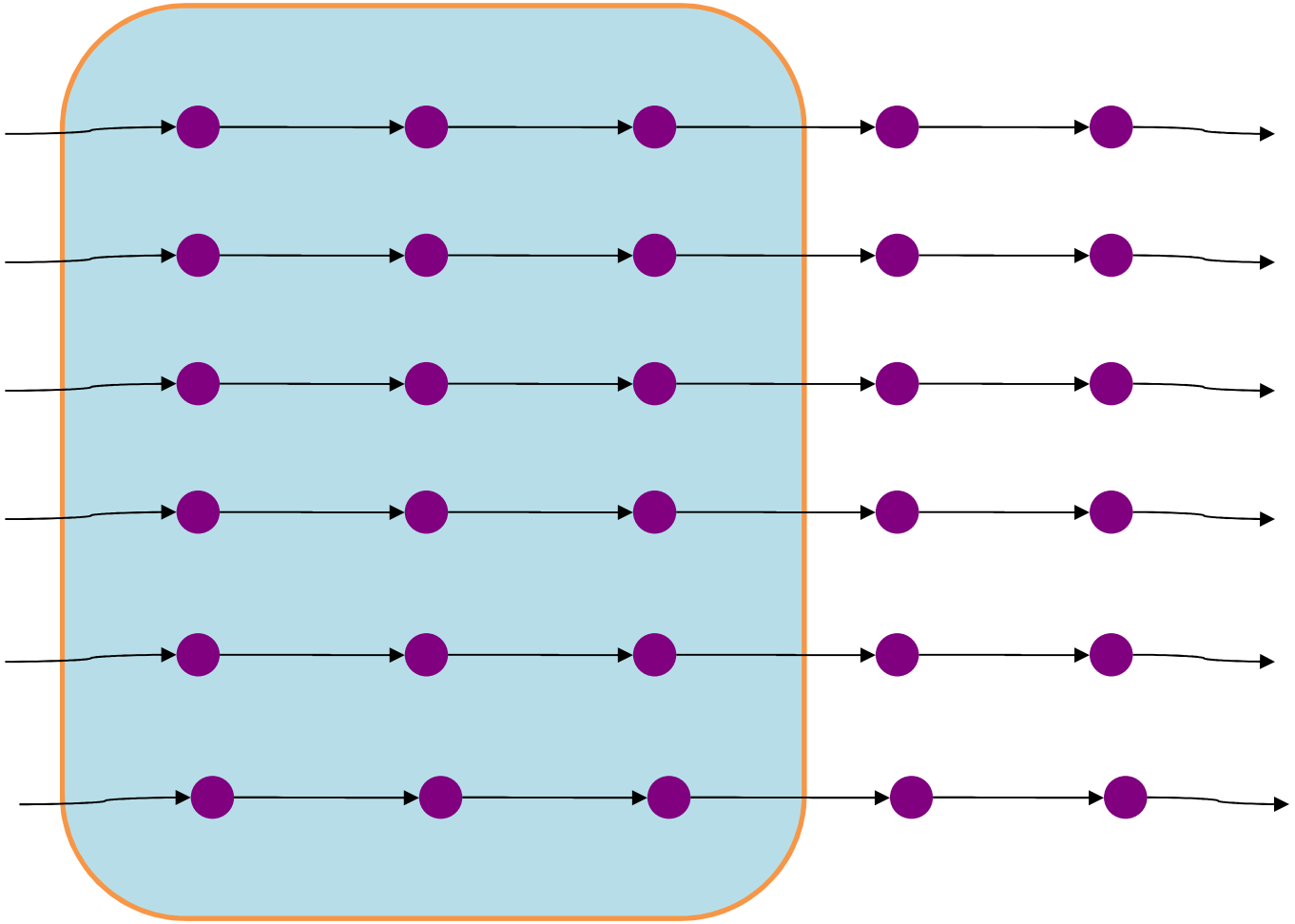
Kripke 结构



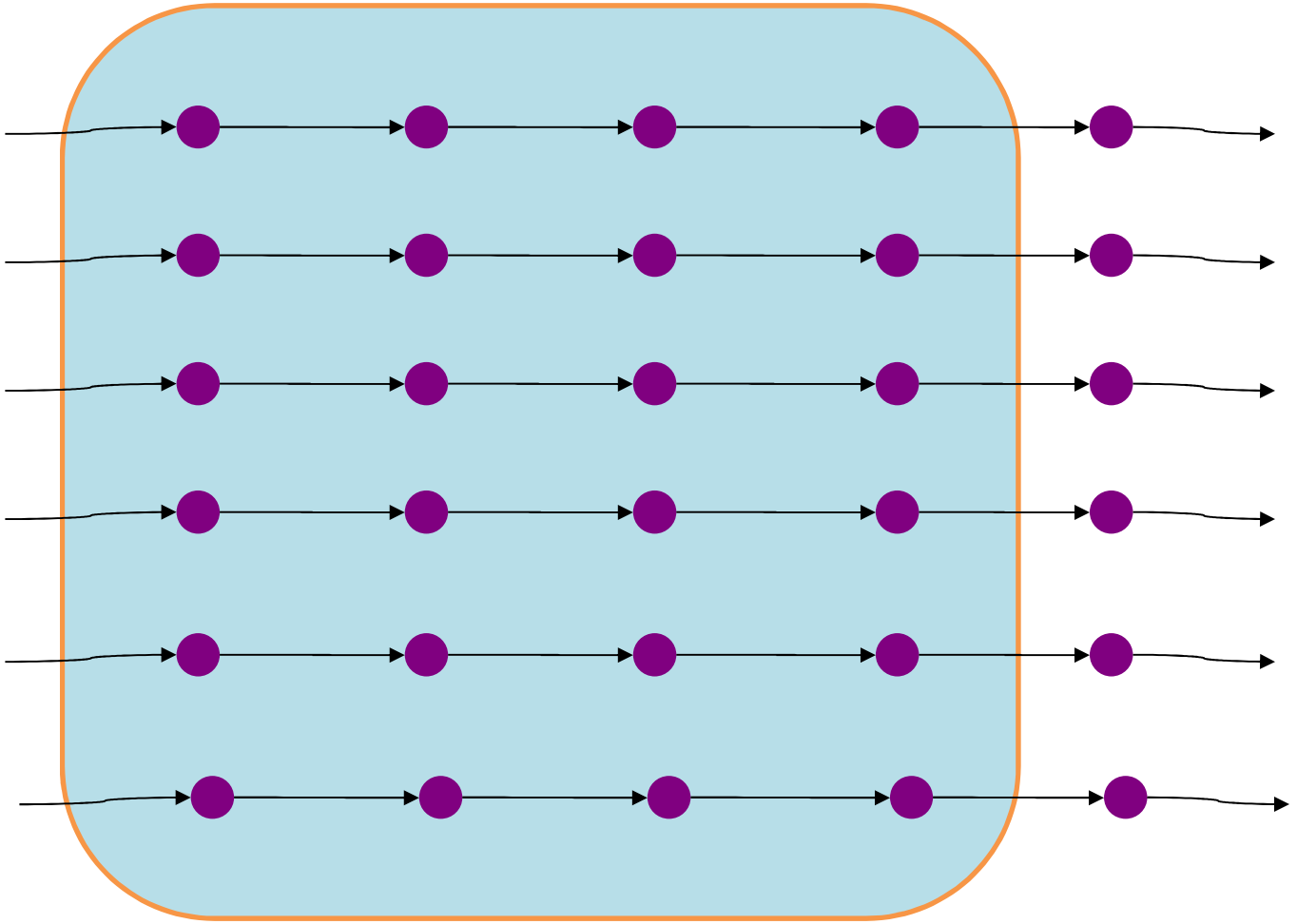
无穷路径



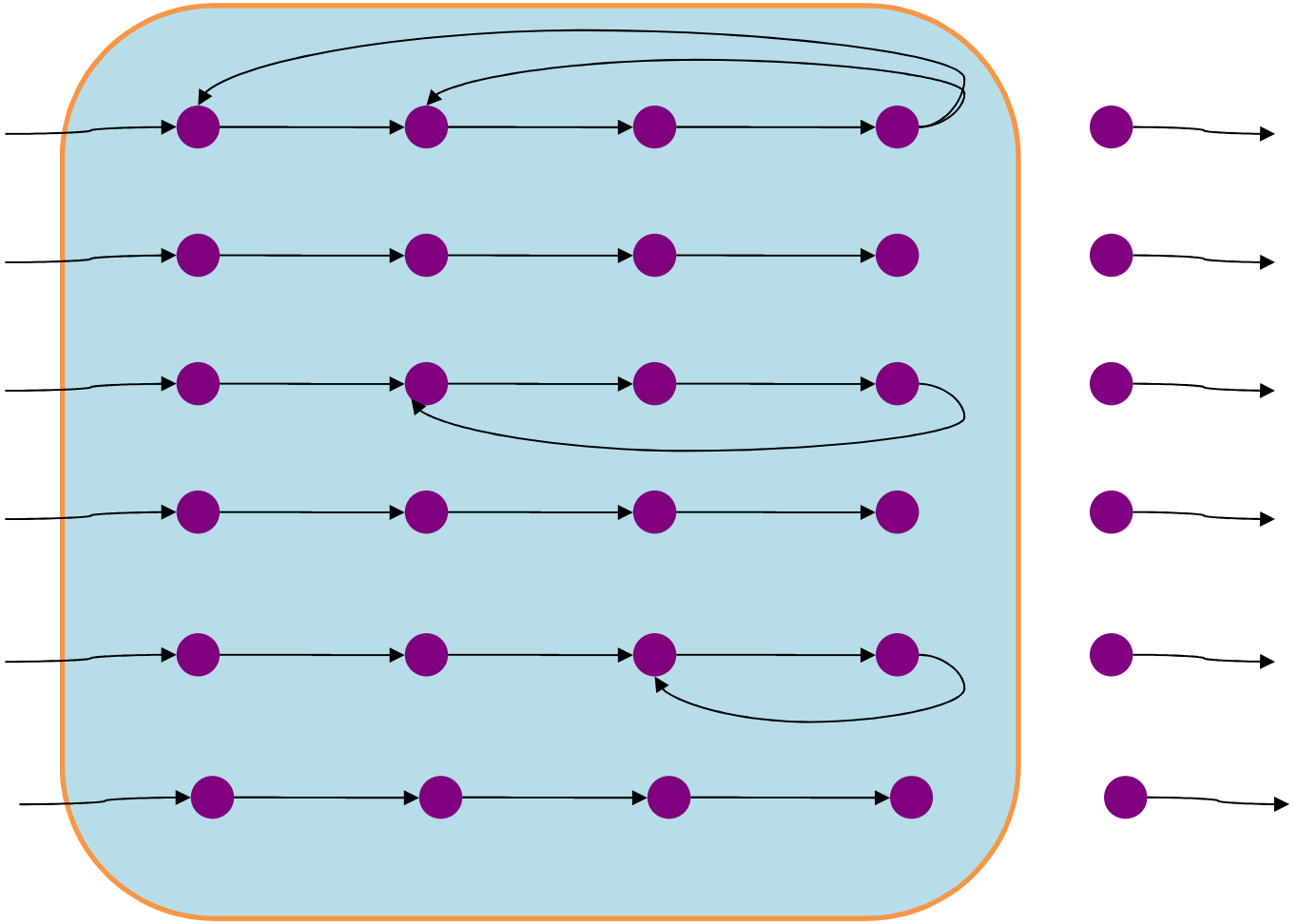
有穷路径



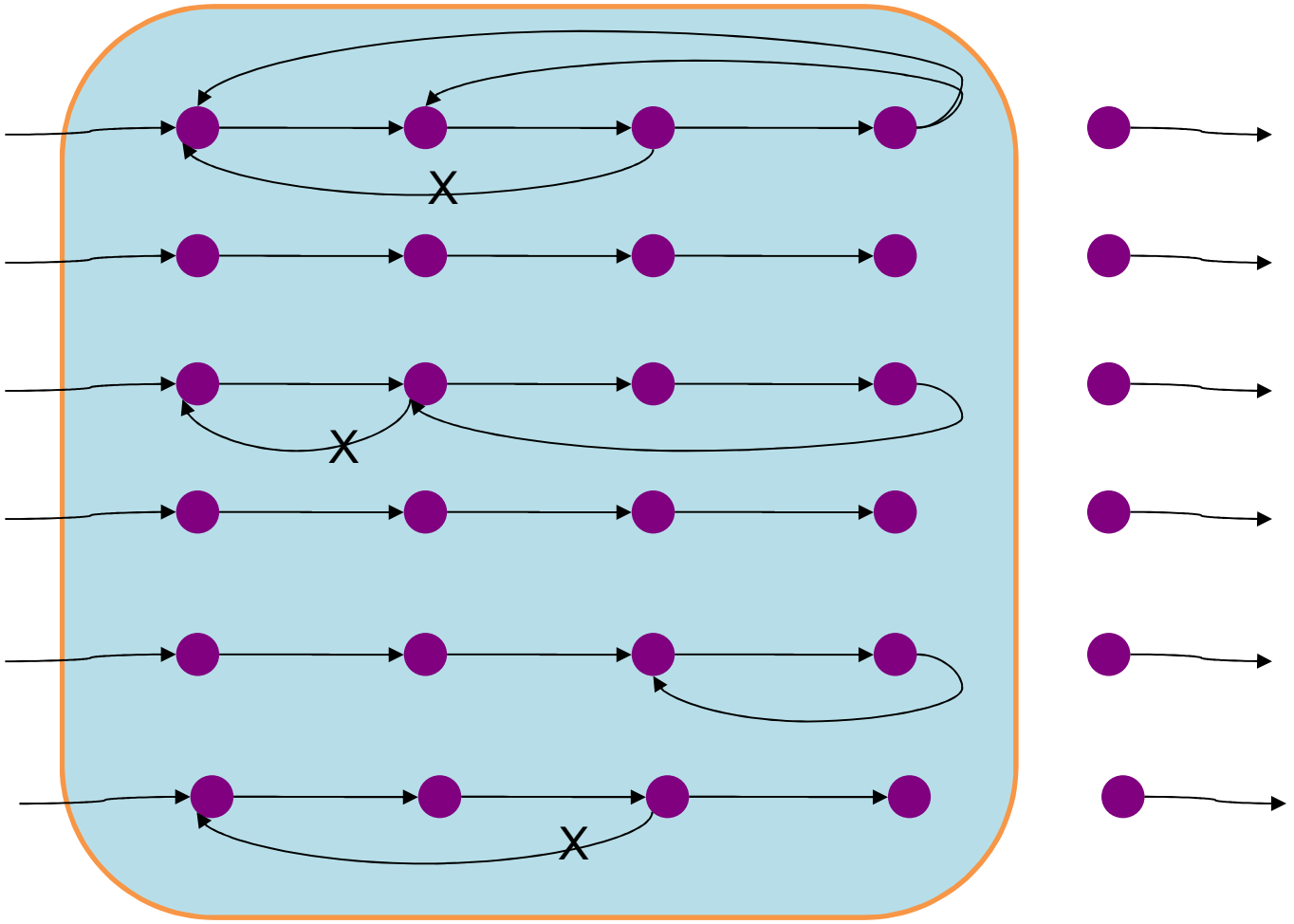
有穷路径



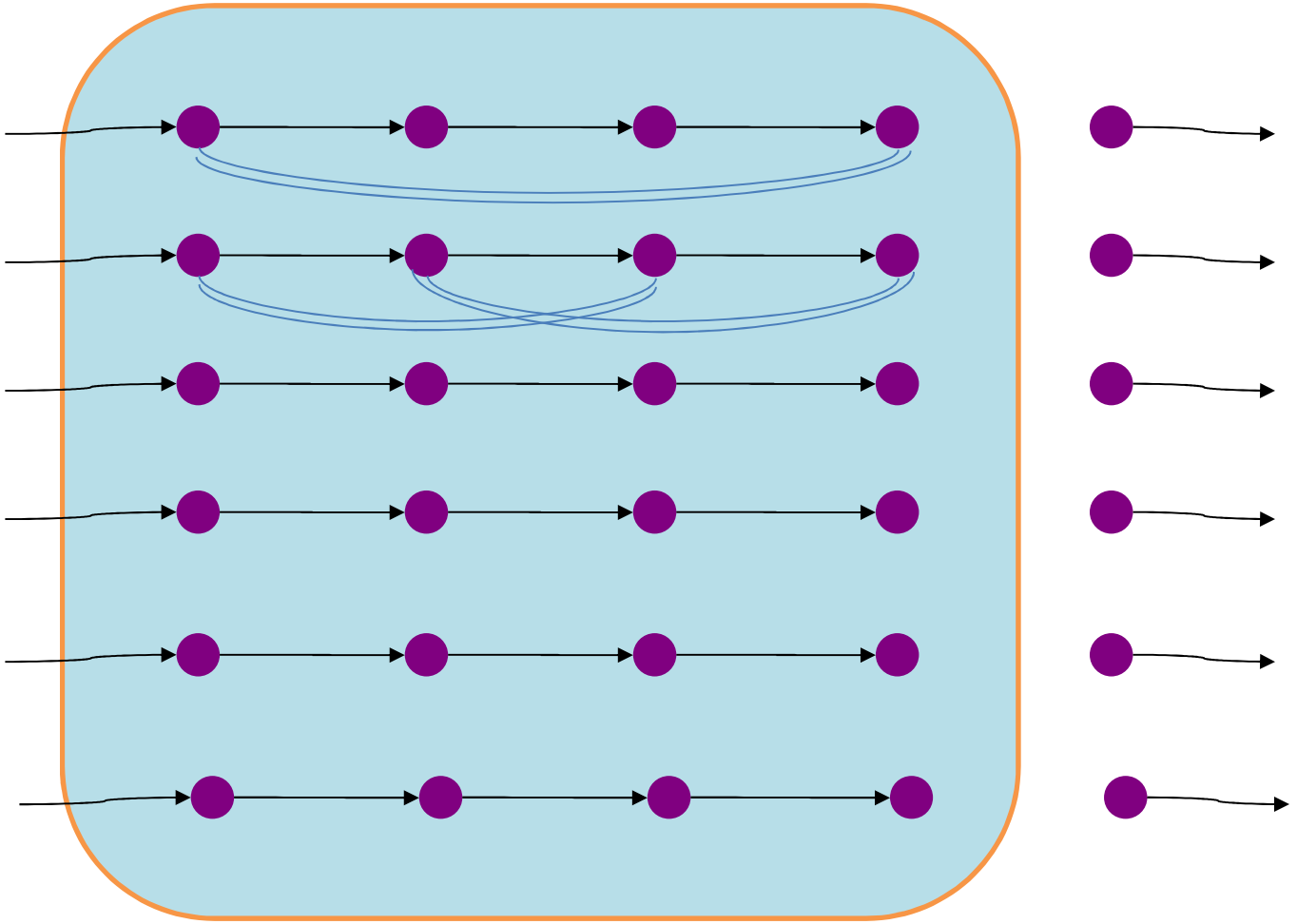
有穷路径(LTL限界语义)



有穷路径(LTL限界语义)



有穷路径



CTL

考虑NNF公式：

Φ == p | $\Phi \wedge \Phi$ | $\Phi \vee \Phi$ | $\neg$p |

AX Φ | A(Φ R Φ) | A(Φ U Φ) |

EX Φ | E(Φ R Φ) | E(Φ U Φ)

语义：$M,u \models \phi$

$M,u \models p$, if $p \in AP$ and $p \in L(u)$

$M,u \models \neg\phi$, if $M,u \not\models \phi$

$M,u \models \phi \vee \psi$, if $M,u \models \phi$ or $M,u \models \psi$

$M,u \models \phi \wedge \psi$, if $M,u \models \phi$ and $M,u \models \psi$

$M,u \models A\ \psi$, if for every path π of u, $(M,\pi \models \psi)$

$M,u \models E\ \psi$, if there is a path π of u, $(M,\pi \models \psi)$

$M,\pi \models X\ \phi$, if $M,\pi_1 \models \phi$

$M,\pi \models \phi U \psi$, if $\exists i \geq 0$, $M,\pi_i \models \psi$ and $\forall j < i$, $M,\pi_j \models \phi$

$M,\pi \models \phi R \psi$, if $\forall i \geq 0$, $(\forall j < i, M,\pi_j \not\models \phi) \rightarrow M,\pi_i \models \psi$

语义：F,G

$M,\pi \models F\psi$, if $\exists i \geq 0$, $M,\pi_i \models \psi$

$M,\pi \models G\psi$, if $\forall i \geq 0$, $M,\pi_i \models \psi$

限界模型

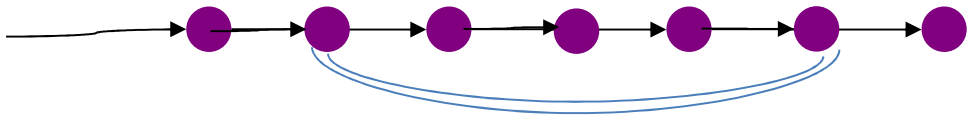
k-path: 长度为k+1的路径

$M = (S,R,I,L)$

$M_k = (S,Ph_k,I,L)$

设 π 为k路径。

rs(π): 路径π中有相同状态



限界语义： $M,u \models_k \phi$

$M,u \models_k p$, if $p \in AP$ and $p \in L(u)$

$M,u \models_k \neg p$, if $p \in AP$ and $p \notin L(u)$

$M,u \models_k \phi \vee \psi$, if $M,u \models_k \phi$ or $M,u \models_k \psi$

$M,u \models_k \phi \wedge \psi$, if $M,u \models_k \phi$ and $M,u \models_k \psi$

$M,u \models_k A\ \psi$, if for every k-path π of u, $(M,\pi \models_k \psi)$

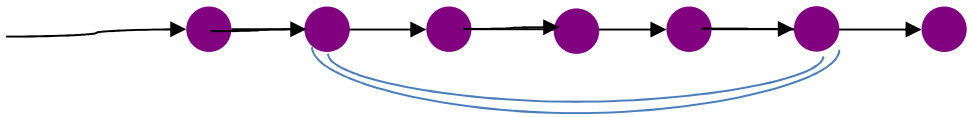
$M,u \models_k E\ \psi$, if there is a k-path π of u, $(M,\pi \models_k \psi)$

$M,\pi \models_k X\ \phi$, if $k \geq 1$ and $M,\pi_1 \models_k \phi$

$M,\pi \models_k \phi U \psi$, if $\exists i \leq k$, $M,\pi_i \models_k \psi$ and $\forall j<i$, $M,\pi_j \models_k \phi$

$M,\pi \models_k \phi R \psi$, if $\forall i \leq k$, $(\forall j<i, M,\pi_j \not\models_k \phi) \rightarrow M,\pi_i \models_k \psi$ and

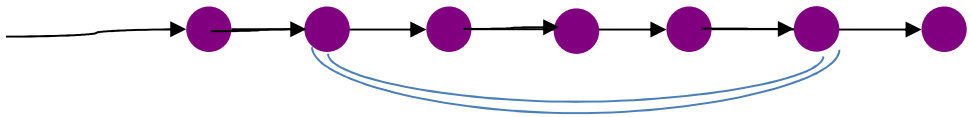
$(\forall i \leq k.(M,\pi_i \not\models_k \phi)) \rightarrow rs(\pi))$



限界语义：F,G

$M,\pi \models_k F\psi$, if $\exists i \leq k$, $M,\pi_i \models_k \psi$

$M,\pi \models_k G\psi$, if $\forall i \leq k$, $(M,\pi_i \models_k \psi)$ and $rs(\pi)$



限界语义： $M \models_k \phi$

Definition

$M \models_k \phi$ if $M,u \models_k \phi$ for every initial state u.

Soundness

For every $i \geq 0$, if $M \models_i \varphi$, then $M \models \varphi$.

Completeness

If $M \models \varphi$, then there is an $i \geq 0$ such that $M \models_i \varphi$.

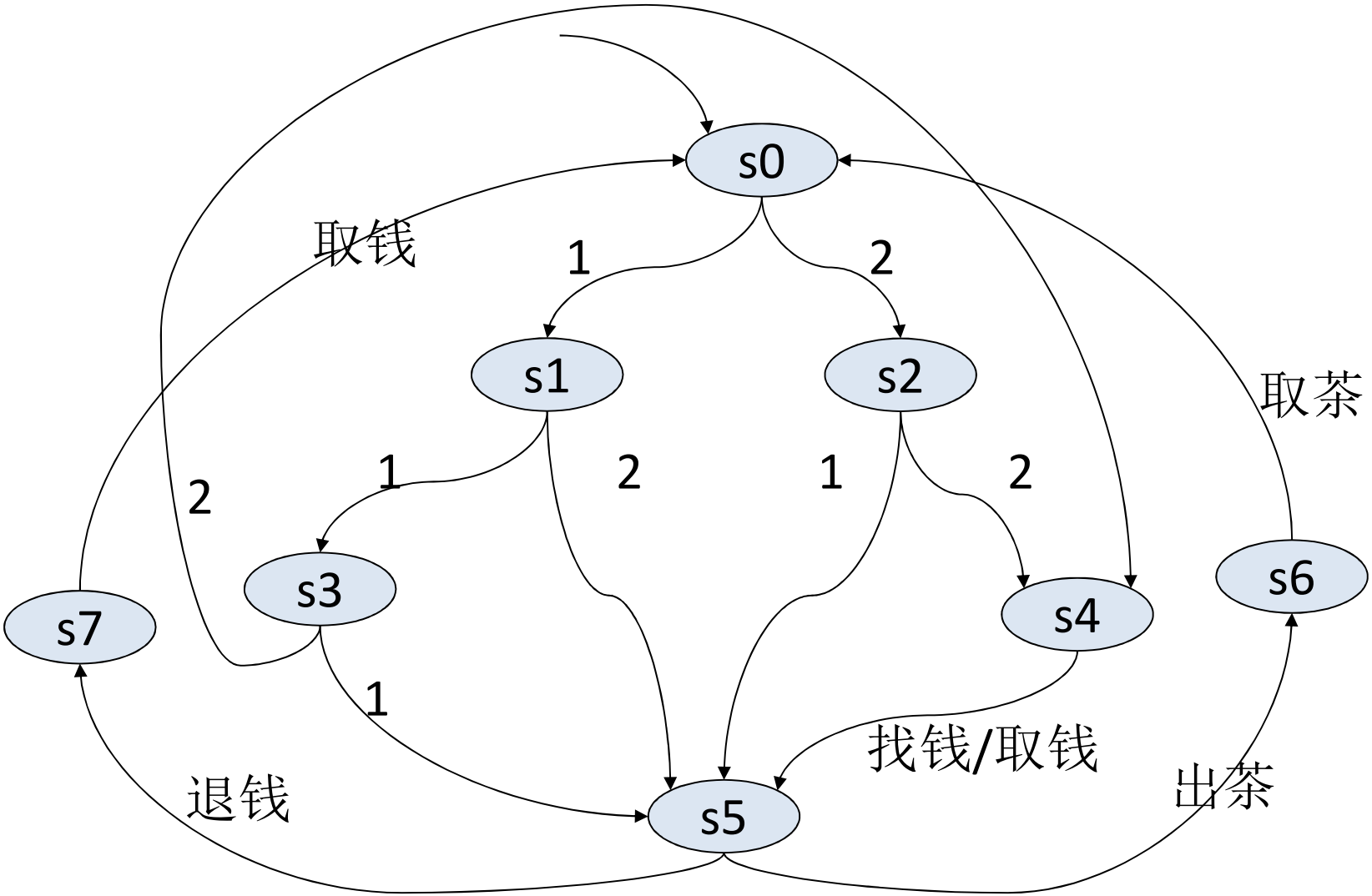
限界正确性检查

(Bounded Correctness Checking)

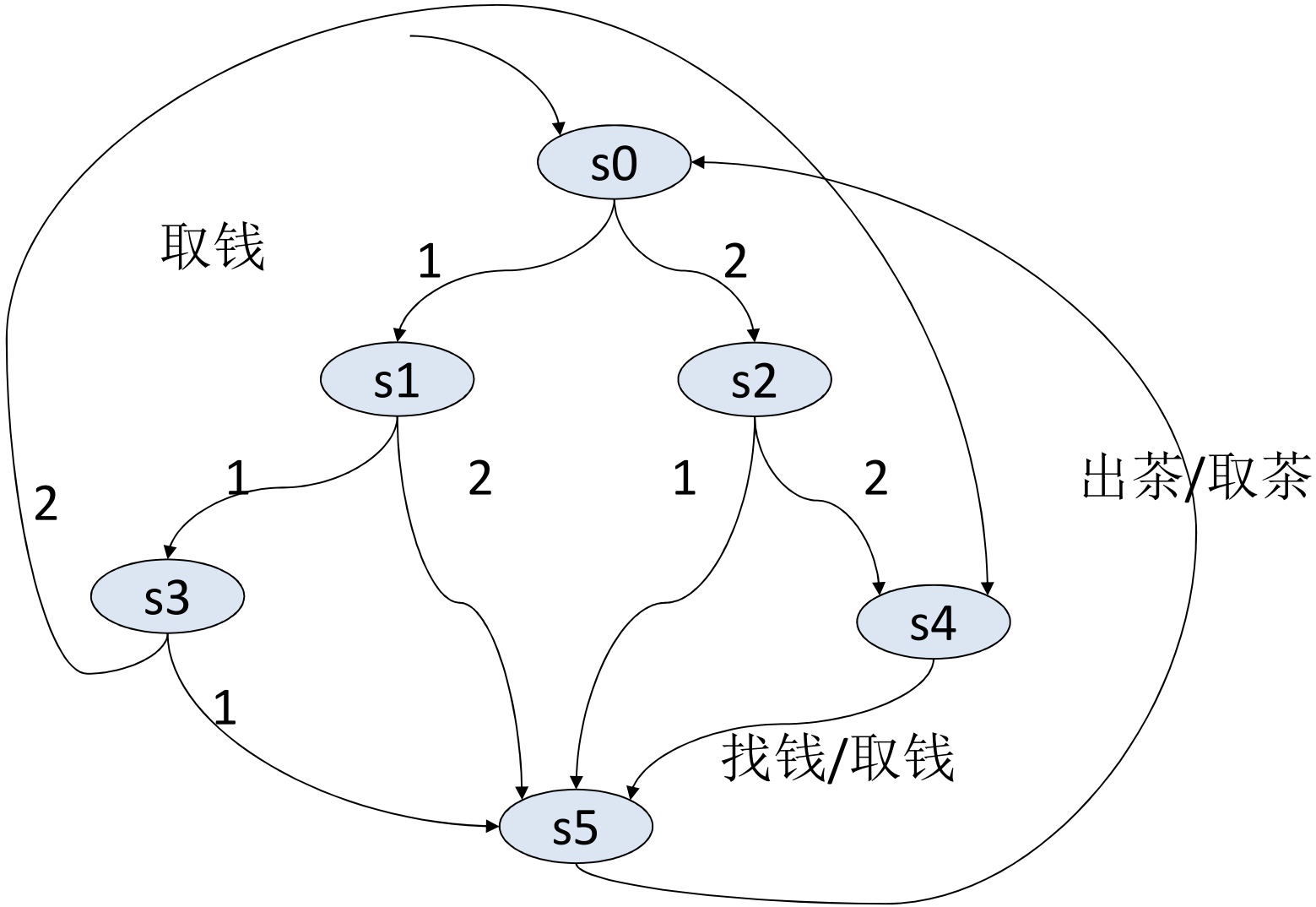
1. k=0;
2. if $M|=_k \varphi$, then report $M|= \varphi$;
3. if $M,s|=_k \neg\varphi$ for some $s\in I$, then report $M|\not= \varphi$;
4. k=k+1; goto step 2;

例子

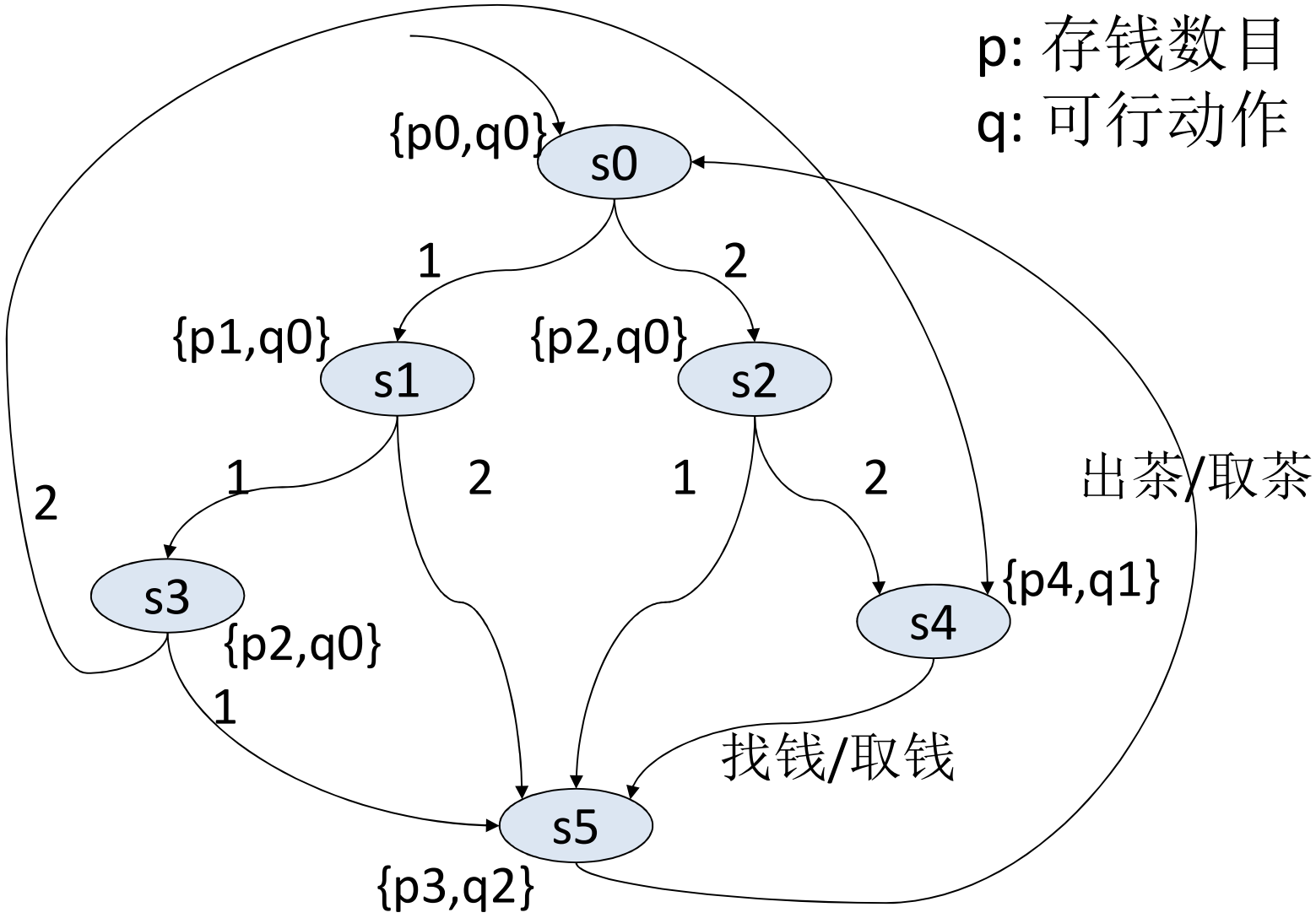
例子：自动售茶机模型



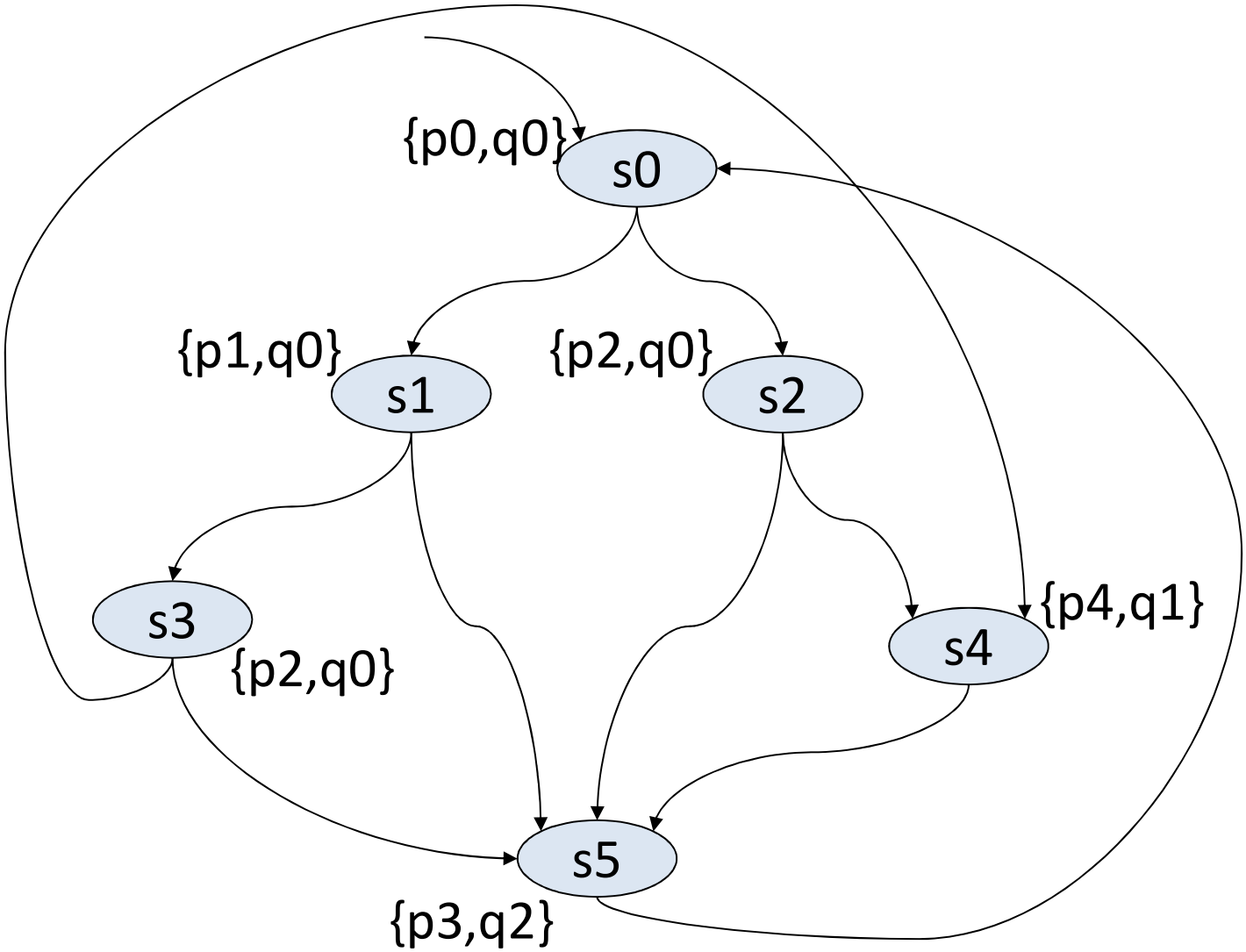
简化自动售茶机模型



简化自动售茶机模型


p: 存钱数目
q: 可行动作
{p0,q0}
s0
1
2
{p1,q0}
{p2,q0}
s1
s2
1
2
1
2
2
出茶/取茶
s3
{p4,q1}
s4
{p2,q0}
1
找钱/取钱
s5
{p3,q2}

自动售茶机模型(标号Kripke结构)



例子

设M为简化自动售茶机模型。

用限界语义验证M是否满足E(q0 U q2)和AG(q0 $\vee$ q2)。

分别给出最小可以确定以上公式是否满足的界。

E(q0 U q2) vs A(¬q0 R ¬q2)

- Ph_0: 我们有
- s0 $M, s0\ s1\ s5 \models_2 (q0\ U\ q2)$
- s1/s2/s3/…
- Ph_1: 因此
- s0 s1; M_2 满足 E(q0 U q2)
- s0 s2; … M 满足 E(q0 U q2)
- Ph_2:
- s0 s1 s3; 由于 M_0 不满足 E(q0 U q2)
- s0 s1 s5; M_1 不满足 E(q0 U q2)
- s0 s2 s4;
- s0 s2 s5; … k=2是最小可确定E(q0 U q2)是否满足的界

AG(q0∨q2) vs EF(¬q0∧¬q2)

- Ph_0: 我们有
- s0 M, s0 s2 s4 $|=_2$ F(¬q0∧¬q2)
- s1/s2/s3/… 因此
- Ph_1: M_2,s0 满足 EF(¬q0∧¬q2)
- s0 s1; M,s0 满足 EF(¬q0∧¬q2)
- s0 s2; … M 不满足 AG(q0∨q2)
- Ph_2:
- s0 s1 s3; 由于 M_0 不满足 EF(¬q0∧¬q2)
- s0 s1 s5; M_1 不满足 EF(¬q0∧¬q2)
- s0 s2 s4;
- s0 s2 s5; … k=2是最小可确定AG(q0∨q2)是否满足的界

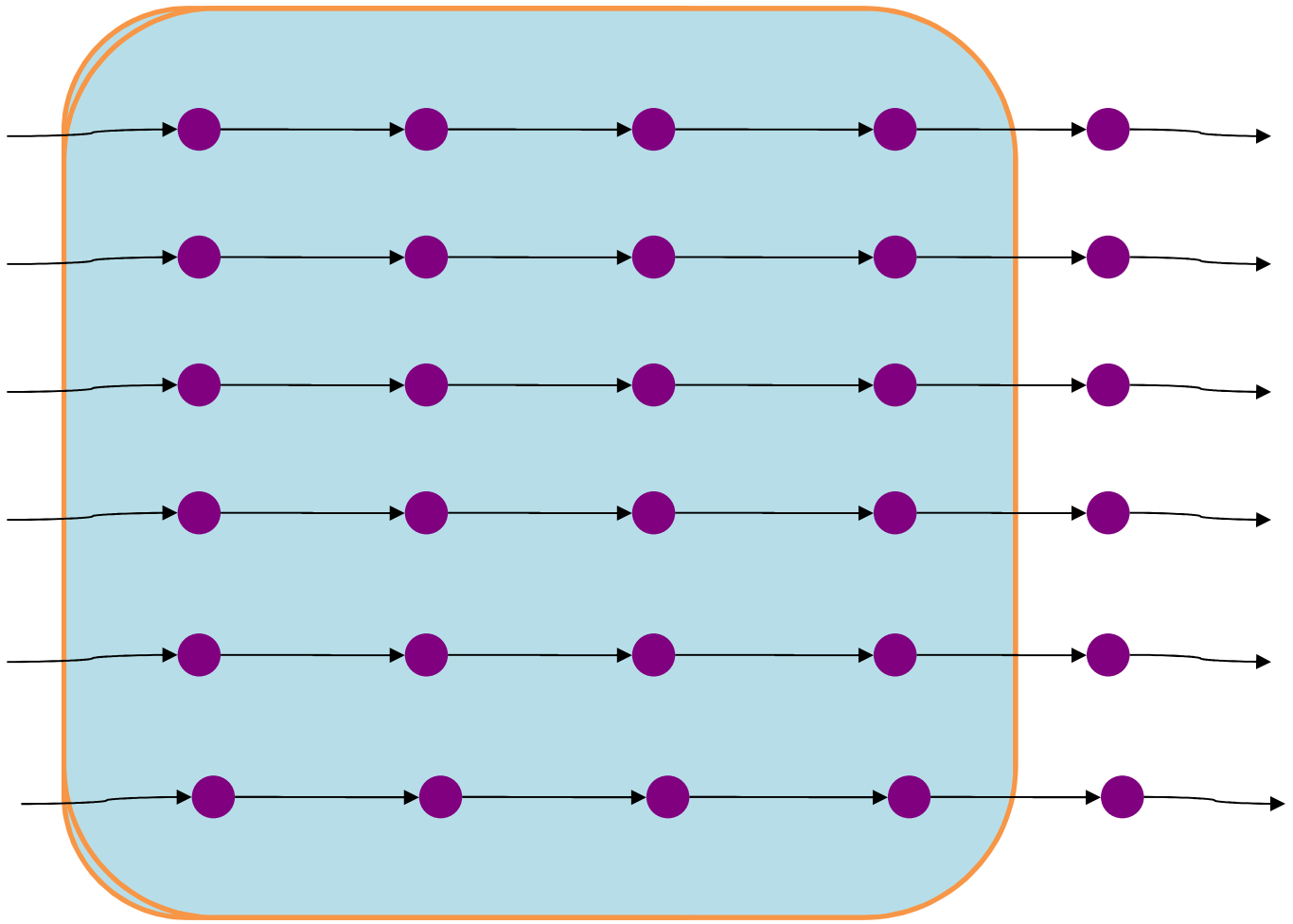
限界模型(s0…)

- Ph_0:
- s0 ;…
- Ph_1:
- s0 s1;
- s0 s2; …
- Ph_2:
- s0 s1 s3;
- s0 s1 s5;
- s0 s2 s4;
- s0 s2 s5; …
- Ph_3:
- s0 s1 s3 s4;
- s0 s1 s3 s5;
- s0 s1 s5 s0;
- s0 s2 s4 s5;
- s0 s2 s5 s0;
- …
- Ph_4:
- s0 s1 s3 s4 s5;
- s0 s1 s3 s5 s0;
- s0 s1 s5 s0 s1;
- s0 s1 s5 s0 s2;
- s0 s2 s4 s5 s0;
- s0 s2 s5 s0 s1;
- s0 s2 s5 s0 s2;
- …

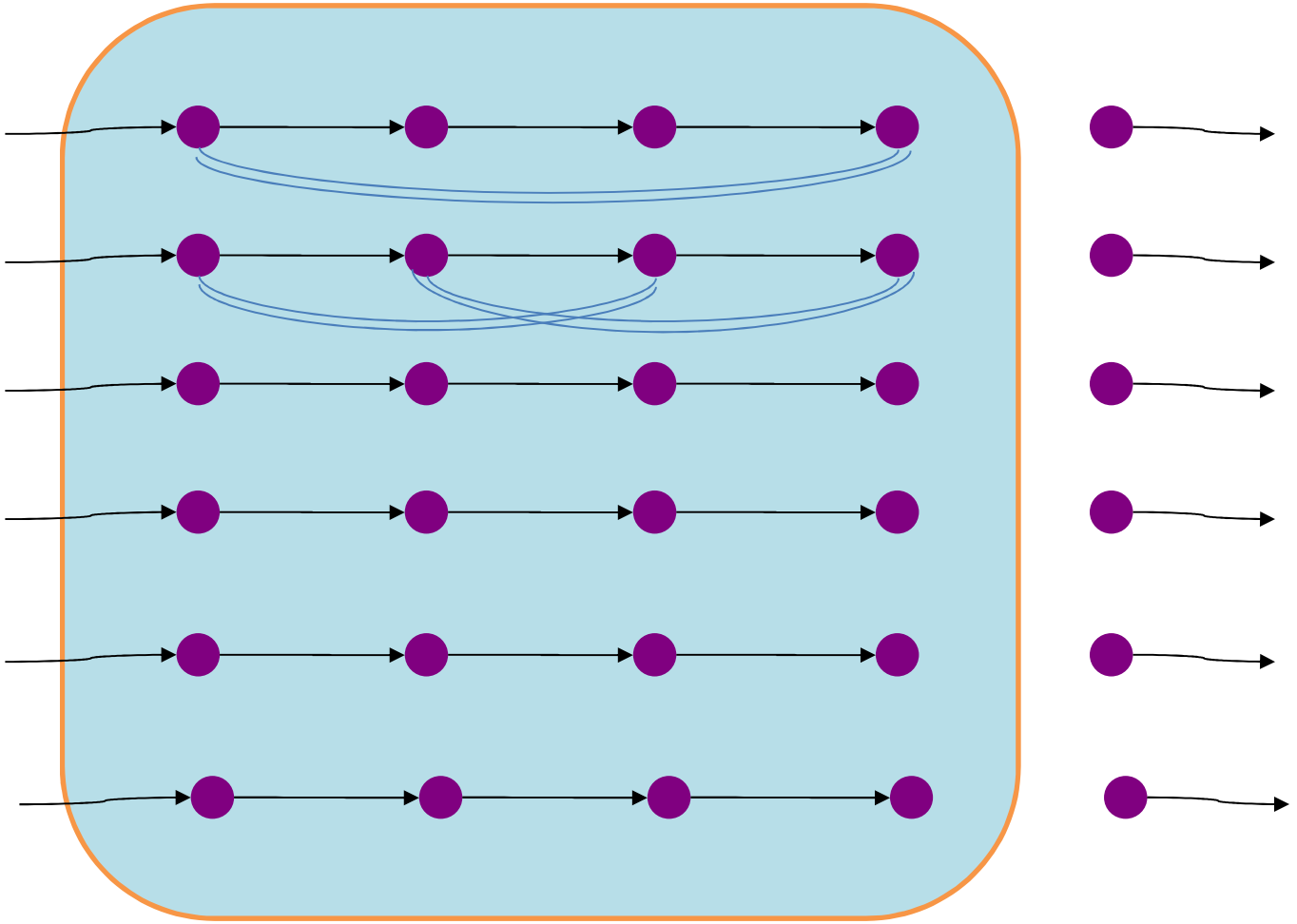
限界正确性检查 – 小结

- 从模型的局部考察一个性质是否满足
- 对一些不满足的性质可能很快知道问题
- 对一些满足的性质也可能很快知道结论

限界正确性检查



限界正确性检查



限界正确性检查

- $M,s \models \varphi$ ， 限界模型 $M_0, M_1, \ldots$
  问题：是否存在k ， $M,s \models_k \varphi$?

- 存在k, $M,s \models_k \varphi$, 则 $M,s \models \varphi$

可靠性

==> K 较小时，
较快验证系统性质

限界正确性检查

- $M,s \models \varphi$ ， 限界模型 $M_0, M_1, \ldots$
  问题：是否存在k ， $M,s \models_k \neg\varphi$?

- 存在k, $M,s \models_k \neg\varphi$, 则 $M,s \models \neg\varphi$

  则 $M,s \not\models \varphi$

可靠性

K 较小时，
较快查出系统问题

限界正确性检查与限界模型检测

限界正确性检查

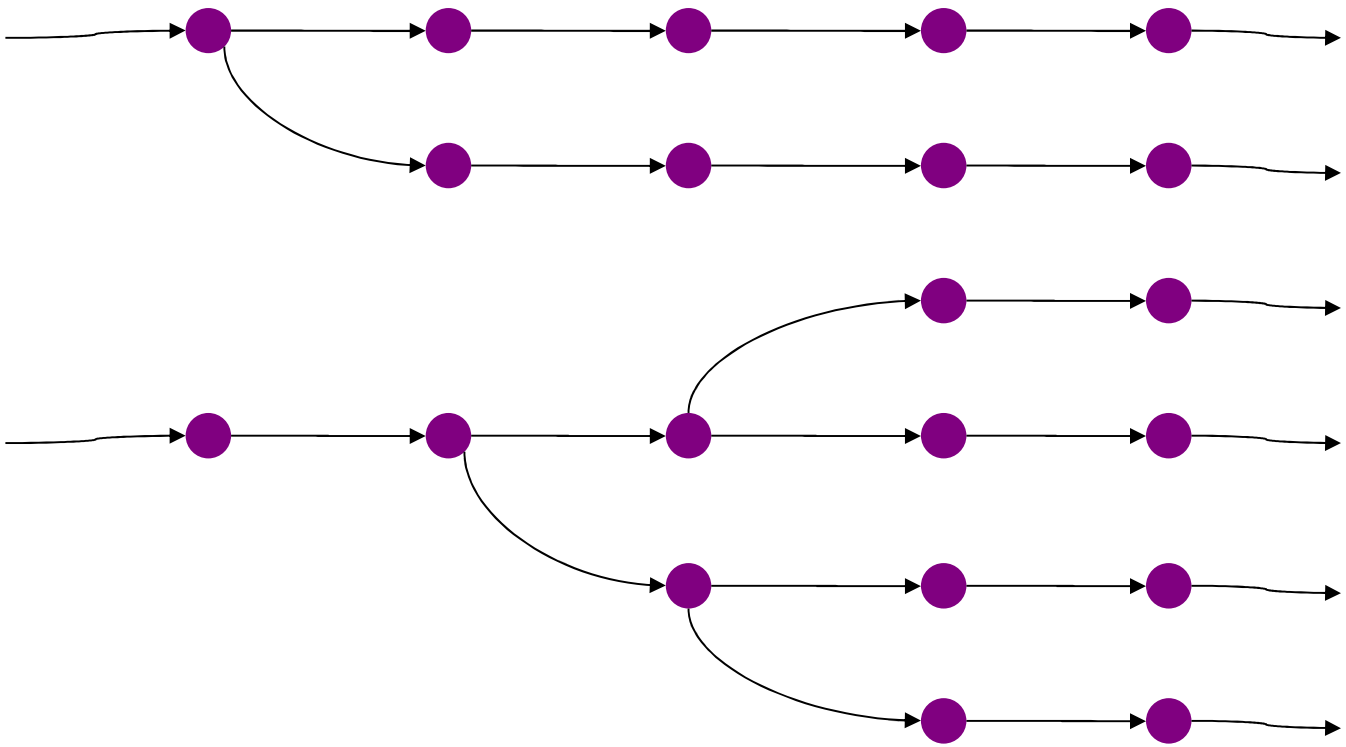
- 性质是否满足都根据$|=_k$的可靠性
- 是否一定能得到结论根据完备性

限界模型检测

- 不满足的性质的确认根据$|=_{E,k}$的可靠性
- 满足的性质的确认根据完备性与完备阈值

(III) Fixpoint Representation of CTL

Computation Tree Logic



Specification

$M \models \varphi$

$I \subseteq [[\varphi]]$

Formulas

$$[[p]] = \{ s \mid p \in L(s) \}$$

$$[[\varphi \wedge \psi]] = [[\varphi]] \cap [[\psi]]$$

$$[[\varphi \vee \psi]] = [[\varphi]] \cup [[\psi]]$$

$$[[\neg\varphi]] = S \setminus [[\varphi]]$$

Minimal Complete Set

EX,EF,EG,ER,EU,

AX,AF,AG,AR,AU

EF is expressible by EU

ER is expressible by EU and EG

Then {EX,EG,EU} is a complete set.

Formulas

$[[EX\ \varphi]] = \{ s \mid s \rightarrow s', s' \in [[\varphi]] \}$

$[[E(\varphi\ U\ \psi)]] = ?$

$[[EG(\psi)]] = ?$

Recursive Equations

$$E(\phi \, U \, \psi) \equiv \psi \vee (\phi \wedge EX(E(\phi \, U \, \psi)))$$

$$E(\phi \, R \, \psi) \equiv \psi \wedge (\phi \vee EX(E(\phi \, R \, \psi)))$$

$$[[E(\phi \, U \, \psi)]] \equiv [[\psi]] \cup ([[\phi \,]] \cap EX([[E(\phi \, U \, \psi)]]))$$

$$[[EG(\psi)]] \equiv [[\psi]] \cap EX([[EG(\psi)]])$$

Recursive Equations

$E(\phi \, U \, \psi) \equiv \psi \vee (\phi \wedge EX(E(\phi \, U \, \psi)))$

$f(Z)=\psi \vee (\phi \wedge EX(Z))$

Then $E(\phi \, U \, \psi)$ is a fixpoint of f.

$f: pow(S) \rightarrow pow(S)$

Fixpoint

$f(Z)=\psi \vee (\phi \wedge EX(Z))$

f is monotonic; f is continuous

pow(S) is a complete lattice.

f has a least and a greatest fixpoint

$\mu Z.f(Z)$, μf

$\nu Z.f(Z)$, νf

Fixpoint

$$\mu f = \cup \{ f^k(\bot) \mid k \in \mathbf{N} \}$$

We prove: $[[E(\phi \, U \, \psi)]] = \mu f$

Then we have:

$E(\phi U \psi) = \mu Z.f(Z) = \mu Z.(\psi \vee (\phi \wedge EX(Z))$

Recursive Equations

$EG(\psi) \equiv \psi \wedge EX(EG(\psi))$

$f(Z)=\psi \wedge EX(Z)$

Then $EG(\psi)$ is a fixpoint of f.

f: pow(S)→ pow(S)

Fixpoint

$f(Z)=\psi \wedge EX(Z)$

f is monotonic; f is continuous

pow(S) is a complete lattice.

f has a least and a greatest fixpoint

$\mu Z.f(Z)$, μf

$\nu Z.f(Z)$, νf

Fixpoint

$$\nu f = \cap \{ f^k(S) \mid k \in N \}$$

We prove: $[[EG(\psi)]] = \nu f$

Then we have:

$EG(\psi) = \nu Z.f(Z) = \nu Z.(\psi \wedge EX(Z))$

EG 和 EU

$\tau_1(Z) = (p \wedge EXZ)$ 是单调的

$\tau_2(Z) = (q \vee (p \wedge EXZ))$ 是单调的

EGp 是 τ_1 的一个不动点

$E(pUq)$ 是 τ_2 的一个不动点

EG 和 EU

需要证明

EGp 是 τ_1 的最大不动点

$E(pUq)$ 是 τ_2 的最小不动点

即

$$s \in \nu Z(p \wedge EXZ) \Rightarrow s \in EGp$$

$$s \in E(pUq) \Rightarrow s \in \mu Z(q \vee (p \wedge EXZ))$$

$s \in E(pUq) \Rightarrow s \in \mu Z.(q \vee (p \wedge EX\ Z))$

$\mu Z(q \vee (p \wedge EXZ)) = \cup \tau_2^i(false)$ 。

若 $s \in E(pUq)$,

则存在 $\pi_0 = s, n \geq 0, \pi = \pi_0\pi_1 \cdots$

使得 $\pi_n \models q$ 且对所有 $j < n$ ， $\pi_j \models p$ 。

若 $n = 0$ 则 $\pi_0 \in \tau_2(false)$ 。

$s \in E(pUq) \Rightarrow s \in \mu Z.(q \vee (p \wedge EX\ Z))$

假设对所有 π 和给定 k,

$\pi_k \models q$ 且对所有 $j < k, \pi_j \models p$ 则 $\pi_0 \in \tau_2^{k+1}(false)$ 。

设 $n = k + 1$ 时有

$\pi_n \models q$ 且对所有 $j < n$ ， $\pi_j \models p$ 。

根据假设，我们有 $\pi_1 \in \tau_2^{k+1}(false)$ 。

由于 $(\pi_0, \pi_1) \in \Delta$ ， $\pi_0 \models p$ ，

根据 τ_2 的定义， $\pi_0 \in \tau_2(\pi_1)$ 。

因此 $\pi_0 \in \tau_2(\tau_2^{k+1}(false)) = \tau_2^{n+1}(false)$ 。

根据归纳原理，$s \in E(pUq) \Rightarrow s \in \mu Z(q \vee (p \wedge EXZ))$

$s \in \nu Z.(p \wedge EX\, Z) \Rightarrow s \in EG\ p$

$\nu Z(p \wedge EXZ) = \cap \tau_1^i(true)$

则存在 n, $\tau_1^n(true) = \tau_1^{n+1}(true)$ 。

设 $s_0 = s \in \nu Z(p \wedge EXZ)$ 。

则 $s_0 \in \cap \tau_1^n(true)$ 且 $p \in L(s_0)$ 。

同时 $s_0 \in \tau_1^{n+1}(true)$ 。

$s \in \nu Z.(p \wedge EX\ Z) \Rightarrow s \in EG\ p$

由 τ_1 的定义知

存在状态 s_1 和状态转换 $(s_0, s_1) \in \Delta$

使得 $s_1 \in \cap\tau_1^n(true)$ 且 $p \in L(s_1)$ 。

由此类推知对所有 $i \geq 0$,

存在状态 s_{i+1} 和状态转换 $(s_i, s_{i+1}) \in \Delta$

使得 $s_{i+1} \in \cap\tau_1^n(true)$ 且 $p \in L(s_{i+1})$ 。

根据 EGp 的语义， $s \in EGp$ 。

因此 $s \in \nu Z(p \wedge EX\,Z) \Rightarrow s \in EGp$ 。

Fixpoint Formulation of CTL

$[[p]] = \{ s \mid p \in L(s) \}$

$[[EX\ \varphi]] = \{ s \mid s \rightarrow s', s' \in [[\varphi]] \}$

$E(\phi U \psi) = \mu Z.(\psi \vee (\phi \wedge EX(Z)))$

$E(\phi R \psi) = \nu Z.(\psi \wedge (\phi \vee EX(Z)))$

$EF\psi = \mu Z.(\psi \vee EX(Z))$

$EG\psi = \nu Z.(\psi \wedge EX(Z))$

Fixpoint Formulation of CTL

$[[AX\ \varphi]] = \{ s \mid (s \rightarrow s') \rightarrow s' \in [[\varphi]] \}$

$A(\phi U \psi) = \mu Z.(\psi \vee (\phi \wedge AX(Z)))$

$A(\phi R \psi) = \nu Z.(\psi \wedge (\phi \vee AX(Z)))$

$AF\psi = \mu Z.(\psi \vee AX(Z))$

$AG\psi = \nu Z.(\psi \wedge AX(Z))$

Basic Formulas

$[[p]] = \{ s \mid p \in L(s) \}$

$[[\varphi \vee \psi]] = [[\varphi]] \cup [[\psi]]$

$[[\neg\varphi]] = S \setminus [[\varphi]]$

$[[EX\ \varphi]] = \{ s \mid s \rightarrow s', s' \in [[\varphi]] \}$

$E(\phi U \psi) = \mu Z.(\psi \vee (\phi \wedge EX(Z)))$

$EG\psi = \nu Z.(\psi \wedge EX(Z))$

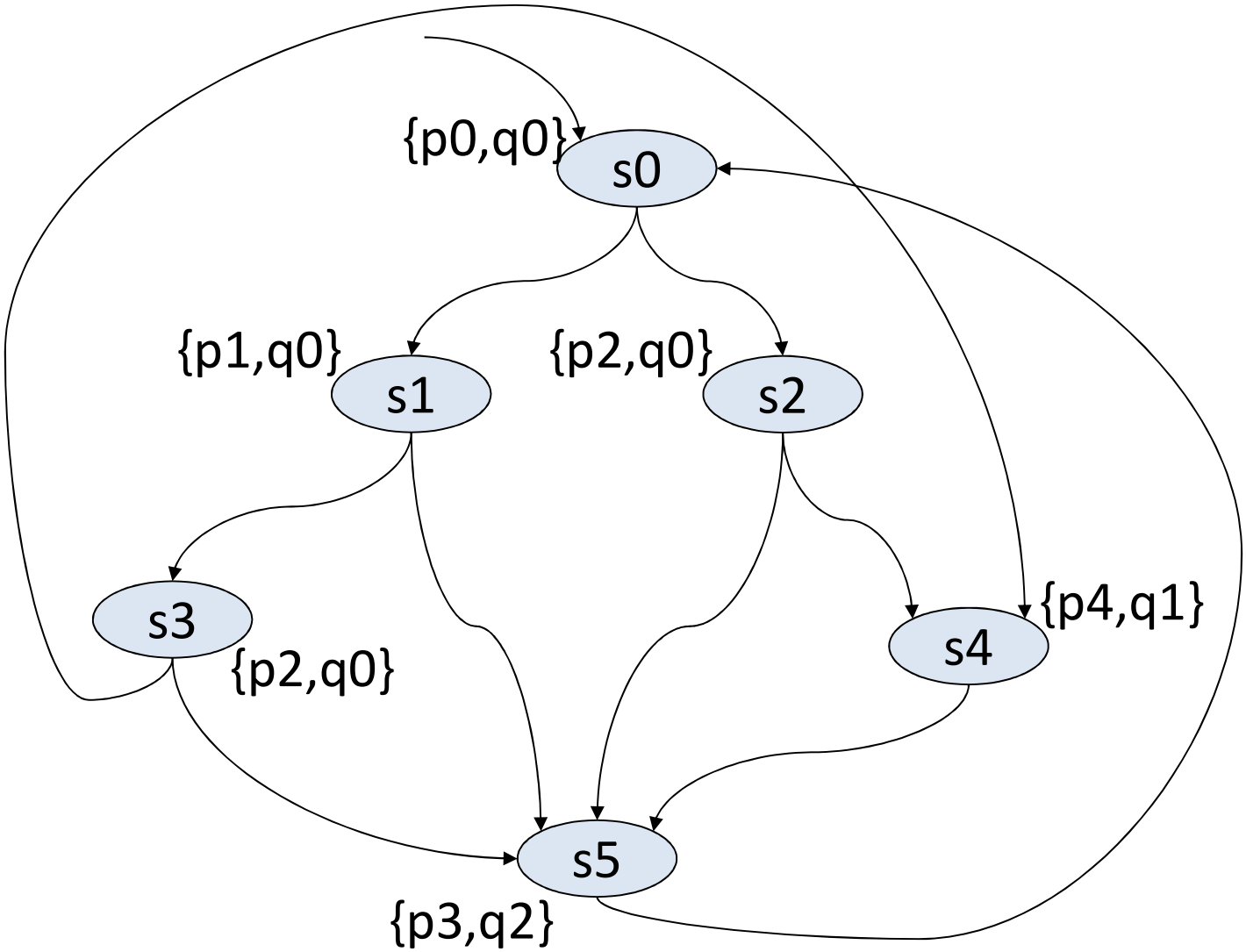
Model Checking

$M \models \phi$

$I \subseteq [[\phi]]$

例子

例子：简化自动售茶机模型



自动售茶机：E(q0Uq2)

- $E(q0Uq2) = \mu Z.(q2 \vee (q0 \wedge EX\ Z))$

- S0=false
- S1=q2 = {s5}
- $S2=\{s5\} \cup (\{s0,\ldots,s3\} \cap \{s1,\ldots,s4\}) = \{s1,s2,s3,s5\}$
- $S3=\{s5\} \cup (\{s0,\ldots,s3\} \cap \{s0,\ldots,s4\}) = \{s0,s1,s2,s3,s5\}$
- $S4=\{s5\} \cup (\{s0,\ldots,s3\} \cap \{s0,\ldots,s5\}) = \{s0,s1,s2,s3,s5\}$

- 该模型满足E(q0Uq2)

自动售茶机： AG(q0∨q2)

- AG(q0∨q2) = νZ.((q0∨q2) ∧ AX Z)

- S0=true
- S1=q0∨q2 = {s0,s1,s2,s3,s5}
- S2={s0,s1,s2,s3,s5}∩{s0,s1,s4,s5} = {s0,s1,s5}
- S3={s0,s1,s2,s3,s5}∩{s4,s5} = {s5}
- S4={s0,s1,s2,s3,s5}∩{s4} = {}
- S5={}

- 该模型不满足AG(q0∨q2)

(IV) μ-Calculus

μ-Calculus

μ-Calculus with actions

Examples of the fixpoint calculation

Formulation of CTL in μ-Calculus

Syntax of μ-Calculus

Let AP be a set of proposition symbols.

Definition

Let p range over AP.

The set Φ of μ-calculus formulas is as follows.

$$\Phi == p \mid Z \mid \Phi \wedge \Phi \mid \Phi \vee \Phi \mid \neg \Phi \mid$$
$$<.> \Phi \mid [.] \Phi \mid \mu Z.\Phi \mid \nu Z.\Phi$$

Variables are in the scope of even number of neg.

Semantics

e: VAR → pow(S)

$[[p]]e = \{ s \mid p \in L(s) \}$

$[[Z]]e = e(Z)$

$[[\langle . \rangle \varphi]]e = \{ s \mid s \rightarrow s', s' \in [[\varphi]]e \}$

$[[\,[.]\varphi]]e = \{ s \mid (s \rightarrow s') \Rightarrow (s' \in [[\varphi]]e) \}$

$[[\varphi \wedge \psi]]e = [[\varphi]]e \cap [[\psi]]\, e$

$[[\varphi \vee \psi]]e = [[\varphi]]e \cup [[\psi]]e$

$[[\neg\varphi\,]]e = S \setminus [[\varphi]]e$

$[[\mu Z.\phi]]e = \cap\{ Y \subseteq S \mid [[\phi]]e(Z/Y) \subseteq Y \}$

$[[\nu Z.\phi]]e = \cup\{ Y \subseteq S \mid Y \subseteq [[\phi]]e(Z/Y) \}$

Closed Formulas

Formulas without free variables.

The semantics of such a formula does not depend on the initial assignment e.

$[[\varphi]] = [[\varphi]]e$ for any e

Example

$\nu Z.(p \wedge [.][.]Z)$

p is true at all even places along all paths

Satisfiability

The complexity is EXPTIME-complete.

Applications of μ-Calculus

μ-Calculus as a Specification Language

System Models: Kripke Structures (K)

System Specifications: Closed Formulas (φ)

$K \models \varphi$

$I \subseteq [[\varphi]]$

Model Checking

Definition

Given a model K and a formula ϕ.

The model checking problem is the problem of checking whether K |= ϕ holds.

Model Checking

The complexity of model checking is in NP $\cap$ coNP.

μ-Calculus with Actions

Syntax of μ-Calculus

Let AP be a set of proposition symbols.

Definition

Let p range over AP.

The set Φ of μ-calculus formulas is as follows.

$$\Phi == p \mid Z \mid \Phi \wedge \Phi \mid \Phi \vee \Phi \mid \neg \Phi \mid$$

$$\langle a \rangle \Phi \mid [a]\Phi \mid \mu Z.\Phi \mid \nu Z.\Phi$$

Variables are in the scope of even number of neg.

Semantics

Interpreted on

Doubly Labeled Transition Systems

Labeled Transition Systems

Definition

A LTS is a quadruple $<\Sigma,S,\Delta,I>$

- Σ : A finite set of symbols
- S : A finite set of states
- $\Delta \subseteq S \times \Sigma \times S$: A transition relation
- $I \subseteq S$: A set of initial states

Double-Labeled Transition Systems

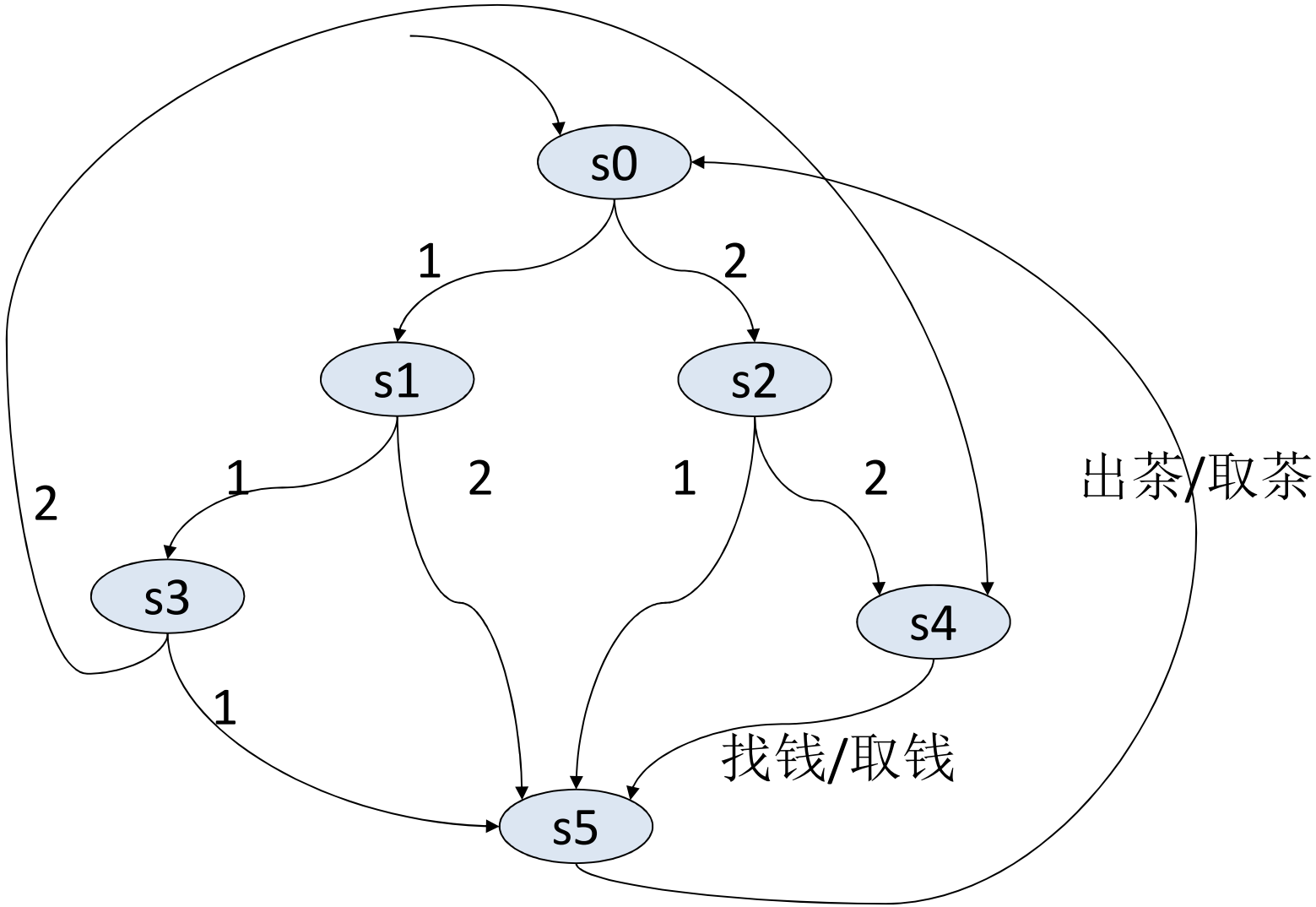
AP: A set of propositions.

Definition

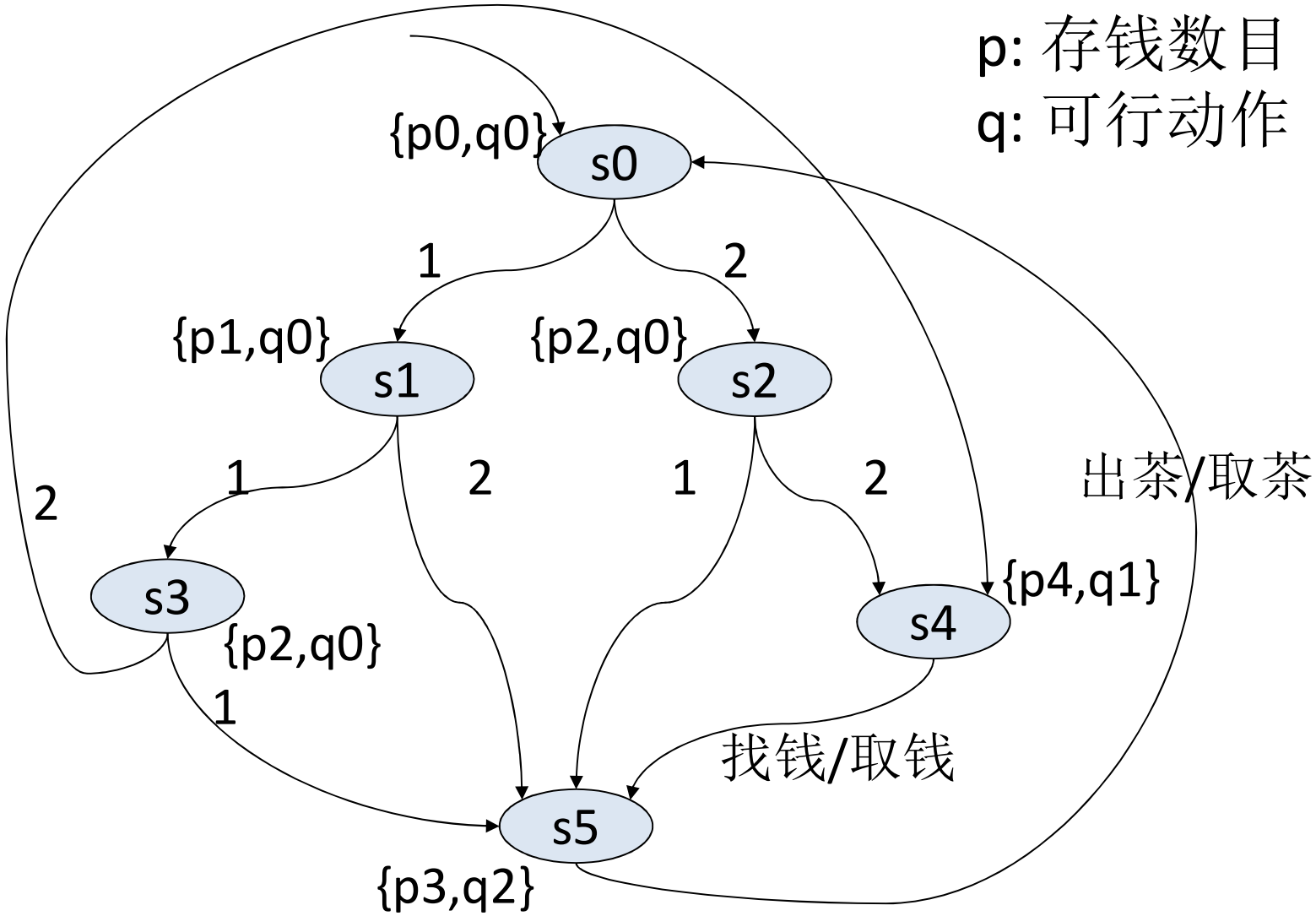
A DLTS is a quintuple $M=<\Sigma,S,\Delta,I,L>$

- Σ : A set of action symbols
- S : A finite set of states
- $\Delta \subseteq$ S x Σ x S : A total transition relation
- $I \subseteq$ S : A set of initial states
- L: S $\rightarrow$ 2^{AP} is a labeling function

例子：简化自动售茶机模型



例子：双标号自动售茶机模型



Semantics

Let e: VAR → pow(S)

$[[p]]e = \{ s \mid p \in L(s) \}$

$[[Z]]e = e(Z)$

$[[\langle a \rangle \varphi]]e = \{ s \mid s \xrightarrow{a} s', s' \in [[\varphi]]e \}$

$[[\,[a]\varphi]]e = \{ s \mid (s \xrightarrow{a} s') \Rightarrow (s' \in [[\varphi]]e) \}$

$[[\varphi \wedge \psi]]e = [[\varphi]]e \cap [[\psi]]\, e$

$[[\varphi \vee \psi]]e = [[\varphi]]e \cup [[\psi]]e$

$[[\neg\varphi\,]]e = S \setminus [[\varphi]]e$

$[[\mu Z.\phi]]e = \cap\{ Y \subseteq S \mid [[\phi]]e(Z/Y) \subseteq Y \}$

$[[\nu Z.\phi]]e = \cup\{ Y \subseteq S \mid Y \subseteq [[\phi]]e(Z/Y) \}$

Applications of μ-Calculus

μ-Calculus as a Specification Language

System Models: DLTS (M)

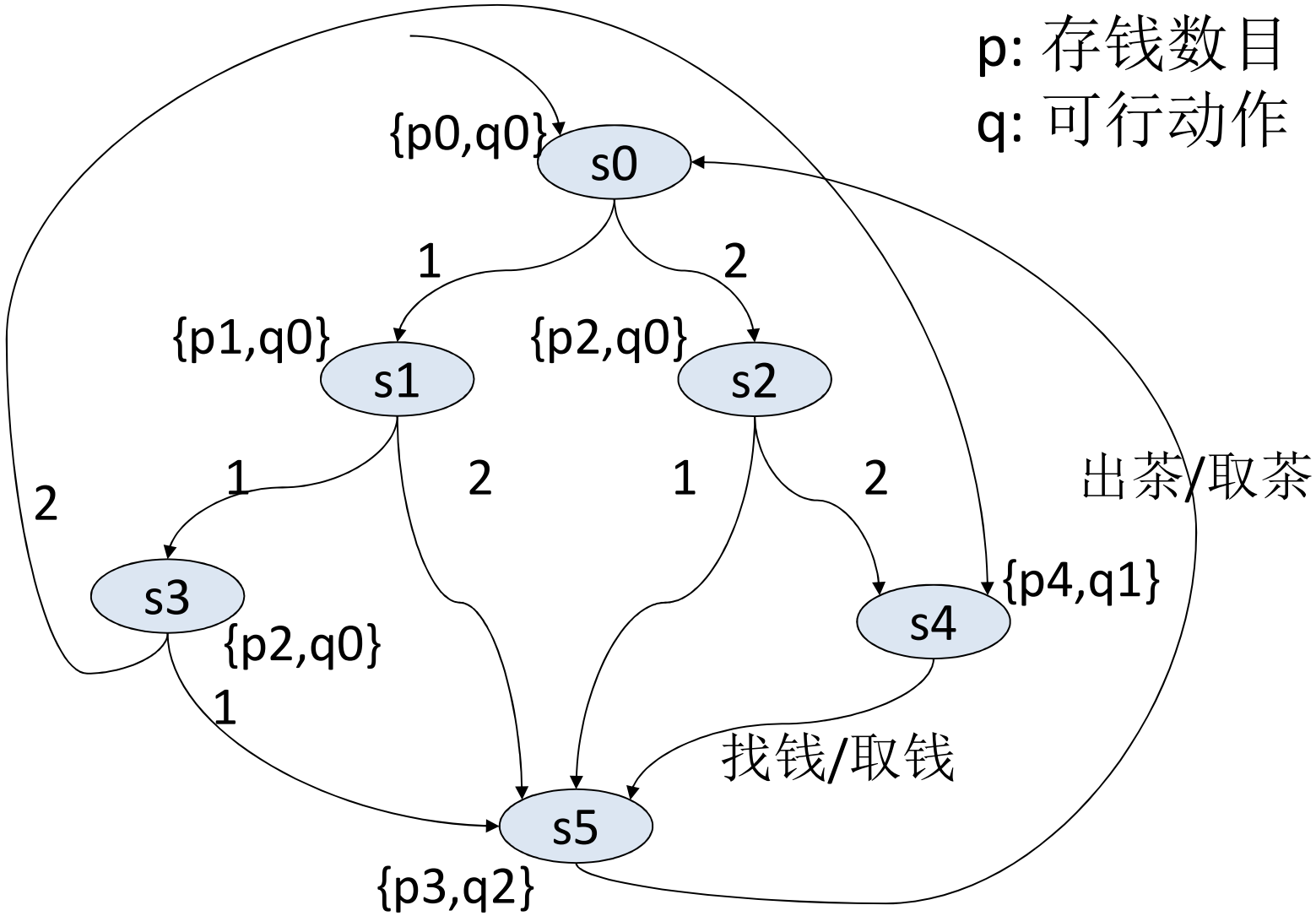
System Specifications: Closed Formulas (φ)

$M \models \varphi$

$I \subseteq [[\varphi]]$

例子

例子：双标号自动售茶机模型



自动售茶机：μX.(q2∨<1>X)

- S0=false
- S1=q2∨<1>{} = {s5}
- S2={s5}∪{s2,s3} = {s2,s3,s5}
- S3={s5}∪{s1,s2,s3} = {s1,s2,s3,s5}
- S4={s5}∪{s0,s1,s2,s3} = {s0,s1,s2,s3,s5}
- S5={s5}∪{s0,s1,s2,s3} = {s0,s1,s2,s3,s5}

自动售茶机：νX.(¬q2∧[1]X)

Negation of μX.(q2∨<1>X)

- S0=true
- S1=¬q2 = {s0,s1,s2,s3,s4}
- S2={s0,s1,s2,s3,s4}∩{s0,s1,s4,s5} = {s0,s1,s4}
- S3={s0,s1,s2,s3,s4}∩{s0,s4,s5} = {s0,s4}
- S4={s0,s1,s2,s3,s4}∩{s4,s5} = {s4}
- S5={s0,s1,s2,s3,s4}∩{s4,s5} = {s4}

自动售茶机：μX.(q2∨[1]X)

- S0=false
- S1=q2∨[1]{} = {s5}∪{s4,s5} = {s4,s5}
- S2={s5}∪{s2,s3,s4,s5} = {s2,s3,s4,s5}
- S3={s5}∪{s1,s2,s3,s4,s5} = {s1,s2,s3,s4,s5}
- S4={s5}∪{s0,s1,s2,s3,s4,s5} = {s0,s1,s2,s3,s4,s5}
- S5={s0,s1,s2,s3,s4,s5}

双标号与单标号

- μX.(q2∨<1>X)
- μX.(<取茶>TRUE∨<1>X)

Formulation of CTL in μ-Calculus (1)

$TR(p) = p$

$TR(\phi \wedge \psi) = TR(\phi) \wedge TR(\psi)$

$TR(\phi \vee \psi) = TR(\phi) \vee TR(\psi)$

$TR(\neg\phi) = \neg TR(\phi)$

$TR(EX\ \varphi) = \langle . \rangle TR(\phi)$

$TR(E(\phi U \psi)) = \mu Z.(TR(\psi) \vee (TR(\phi) \wedge \langle . \rangle (Z)))$

$TR(E(\phi R \psi)) = \nu Z.(TR(\psi) \wedge (TR(\phi) \vee \langle . \rangle (Z)))$

$TR(EF\psi) = \mu Z.(TR(\psi) \vee \langle . \rangle (Z))$

$TR(EG\psi) = \nu Z.(TR(\psi) \wedge \langle . \rangle (Z))$

Formulation of CTL in μ-Calculus (2)

$TR(AX\ \varphi) = [.]\ TR(\phi)$

$TR(A(\phi U \psi)) = \mu Z.(TR(\psi) \vee (TR(\phi) \wedge [.](Z)))$

$TR(A(\phi R \psi)) = \nu Z.(TR(\psi) \wedge (TR(\phi) \vee [.](Z)))$

$TR(AF\psi) = \mu Z.(TR(\psi) \vee [.](Z))$

$TR(AG\psi) = \nu Z.(TR(\psi) \wedge [.](Z))$

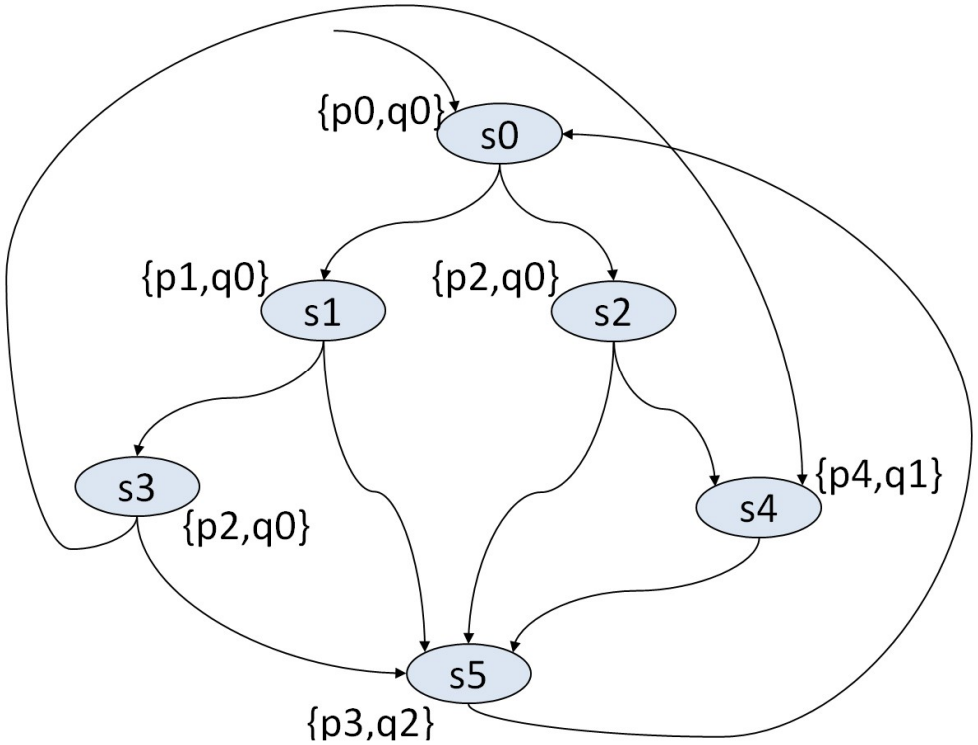
(V) Summary

- Computation Tree Logic (CTL)
- Bounded Semantics of CTL
- Fixpoint Representation of CTL Formulas
- μ-Calculus

练习1

设M为简化自动售茶机模型。

用限界语义验证M是否满足
A(q0 U q2)和
EG(q0 ∨ q2)，
分别给出最小的
可以确定以上公式
是否满足的界。



练习2

设M为简化自动售茶机模型。

用简化自动机模型M计算
[[A(q0 U q2)]]和
[[EG(q0∨q2)]]，
并讨论该模型
是否满足
A(q0 U q2)和
EG(q0∨q2)。