A Language for Distributed Strategies

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Generalise domain theory. To tackle anomalies like non-deterministic dataflow, and repair the divide between denotational and operational semantics.

Concurrent games, with behaviour based on event structures, rather than trees. The extra generality reveals new structure and a mathematical robustness to the concept of strategy—showing strategies are (special) profunctors.

Concurrent strategies support operations yielding an expressive higher-order concurrent process language. Its operational semantics requires we take internal (neutral) moves seriously.

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Event structures

An **event structure** comprises $(E, \leq, \mathrm{Con})$, consisting of a set of *events* E

- partially ordered by $\leq$, the **causal dependency relation**, and

- a nonempty family Con of finite subsets of E, the **consistency relation**,

which satisfy

$$\{e' \mid e' \leq e\} \text{ is finite for all } e \in E,$$

$$\{e\} \in \mathrm{Con} \text{ for all } e \in E,$$

$$Y \subseteq X \in \mathrm{Con} \Rightarrow Y \in \mathrm{Con}, \quad \text{and}$$

$$X \in \mathrm{Con} \ \& \ e \leq e' \in X \Rightarrow X \cup \{e\} \in \mathrm{Con}.$$

Say e, e' are **concurrent** if $\{e, e'\} \in \mathrm{Con} \ \& \ e \not\leq e' \ \& \ e' \not\leq e$.
In games the relation of **immediate dependency** $e \rightarrowtriangle e'$, meaning e and e' are distinct with $e \leq e'$ and no event in between, will play an important role.

Configurations of an event structure

The **configurations**, $\mathcal{C}^\infty(E)$, of an event structure E consist of those subsets $x \subseteq E$ which are

Consistent: $\forall X \subseteq_{\mathrm{fin}} x.\ X \in \mathrm{Con}$ and

Down-closed: $\forall e, e'.\ e' \leq e \in x \Rightarrow e' \in x$.

$x \subseteq x'$, *i.e.* x is a sub-configuration of x', means that x is a sub-history of x'.

Often concentrate on the **finite configurations** $\mathcal{C}(E)$.

Maps of event structures

A **map** of event structures $f : E \to E'$ is a partial function $f : E \rightharpoonup E'$ such that for all $x \in \mathcal{C}(E)$

$$fx \in \mathcal{C}(E') \text{ and } e_1, e_2 \in x \ \& \ f(e_1) = f(e_2) \Rightarrow e_1 = e_2 \,.$$

Note that when f is total it restricts to a bijection $x \cong fx$, for any $x \in \mathcal{C}^\infty(E)$.

Maps *preserve concurrency*, and *locally reflect causal dependency:*

$$e_1, e_2 \in x \ \& \ f(e_1) \leq f(e_2) \text{ (both defined) } \Rightarrow e_1 \leq e_2 \,.$$

Defined part of a map

A partial map

$$f : E \to E'$$

of event structures has **partial-total factorization** as a composition

$$E \xrightarrow{p} E{\downarrow}V \xrightarrow{t} E'$$

where $V =_{\mathrm{def}} \{e \in E \mid f(e) \text{ is defined}\}$ is the domain of definition of f;

$E{\downarrow}V =_{\mathrm{def}} (V, \leq_V, \mathrm{Con}_V)$, where

$v \leq_V v'$ iff $v \leq v'$ & $v, v' \in V$ and $X \in \mathrm{Con}_V$ iff $X \in \mathrm{Con}$ & $X \subseteq V$;

the *partial* map $p : E \to E{\downarrow}V$ acts as identity on V and is undefined otherwise;

and the *total* map $t : E{\downarrow}V \to E'$, called the **defined part** of f, acts as f.

Pullbacks of total maps event structures

Total maps $f : A \to C$ and $g : B \to C$ have pullbacks in the category of event structures:

$$\begin{tikzcd} & P \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\ A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & C\,. & \end{tikzcd}$$

Finite configurations of P correspond to the composite bijections

$$\theta : x \cong fx = gy \cong y$$

between configurations $x \in \mathcal{C}(A)$ and $y \in \mathcal{C}(B)$ s.t. $fx = gy$ for which the transitive relation generated on θ by $(a, b) \leq (a', b')$ if $a \leq_A a'$ or $b \leq_B b'$ is a partial order.

Concurrent games and strategies

A **game** is represented by an event structure A in which an event $a \in A$ carries a polarity $pol(a)$, $+$ (Player) and $-$ (Opponent).

A (nondeterministic concurrent) **strategy in** a game A is represented by a total map of event structures $\sigma : S \to A$ which preserves polarities and is

Receptive: $\sigma x \subseteq^- y$ implies there is a *unique* $x' \in \mathcal{C}(S)$ such that

$$x \subseteq x' \;\&\; \sigma x' = y \; : \quad \begin{array}{ccc} x & \subseteq & x' \\ \downarrow & & \downarrow \\ \sigma x & \subseteq^- & y \end{array}$$

A strategy should be receptive to all possible moves of opponent.

Innocent: if $s \rightarrowtriangle_S s'$ & $(pol(s) = + \; or \; pol(s') = -)$ then $\sigma(s) \rightarrowtriangle_A \sigma(s')$.
A strategy should only adjoin immediate causal dependencies $\ominus \rightarrowtriangle \oplus$.
[The relation $\rightarrowtriangle$ stands for immediate causal dependence]

Strategies between games

Following Conway & Joyal, a **strategy from** a game A **to** a game B, written $\sigma : A \mathrel{+\!\!\!\rightarrow} B$, is a strategy σ in the game $A^\perp \| B$, based on:

(Simple) Parallel composition $A\|B$, by juxtaposition. Its unit is the empty game $\emptyset$.

Dual $A^\perp$, of an event structure with polarity A: a copy of the event structure A with a reversal of polarities.

The conditions on strategies exactly ensure that copy-cat is identity w.r.t. composition.

Example of a strategy: copy-cat strategy from A to A

$$\gamma_A : \mathbb{C}\mathbb{C}_A \to A^{\perp} \| A$$

$$\mathbb{C}\mathbb{C}_A$$

$$\begin{tikzcd}
A^{\perp} & & & A \\
\overline{a}_2 & \ominus \arrow[r, dashed] & \oplus & a_2 \\
\overline{a}_1 & \oplus \arrow[u] & \ominus \arrow[u] \arrow[l, dashed] & a_1
\end{tikzcd}$$

Composition of strategies $\sigma : S \to A^{\perp} \| B$, $\tau : T \to B^{\perp} \| C$

Via pullback. Ignoring polarities, the composite partial map

$$
\begin{tikzcd}
 & P \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee"] & \\
S \| C \arrow[dr, "\sigma \| C"'] & & A \| T \arrow[dl, "A \| \tau"] \\
 & A \| B \| C \arrow[d] & \\
 & A \| C &
\end{tikzcd}
$$

has defined part, which yields $\tau \odot \sigma : T \odot S \to A^{\perp} \| C$ once reinstate polarities.

An alternative characterization of strategies

Defining a partial order — *the Scott order* — on configurations of A

$$y \sqsubseteq_A x \;\; \text{iff} \;\; y \supseteq^- \cdot \subseteq^+ \cdot \supseteq^- \cdots \supseteq^- \cdot \subseteq^+ x$$

we obtain a factorization system $((\mathcal{C}(A), \sqsubseteq_A), \supseteq^-, \subseteq^+)$, *i.e.* $\exists! z.$

$$\begin{tikzcd} & & x \\ y \arrow[urr, dashed, phantom, "\sqsubseteq" sloped] & \supseteq^- & z \arrow[u, phantom, "\subseteq^+" sloped] \end{tikzcd} .$$

Proposition $z \in \mathcal{C}(\mathbb{C}_A)$ iff $z_2 \sqsubseteq_A z_1$.

Theorem Strategies $\sigma : S \to A$ correspond to discrete fibrations

$$\sigma\text{``} : (\mathcal{C}(S), \sqsubseteq_S) \to (\mathcal{C}(A), \sqsubseteq_A)\,, \quad i.e. \;\; \exists! x'. \quad \begin{tikzcd} x' \arrow[d, dotted, "\sigma\text{``}"'] \arrow[r, dotted, phantom, "\sqsubseteq_S"] & x \arrow[d, "\sigma\text{``}"] \\ y \arrow[r, phantom, "\sqsubseteq_A"] & \sigma\text{``}(x) \end{tikzcd} ,$$

which preserve $\supseteq^-$, $\subseteq^+$ and $\emptyset$.

$\rightsquigarrow$ *A lax functor from strategies to profunctors ...*

A bicategory of games

A bicategory, $\mathbf{Games}$, whose

objects are event structures with polarity—the games,

arrows are strategies $\sigma : A \multimap B$

2-cells are (the obvious) maps of (pre-)strategies.

The vertical composition of 2-cells is the usual composition of maps of spans. Horizontal composition is given by the composition of strategies $\odot$ (which extends to a functor on 2-cells via the universality of pullback).

The bicategory is rich in structure, in particular, it is compact-closed (so has a trace, a feedback operation)—though compact-closure is disturbed by extensions such as winning conditions.
Duality: $\sigma : A \multimap B$ *corresponds to* $\sigma^{\perp} : B^{\perp} \multimap A^{\perp}$.

Special cases

Ingenuous strategies Deterministic concurrent strategies coincide with the $receptive$ ingenuous strategies of and Melliès and Mimram.

Stable spans and stable functions The sub-bicategory of $\mathbf{Games}$ where the events of games are purely $+$ve is equivalent to the bicategory of 'stable spans' used in nondeterministic dataflow; feedback is given by trace.
When deterministic we obtain a sub-bicategory equivalent to Berry's **dI-domains and stable functions**.

Closure operators A deterministic strategy in A determines a closure operator on $\mathcal{C}^{\infty}(A)^{\top}$ of Abramsky and Melliès.

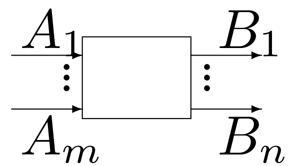
Simple games *"Simple games"* of game semantics arise when we restrict $\mathbf{Games}$ to objects and deterministic strategies which are 'tree-like'—alternating polarities, with conflicting branches, beginning with opponent moves.

Constructions on strategies

Types: Games A, B, C, ... with operations $A^\perp$, $A\|B$, sums $\sum_{i\in I} A_i$, recursively-defined types, ...

A **term**

$$x_1 : A_1, \cdots, x_m : A_m \vdash \; t \; \dashv y_1 : B_1, \cdots, y_n : B_n \;,$$



denotes a strategy $A_1\|\cdots\|A_m \;\xrightarrow{+}\; B_1\|\cdots\|B_n$.

Duality of input and output:

$$\frac{\Gamma, x : A \vdash t \dashv \Delta}{\Gamma \vdash t \dashv x : A^\perp, \Delta}$$

because t denotes a strategy in $(\Gamma^\perp\|A^\perp)\|\Delta \cong \Gamma^\perp\|(A^\perp\|\Delta)$.

Composition

Composition of strategies:

$$\frac{\Gamma \vdash t \dashv \Delta \qquad \Delta \vdash u \dashv \mathrm{H}}{\Gamma \vdash \exists \Delta.\, [\, t \parallel u \,] \dashv \mathrm{H}}$$

When Δ is empty, this yields simple parallel composition $t \| u$.

Via

$$\frac{\dfrac{\mathrm{H}^{\perp} \vdash u \dashv \Delta^{\perp} \qquad \Delta^{\perp} \vdash t \dashv \Gamma^{\perp}}{\mathrm{H}^{\perp} \vdash \exists \Delta^{\perp}.\, [\, u \parallel t \,] \dashv \Gamma^{\perp}}}{\Gamma \vdash \exists \Delta^{\perp}.\, [\, u \parallel t \,] \dashv \mathrm{H}}\,,$$

obtain the equivalent $\exists \Delta^{\perp}.\, [\, u \parallel t \,] \cong \exists \Delta.\, [\, t \parallel u \,]$.

Hom-set terms: special cases

Copy-cat on A, $\quad x : A \vdash \; y \sqsubseteq_A x \; \dashv y : A \qquad$ or $\qquad \vdash y \sqsubseteq_A x \dashv x : A^{\perp}, y : A\,.$

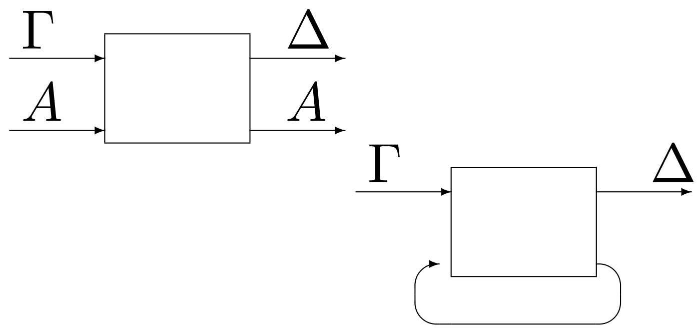
$$\oplus \;\leftarrow\; \ominus$$
$$\ominus \;\rightarrow\; \oplus$$

Copy-cat on $A^{\perp}$, $\quad x : A^{\perp}, y : A \vdash \; x \sqsubseteq_A y \; \dashv \,.$

$$\oplus \;\rightarrow\; \ominus$$
$$\ominus \;\leftarrow\; \oplus$$

For

$$\Gamma, x : A \vdash \; t \; \dashv y : A, \Delta\,,$$



$$\Gamma \vdash \; \exists x : A^{\perp}, y : A.\,[\, t \parallel x \sqsubseteq_A y \,] \; \dashv \Delta$$

denotes the **trace** adjoining causal links to achieve feedback.
Cf. non-deterministic dataflow.

Pullback of strategies

$$\frac{\Gamma \vdash t_1 \dashv \Delta \quad \Gamma \vdash t_2 \dashv \Delta}{\Gamma \vdash t_1 \wedge t_2 \dashv \Delta}$$

In the case where t_1 and t_2 denote the respective strategies $\sigma_1 : S_1 \to \Gamma^\perp \| \Delta$ and $\sigma_1 : S_1 \to \Gamma^\perp \| \Delta$ the strategy $t_1 \wedge t_2$ denotes the pullback

$$\begin{tikzcd}
 & S_1 \wedge S_2 \arrow[dl] \arrow[dr] \arrow[dd, dotted, "\sigma_1 \wedge \sigma_2"] & \\
S_1 \arrow[dr, "\sigma_1"'] & & S_2 \arrow[dl, "\sigma_2"] \\
 & \Gamma^\perp \| \Delta \,. &
\end{tikzcd}$$

Nondeterministic sum of strategies

In the **sum of strategies**, in the same game, the strategies are glued together on their initial Opponent moves (to maintain receptivity) and only commit to a component with the occurrence of a Player move.

$$\frac{\Gamma \vdash t_i \dashv \Delta \quad i \in I}{\Gamma \vdash \|_{i \in I}\, t_i \dashv \Delta}$$

We use $\perp$ for the **empty nondeterministic sum**, when the rule above specialises to

$$\Gamma \vdash \perp \dashv \Delta\,,$$

which denotes the minimum strategy in the game $\Gamma^{\perp} \| \Delta$—it comprises the initial segment of the game $\Gamma^{\perp} \| \Delta$ consisting of all the initial Opponent events of A.

Sum types, injections and projections

The **sum-type** $\Sigma_{i\in I} A_i$ denotes the sum of the event structures A_i, $i \in I$, the juxtaposition of the A_i but where moves from different components are inconsistent.

Special cases of hom-set terms, *viz.*

$$x : A_j \vdash \; y \sqsubseteq_{\Sigma_{i\in I} A_i} jx \; \dashv y : \Sigma_{i\in I} A_i \,,$$

with $j \in I$, denote the **injection** strategy $A_j \rightarrow\!\!\!\!\!+ \; \Sigma_{i\in I} A_i$, and

$$x : \Sigma_{i\in I} A_i \vdash \; jy \sqsubseteq_{\Sigma_{i\in I} A_i} x \; \dashv y : A_j$$

the **projection** strategy $\Sigma_{i\in I} A_i \rightarrow\!\!\!\!\!+ \; A_j$.

Hom-set terms: general case

$$\frac{\Gamma \vdash p' : C \qquad \Delta \vdash p : C}{\Gamma \vdash \ p \sqsubseteq_C p' \ \dashv \Delta} \quad p[\emptyset_\Delta] \sqsubseteq_C p'[\emptyset_\Gamma] \quad (\dagger)$$

introduces a term standing for the hom-set $(\mathcal{C}(C), \sqsubseteq_C)(p, p')$. It relies on **configuration expressions** p, p' and their typings, $e.g.$

$x : A \vdash x : A$, $\Gamma \vdash \emptyset : A$, $\Gamma \vdash \{a\} : A$, a is an initial event of A,

$\Gamma, x' : A/a \vdash \{a\} \cup x' : A$, where a is an initial event of in A,

$x : A, y : B \vdash (x, y) : A \| B$.

Configuration expressions denote *affine* maps of event structures.

$(\dagger)$ *The hom-set rule's side condition says that the expressions are in the Scott order when all variables are assigned the empty configuration.*

Further examples of uses of hom-set terms

- $\vdash\; y \sqsubseteq_A \emptyset \;\dashv y : A$ denotes $\bot_A$, **the minimum strategy** in the game A.

- Assume $\vdash t \dashv y : B$. When $x : A \vdash p : B$ denotes a map $f : A \to B$,

$$\vdash\; \exists y : B.\, [\, t \parallel p \sqsubseteq_B y\,] \;\dashv x : A$$

denotes the **pullback** $f^* t$ of the strategy t across the map $f : A \to B$.
There's a similar definition of $f_! t$.

- "**Lambda abstraction $\lambda x : A.\, t$ of type $A^\perp \| B$**":

$$\dfrac{\dfrac{\Gamma, x : A \vdash t \dashv y : B}{\Gamma \vdash t \dashv x : A^\perp, y : B} \qquad \dfrac{\dfrac{}{x : A^\perp, y : B \vdash (x,y) : A^\perp \| B} \qquad \dfrac{}{z : A^\perp \| B \vdash z : A^\perp \| B}}{x : A^\perp, y : B \vdash\; z \sqsubseteq_{A^\perp \| B} (x,y) \;\dashv z : A^\perp \| B}}{\Gamma \vdash\; \exists x : A^\perp, y : B.\, [\, t \parallel z \sqsubseteq_{A^\perp \| B} (x,y))\,] \;\dashv z : A^\perp \| B}$$

The duplication comonoid

To turn trace into a recursion operator we duplicate arguments through

$$\delta_A : A \mathrel{+\!\!\!\!\rightarrow} A \| A$$

which with $\perp : A \mathrel{+\!\!\!\!\rightarrow} \emptyset$ forms a comonoid. The general defn is involved, but $e.g.$,

if $A = \oplus$, δ_A is

$$\begin{tikzcd} & \oplus \\ \ominus \arrow[ur] \arrow[dr] & \\ & \oplus \end{tikzcd},$$

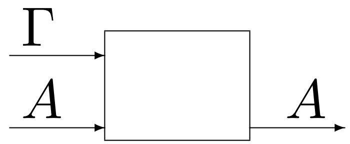
if $A = \ominus$, δ_A is

$$\begin{tikzcd} \oplus \arrow[d, no head, squiggly] & \ominus \arrow[l] \\ \oplus & \ominus \arrow[l] \end{tikzcd}.$$

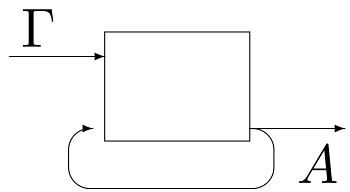
$\rightsquigarrow$ **duplication terms** such as $x : A \vdash \delta(x, y_1, y_2) \dashv y_1 : A, y_2 : A$.

Recursion from duplication and trace

Given



composing with δ_A and forming the trace



yields a recursion operator, provided the body σ respects δ:

$$\delta_A \odot \sigma \;\cong\; (\sigma \| \sigma) \odot \delta_{\Gamma \| A}\,.$$

Issues

The operations form the basis of a higher-order process language.

But,

- what is its operational semantics?

- what are suitable equivalences? ($E.g.$ w.r.t. “may” and “must” testing)

- its expressivity?

These all require we examine the effects of synchronization and the internal, neutral events it produces, more carefully.

Partial strategies—not quite right definition

A **partial strategy** in a game A (in which all events have $+$ve or $-$ve polarity) comprises a (partial) map $\sigma : S \to A$ of event structures with polarity (in which S may also have neutral events) which

(i) is *receptive*;

(ii) has *domain of definition the non-neutral events of* S;

(ii) partial-total factorization in which the *defined part* σ_0 *is a strategy*:

$$\begin{tikzcd} S \arrow[r] \arrow[d, "\sigma"'] & S_0 \arrow[dl, "\sigma_0"] \\ A & \end{tikzcd}$$

The operations on strategies extend to partial strategies—**though only if the use of pb is replaced by a stricter variant of pb when working with partial maps.**

A **partial strategy** in a game A (in which all events have +ve or −ve polarity) comprises a total map

$$\sigma : S \to N \| A$$

of event structures with polarity (in which S may also have neutral events), preserving polarity, where

(i) N is an event structure consisting solely of neutral events;

(ii) σ is *receptive*,

(ii) in the partial-total factorization of the composition of $S \xrightarrow{\sigma} N \| A$ with the projection $N \| A \to A$

$$\begin{tikzcd} S \arrow[r] \arrow[d, "\sigma"'] & S_0 \arrow[d, "\sigma_0"] \\ N \| A \arrow[r] & A \end{tikzcd}$$

the **defined part** σ_0 is a strategy.

Strategies are those partial strategies in which N is empty. The operations on strategies extend to partial strategies; the defined part of an operation equals the operation on the defined parts. In composition we don't hide the result of synchronisations, but keep them as neutral events.

Transition semantics

Transitions are associated with three kinds of actions: an action o associated with a hidden neutral action; an initial event located in the left environment; and an initial event of the right environment.

$$
\begin{array}{ccccc}
\Gamma & \vdash & t & \dashv & \Delta \\
 & & \Big\downarrow{\scriptstyle o} & & \\
\Gamma & \vdash & t' & \dashv & \Delta\,;
\end{array}
$$

$$
\begin{array}{ccccc}
\Gamma, x : A & \vdash & t & \dashv & \Delta \\
 & & \Big\downarrow{\scriptstyle x:\, a\, :x'} & & \\
\Gamma, x' : A/a & \vdash & t' & \dashv & \Delta
\end{array}
\qquad
\begin{array}{ccccc}
\Gamma & \vdash & t & \dashv & y : B, \Delta \\
 & & \Big\downarrow{\scriptstyle y:\, b\, :y'} & & \\
\Gamma & \vdash & t' & \dashv & y' : B/b, \Delta\,.
\end{array}
$$

Rules for composition

$$\dfrac{\begin{array}{ccc} \Gamma \vdash & t & \dashv y : B, \Delta \\ & \Big\downarrow {\scriptstyle y:\, b\, :y'} & \\ \Gamma \vdash & t' & \dashv y' : B/b, \Delta \end{array} \qquad \begin{array}{ccc} \Delta^{\perp}, y : B^{\perp} \vdash & u & \dashv \mathrm{H} \\ & \Big\downarrow {\scriptstyle y:\, \overline{b}\, :y'} & \\ \Gamma, y' : B^{\perp}/\overline{b} \vdash & u' & \dashv \mathrm{H} \end{array}}{\begin{array}{ccc} \Gamma \vdash & \exists y : B, \Delta. [\, t \parallel u \,] & \dashv \mathrm{H} \\ & \Big\downarrow {\scriptstyle o} & \\ \Gamma \vdash & \exists y' : B/b, \Delta. [\, t' \parallel u' \,] & \dashv \mathrm{H} \end{array}}$$

Rules for composition (cont)

Below α stands for o or an action on the left of the form $x : a : x'$, and β for o or an action on the right of the form $y : b : y'$.

$$\frac{\begin{array}{ccc} \Gamma \vdash & t & \dashv \Delta \\ & \Big\downarrow \alpha & \\ \Gamma' \vdash & t' & \dashv \Delta \end{array}}{\begin{array}{ccc} \Gamma \vdash & \exists\Delta.\,[\,t \parallel u\,] & \dashv \mathrm{H} \\ & \Big\downarrow \alpha & \\ \Gamma' \vdash & \exists\Delta.\,[\,t' \parallel u\,] & \dashv \mathrm{H} \end{array}} \qquad \frac{\begin{array}{ccc} \Delta \vdash & u & \dashv \mathrm{H} \\ & \Big\downarrow \beta & \\ \Delta \vdash & u' & \dashv \mathrm{H}' \end{array}}{\begin{array}{ccc} \Gamma \vdash & \exists\Delta.\,[\,t \parallel u\,] & \dashv \mathrm{H} \\ & \Big\downarrow \beta & \\ \Gamma \vdash & \exists\Delta.\,[\,t \parallel u'\,] & \dashv \mathrm{H}' \end{array}}$$

Rules for hom-sets

Provided a is an initial event of A for which $p[\{a\}/x][\emptyset] \sqsubseteq_C p'[\{a\}/x][\emptyset]$,

$$
\begin{array}{lcr}
\Gamma, x : A \ \vdash & p \sqsubseteq_C p' & \dashv \ \Delta \\
& \Big\downarrow {\scriptstyle x:\, a\, : x'} & \\
\Gamma, x' : A/a \ \vdash & p[\{a\} \cup x'/x] \sqsubseteq_C p'[\{a\} \cup x'/x] & \dashv \ \Delta\,.
\end{array}
$$

The variable x will only appear in one of p and p', though because of duality in forming terms we cannot be sure which.

Rules for hom-sets (cont)

Provided b is an initial event of B for which $p[\{b\}/y][\emptyset] \sqsubseteq_C p'[\{b\}/y][\emptyset]$,

$$\begin{array}{lccr} \Gamma \ \vdash & p \sqsubseteq_C p' & & \dashv \ y : B, \Delta \\ & \Big\downarrow {\scriptstyle y:\, b\, :y'} & & \\ \Gamma \ \vdash & p[\{b\} \cup y'/y] \sqsubseteq_C p'[\{b\} \cup y'/y] & & \dashv \ y' : B/b, \Delta\,. \end{array}$$

Example of hom-set rule in action

Let

$$A = \oplus \| A'$$

where $a = \oplus$. Then $A/a = A'$.

$$\ominus \xrightarrow{\hspace{8cm}} \oplus$$

$$\begin{array}{lccl}
x : A \ \vdash & y \sqsubseteq_A x & \dashv \ y : A & \\
 & \big\downarrow {\scriptstyle x :\, a \,: x'} & & \\
x' : A' \ \vdash & y \sqsubseteq_A \{a\} \cup x' & \dashv \ y : A & \text{as } \emptyset \sqsubseteq_A \{a\} \\
 & \big\downarrow {\scriptstyle y :\, a \,: y'} & & \\
x' : A' \ \vdash & \{a\} \cup y' \sqsubseteq_A \{a\} \cup x' & \dashv \ y' : A' & \text{as } \{a\} \sqsubseteq_A \{a\}
\end{array}$$

Pullback:

$$\dfrac{\begin{array}{ccc}\Gamma \vdash & t_1 & \dashv \Delta \\ & \big\downarrow z\!:c\!:\!z' & \\ \Gamma' \vdash & t_1' & \dashv \Delta'\end{array} \qquad \begin{array}{ccc}\Gamma \vdash & t_2 & \dashv \Delta \\ & \big\downarrow z\!:c\!:\!z' & \\ \Gamma' \vdash & t_2' & \dashv \Delta'\end{array}}{\begin{array}{ccc}\Gamma \vdash & t_1 \wedge t_2 & \dashv \Delta \\ & \big\downarrow z\!:c\!:\!z' & \\ \Gamma' \vdash & t_1' \wedge t_2 & \dashv \Delta'\end{array}}$$

$$\dfrac{\begin{array}{ccc}\Gamma \vdash & t_1 & \dashv \Delta \\ & \big\downarrow o & \\ \Gamma \vdash & t_1' & \dashv \Delta\end{array}}{\begin{array}{ccc}\Gamma \vdash & t_1 \wedge t_2 & \dashv \Delta \\ & \big\downarrow o & \\ \Gamma \vdash & t_1' \wedge t_2 & \dashv \Delta\end{array}} \qquad \dfrac{\begin{array}{ccc}\Gamma \vdash & t_2 & \dashv \Delta \\ & \big\downarrow o & \\ \Gamma \vdash & t_2' & \dashv \Delta\end{array}}{\begin{array}{ccc}\Gamma \vdash & t_1 \wedge t_2 & \dashv \Delta \\ & \big\downarrow o & \\ \Gamma \vdash & t_1 \wedge t_2' & \dashv \Delta\end{array}}$$

Sums

$$\dfrac{\begin{array}{ccc} \Gamma \vdash & t_i & \dashv \Delta \\ & \big\downarrow \epsilon & \\ \Gamma' \vdash & t'_i & \dashv \Delta',\ i \in I \end{array}}{\begin{array}{ccc} \Gamma \vdash & \|_{i \in I}\, t_i & \dashv \Delta \\ & \big\downarrow \epsilon & \\ \Gamma' \vdash & \|_{i \in I}\, t'_i & \dashv \Delta' \end{array}}\ \epsilon \text{ is } -\text{ve} \qquad \dfrac{\begin{array}{ccc} \Gamma \vdash & t_j & \dashv \Delta \\ & \big\downarrow o & \\ \Gamma' \vdash & t'_j & \dashv \Delta' \end{array}}{\begin{array}{ccc} \Gamma \vdash & \|_{i \in I}\, t_i & \dashv \Delta \\ & \big\downarrow o & \\ \Gamma' \vdash & (\|_{i \in I}\, t_i)[t'_j/j] & \dashv \Delta' \end{array}}\ j \in I$$

An action ϵ is $+/-$ve if on the right and its event is $+/-$ve, or on the left with event $-/+$ve.

Sums (cont)

$$
\dfrac{\begin{array}{c} \Gamma \vdash t_j \dashv \Delta \\ \big\downarrow \epsilon \\ \Gamma' \vdash t'_j \dashv \Delta' \end{array}}{\begin{array}{c} \Gamma \vdash \|_{i \in I}\, t_i \dashv \Delta \\ \big\downarrow \epsilon \\ \Gamma' \vdash t'_j \dashv \Delta' \end{array}} \; j \in I \;\&\; \epsilon \text{ is } +\text{ve}
$$

Rules for δ

Provided b is an initial $-$ve event of B,

$$\begin{array}{ccc} \Gamma \vdash & \delta_C(p, q_1, q_2) & \dashv y : B, \Delta \\ & \Big\downarrow{\scriptstyle y:\, b\, :y'} & \\ \Gamma \vdash & \delta_C(p, q_1, q_2)[\{b\} \cup y'/y] & \dashv y' : B/b, \Delta\,. \end{array}$$

Dually, provided a is an initial $+$ve event of A,

$$\begin{array}{ccc} \Gamma, x : A \vdash & \delta_C(p, q_1, q_2) & \dashv \Delta \\ & \Big\downarrow{\scriptstyle x:\, a\, :x'} & \\ \Gamma, x' : A/a \vdash & \delta_C(p, q_1, q_2)[\{a\} \cup x'/x] & \dashv \Delta\,. \end{array}$$

Rules for δ (cont)

In typed judgements of $\delta_C(p, q_1, q_2)$ a variable can appear free in at most one of p, q_1, q_2. Write, for example, $y \in \mathrm{fv}(p)$ for y is a free variable of p, and $q_1(y : b) \in p[\emptyset]$ to mean the image of b under the map q_1 denotes is in the configuration denoted by $p[\emptyset]$.

Provided b is an initial +ve event of B, $y \in \mathrm{fv}(q_1)$ and $q_1(y : b) \in p[\emptyset]$,

$$
\begin{array}{lcr}
\Gamma \vdash & \delta_C(p, q_1, q_2) & \dashv y : B, \Delta \\
 & \Big\downarrow {\scriptstyle y : b : y'} & \\
\Gamma \vdash & \delta_C(p, q_1, q_2)[\{b\} \cup y'/y] & \dashv y' : B/b, \Delta \,.
\end{array}
$$

Similarly for q_2. And dually.

Rules for δ (cont)

Provided b is an initial +ve event of B, $y \in \mathrm{fv}(p)$ and $p(y : b) \in q_1[\emptyset]$,

$$\begin{array}{lccc}
\Gamma \ \vdash & \delta_C(p, q_1, q_2) & \dashv & y : B, \Delta \\
& \Big\downarrow {\scriptstyle y : b : y'} & & \\
\Gamma \ \vdash & \delta_C(p, q_1, q_2)[\{b\} \cup y'/y] & \dashv & y' : B/b, \Delta \,.
\end{array}$$

Similarly for q_2. And dually.

Derivations and events

The rules preserve the following property:

$Derivations$ in the operational semantics

$$\frac{\vdots}{\begin{array}{ccc} \Gamma \;\vdash & t & \dashv\; \Delta \\ & \downarrow \epsilon & \\ \Gamma' \;\vdash & t' & \dashv\; \Delta' \end{array}},$$

up to α-equivalence, in which t denotes the partial strategy $\sigma : S \to \Gamma^{\perp} \| \Delta$, are in 1-1 correspondence with initial events s in S such that $\sigma(s) = ev(\epsilon)$ when $ev(\epsilon) \neq o$ or s is neutral when $ev(\epsilon) = o$.

Beyond linear strategies

But concurrent strategies are *linear*, so disallow backtracking. One reason to introduce *symmetry* in games and strategies. Then can support (co)monads *up to symmetry* for copying, to break linearity, and achieve cartesian-closed bicategories of strategies, *e.g.* generalising the categories of AJM and HO games. There are relations with *homotopy*.

In games with symmetry the Scott partial order $\sqsubseteq$ becomes the Scott category, with non-trivial hom-sets. All the operations mentioned generalise to games with symmetry. There remains the issue of how to adjoin (co)monads systematically to the metalanguage.

Other issues: *May* and *Must* testing and equivalences. Concurrent strategies extend to *winning conditions, pay-off, imperfect-information, probabilistic* and *quantum games*. Extensions to the metalanguage. Encoding of the differential lambda calculus?

Stopping configurations

For 'may' and 'must' equivalence it is not necessary to use partial strategies; it's sufficient to carry with a strategy the extra structure of 'stopping' configurations ($=$ images of $+/0$-maximal configurations in a partial strategy). Composition on strategies extends to composition on strategies with stopping configurations.

Let $\sigma : S \to N \| A$ be a partial strategy. Its defined part is a strategy σ_0. Define the (possibly) *stopping* configurations in $\mathcal{C}^\infty(S_0)$ to be

$$\mathrm{Stop}(\sigma) =_{\mathrm{def}} \{\mathrm{p\,x} \mid \mathrm{x} \in \mathcal{C}^\infty(\mathrm{S}) \text{ is } +/0\text{-maximal}\} .$$

The operation $\sigma \mapsto (\sigma_0, \mathrm{Stop}(\sigma))$ preserves composition.

Deadlocks in composition

Composition of strategies can introduce deadlock which is presently undetected:

Example 1 Deadlock may arise in a composition $\tau \odot \sigma$ through $\sigma : A \nrightarrow B$ and $\tau : B \nrightarrow C$ imposing incompatible causal dependencies between events in B.

Example 2 For games $B = \oplus \| \oplus$ and $C = \oplus$,

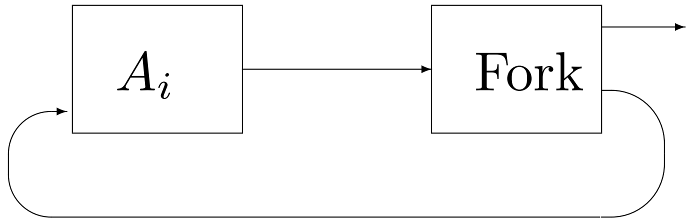
strategy $\sigma_1 : \emptyset \nrightarrow B$ nondeterministically chooses the right or left move in B,

strategy $\sigma_2 : \emptyset \nrightarrow B$ chooses just the right move in B,

strategy $\tau : B \nrightarrow C$ yields output in C if gets the right event of B as input.

The two strategy compositions $\tau \odot \sigma_1$ and $\tau \odot \sigma_2$ are indistinguishable.

Nondeterministic dataflow—the Brock-Ackerman anomaly

$C[A_i] =$ 

Both nondeterministic processes

$$A_1 = O + OIO \qquad \text{and} \qquad A_2 = O + IOO$$

have the same I/O relation, comprising

$$(\varepsilon, O),\ (I, O),\ (I, OO)\ .$$

But

$$C[A_1] = O + OO \qquad \text{and} \qquad C[A_2] = O\ .$$