Identifying isomorphic propositions

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Motivation

Definitionally equivalent

"Definitional equality is the equivalence relation generated by abbreviatory definitions" [Martin-Löf, 1980]

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Isomorphism

$$A \equiv B \quad \text{iff} \quad \exists f, g \text{ s.t. } \begin{cases} f \circ g = Id_A \\ g \circ f = Id_B \end{cases}$$

$$\text{e.g.} \qquad A \wedge B \equiv B \wedge A$$

Isomorphism is stronger that "definitionally equivalent"

e.g. $A \wedge B \equiv B \wedge A$ but $\langle \mathbf{r}, \mathbf{s} \rangle \not=_{\text{def}} \langle \mathbf{s}, \mathbf{r} \rangle$

So if $\mathbf{r} : (A \wedge B) \Rightarrow C$ and $\mathbf{s} : B \wedge A$

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Our goal is to identify isomorphic types

Outline

Part 1: A type-isomorphic lambda-calculus

- Defining the system
- Normalisation
- Future and ongoing work

Part 2: Relation with probabilistic calculi

- General technique
- Application to our particular case

The basic setting

- Simply types with conjunction and implication

$$A, B, C ::= X \mid A \Rightarrow B \mid A \wedge B$$

- An equivalence relation between types (based on the isomorphisms)
  1. $A \wedge B \equiv B \wedge A$
  2. $A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$
  3. $A \Rightarrow (B \wedge C) \equiv (A \Rightarrow B) \wedge (A \Rightarrow C)$
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Associative and commutative conjunction

$$\frac{\Gamma \vdash \mathbf{r} : A \quad \Gamma \vdash \mathbf{s} : B}{\Gamma \vdash \ \langle \mathbf{r}, \mathbf{s} \rangle \ : A \wedge B} \ (\wedge_i)$$

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$$\langle \mathbf{r}, \langle \mathbf{s}, \mathbf{t} \rangle \rangle \leftrightarrows \langle \langle \mathbf{r}, \mathbf{s} \rangle, \mathbf{t} \rangle$$

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What about the elimination?

$$\frac{\Gamma \vdash \langle \mathbf{r}, \mathbf{s} \rangle : A \wedge B}{\Gamma \vdash \pi_1 \langle \mathbf{r}, \mathbf{s} \rangle : A} \ (\wedge_e)$$

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Moreover $\langle \mathbf{r}, \mathbf{s} \rangle = \langle \mathbf{s}, \mathbf{r} \rangle$ so $\pi_1 \langle \mathbf{r}, \mathbf{s} \rangle = \pi_1 \langle \mathbf{s}, \mathbf{r} \rangle$!!

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If $\Gamma\vdash \mathbf{r}:A$ then $\pi_A\langle \mathbf{r},\mathbf{s}\rangle \to \mathbf{r}$

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This induces non-determinism

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We are interested in the proof theory and both **r** and **s** are valid proofs of A

"the subject reduction property is more important than the uniqueness of results" [Dowek,Jiang'11]

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Distributivity of implication over conjunction

$$A \Rightarrow (B \wedge C) \quad \equiv \quad (A \Rightarrow B) \wedge (A \Rightarrow C)$$

induces

$$\lambda x^A.(\mathbf{r} + \mathbf{s}) \leftrightarrows \lambda x^A.\mathbf{r} + \lambda x^A.\mathbf{s} \qquad \text{and} \qquad \pi_{A \Rightarrow B}(\lambda x^A.\mathbf{r}) \leftrightarrows \lambda x^A.\pi_B(\mathbf{r})$$

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Example

$$\dfrac{\dfrac{\vdash \lambda x^{A\wedge B}.x : (A \wedge B) \Rightarrow (A \wedge B)}{\vdash \lambda x^{A\wedge B}.x : ((A \wedge B) \Rightarrow A) \wedge ((A \wedge B) \Rightarrow B)}\ (\equiv)}{\vdash \pi_{(A\wedge B)\Rightarrow A}(\lambda x^{A\wedge B}.x) : (A \wedge B) \Rightarrow A}\ (\wedge_e)$$

$$\pi_{(A\wedge B)\Rightarrow A}(\lambda x^{A\wedge B}.x) \leftrightarrows \lambda x^{A\wedge B}.\pi_A(x)$$

Three technical rules

Rules

- If $A \equiv B$, $\mathbf{r} \leftrightarrows \mathbf{r}[A/B]$
- If $\mathbf{r} =_\alpha \mathbf{r}'$, $\mathbf{r} \leftrightarrows \mathbf{r}'$

Example

$$\text{Let } A \equiv B \qquad \overbrace{\lambda x^A.\mathbf{r} + \lambda y^B.\mathbf{s}}^{A \Rightarrow (C_1 \wedge C_2)}$$

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$$\leftrightarrows \lambda x^A.(\mathbf{r} + \mathbf{s}[x/y][A/B])$$

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Rule If $\mathbf{r} : A \wedge B \wedge C$, $\pi_A(\mathbf{r}) \leftrightarrows \pi_A(\pi_{A \wedge B}(\mathbf{r}))$

Example

$$\pi_{A \Rightarrow B} \overbrace{(\lambda x^A.(y^B + x) + \mathbf{r})}^{(A \Rightarrow B) \wedge (A \Rightarrow A) \wedge C}$$

Three technical rules

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Rule If $\mathbf{r} : A \wedge B \wedge C$, $\pi_A(\mathbf{r}) \leftrightarrows \pi_A(\pi_{A \wedge B}(\mathbf{r}))$

Example

$$\pi_{A \Rightarrow B} \overbrace{(\lambda x^A.(y^B + x) + \mathbf{r})}^{(A \Rightarrow B) \wedge (A \Rightarrow A) \wedge C} \leftrightarrows \pi_{A \Rightarrow B}(\pi_{(A \Rightarrow A) \wedge (A \Rightarrow B)}(\lambda x^A.(y^B + x) + \mathbf{r}))$$
$$\hookrightarrow \pi_{A \Rightarrow B}(\lambda x^A.(y^B + x))$$
$$\leftrightarrows \lambda x^A.\pi_B(y^B + x) \;\hookrightarrow \lambda x^A.y^B$$

The full calculus

Types

$A, B, C ::= X \mid A \Rightarrow B \mid A \wedge B$

Equivalences

$$A \wedge B \equiv B \wedge A$$
$$(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$$
$$A \Rightarrow (B \wedge C) \equiv (A \Rightarrow B) \wedge (A \Rightarrow C)$$

Terms

$\mathbf{r}, \mathbf{s}, \mathbf{t} ::= x^A \mid \lambda x^A.\mathbf{r} \mid \mathbf{rs} \mid \mathbf{r} + \mathbf{s} \mid \pi_A(\mathbf{r})$

Reduction rules

$$(\lambda x^A.\mathbf{r})\mathbf{s} \hookrightarrow \mathbf{r}[\mathbf{s}/x]$$
$$\pi_A(\mathbf{r} + \mathbf{s}) \hookrightarrow \mathbf{r} \quad (\text{if } \Gamma \vdash \mathbf{r} : A)$$

$$\mathbf{r} + \mathbf{s} \leftrightarrows \mathbf{s} + \mathbf{r}$$
$$(\mathbf{r} + \mathbf{s}) + \mathbf{t} \leftrightarrows \mathbf{r} + (\mathbf{s} + \mathbf{t})$$
$$\lambda x^A.(\mathbf{r} + \mathbf{s}) \leftrightarrows \lambda x^A.\mathbf{r} + \lambda x^A.\mathbf{s}$$
$$\pi_{A \Rightarrow B}(\lambda x^A.\mathbf{r}) \leftrightarrows \lambda x^A.\pi_B(\mathbf{r})$$

Plus the technical rules

$$\frac{}{\Gamma, x : A \vdash x : A}\ (ax) \qquad \frac{\Gamma \vdash \mathbf{r} : A \quad A \equiv B}{\Gamma \vdash \mathbf{r} : B}\ (\equiv)$$

$$\frac{\Gamma, x : A \vdash \mathbf{r} : B}{\Gamma \vdash \lambda x^A.\mathbf{r} : A \Rightarrow B}\ (\Rightarrow_i) \qquad \frac{\Gamma \vdash \mathbf{r} : A \Rightarrow B \quad \Gamma \vdash \mathbf{s} : A}{\Gamma \vdash \mathbf{rs} : B}$$

$$\frac{\Gamma \vdash \mathbf{r} : A \quad \Gamma \vdash \mathbf{s} : B}{\Gamma \vdash \mathbf{r} + \mathbf{s} : A \wedge B}\ (\wedge_i) \qquad \frac{\Gamma \vdash \mathbf{r} : A \wedge B}{\Gamma \vdash \pi_A(\mathbf{r}) : A}\ (\wedge_e)$$

Theorem (Subject reduction)

If $\Gamma \vdash \boldsymbol{r} : A$ *and* $\boldsymbol{r} \to \boldsymbol{s}$ *then* $\Gamma \vdash \boldsymbol{s} : A$

where $\to$ is $\hookrightarrow$ or $\leftrightarrows$

Normalisation

r is in normal form, if it can only continue reducing by relation $\leftrightarrows$

Normal form

$$\mathrm{Red}(\mathbf{r}) = \{\mathbf{s} \mid \mathbf{r} \leftrightarrows^* \mathbf{r}' \hookrightarrow \mathbf{s}' \leftrightarrows^* \mathbf{s}\}$$

r in normal form if $\mathrm{Red}(\mathbf{r}) = \emptyset$

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Theorem (Strong normalisation)

If $\Gamma \vdash \boldsymbol{r} : A$ *then* $\boldsymbol{r}$ *strongly normalising*

Proof. Reducibility candidates

Neutral terms

Don't wanting to remain neutral

Premise: All terms are neutral, except the abstractions

$$x^A \mid \sout{\lambda x^A.\mathbf{r}} \mid \mathbf{rs} \mid \mathbf{r} + \mathbf{s} \mid \pi_A(\mathbf{r})$$

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$$\underbrace{\pi_{A\Rightarrow B}(\lambda x^A.\mathbf{r})}_{\in\mathcal{N}} \quad \leftrightarrows \quad \underbrace{\lambda x^A.\pi_B(\mathbf{r})}_{\notin\mathcal{N}}$$

$$\overbrace{\lambda x^A.\mathbf{r}+\lambda x^A.\mathbf{s}}^{\in\mathcal{N}} \quad \leftrightarrows \quad \overbrace{\lambda x^A.(\mathbf{r}+\mathbf{s})}^{\notin\mathcal{N}}$$

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$$\overbrace{\lambda x^A.\mathbf{r} + \lambda x^A.\mathbf{s}} \leftrightarrows \overbrace{\lambda x^A.(\mathbf{r} + \mathbf{s})}$$

Premise': All terms are neutral, except those equivalent to abstractions

Inductively:

- If $\mathbf{r} \not\leftrightarrows^* \mathbf{r}_1 + \mathbf{r}_2$ and $\mathbf{r} \not\leftrightarrows^* \lambda x^A.\mathbf{r}'$, then $\mathbf{r} \in \mathcal{N}$
- If $\mathbf{r}_1 \in \mathcal{N}$, $\mathbf{r}_1 + \mathbf{r}_2 \in \mathcal{N}$

Finding an interpretation

The standard interpretation does not work

$$\boxed{[\![A \wedge B]\!] = \{\mathbf{r} \mid \pi_A(\mathbf{r}) \in [\![A]\!] \text{ and } \pi_B(\mathbf{r}) \in [\![B]\!]\}}$$

Counter-example: $\mathbf{r} = x^A + y^B + \Omega \in [\![A \wedge B]\!]$

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How to prove that $\mathbf{r} \in [\![A]\!]$ and $\mathbf{s} \in [\![B]\!]$ implies $\mathbf{r} + \mathbf{s} \in [\![A \wedge B]\!]$?

$\pi_A(\mathbf{r} + \mathbf{s}) \to \mathbf{r}$

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How to prove that $\mathbf{r} \in [\![A]\!]$ and $\mathbf{s} \in [\![B]\!]$ implies $\mathbf{r} + \mathbf{s} \in [\![A \wedge B]\!]$?

$\pi_A(\mathbf{r} + \mathbf{s}) \to \mathbf{r}$ but... $\pi_{A \wedge B}(\underbrace{\mathbf{r}_1 + \mathbf{r}_2}_{A \wedge B} + \underbrace{\mathbf{s}_1 + \mathbf{s}_2}_{B \wedge C}) \to \mathbf{r}_1 + \mathbf{s}_1$

We need something more subtle

Finding an interpretation

Interpreting canonical types

Singleton type $S ::= X \mid A \Rightarrow S$

Lemma

$$\forall A, \qquad A \equiv \bigwedge_{i=1}^{n} S_i$$

Proof (idea) $A \Rightarrow (B \wedge C) \equiv (A \Rightarrow B) \wedge (A \Rightarrow C)$

Finding an interpretation

Singleton type $S ::= X \mid A \Rightarrow S$

Lemma

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Proof (idea) $A \Rightarrow (B \wedge C) \equiv (A \Rightarrow B) \wedge (A \Rightarrow C)$

Canonical form

A^c: canonical form of A (given by the lemma)

$$[\![X]\!] = \mathcal{SN}$$

$$[\![A \Rightarrow S]\!] = \{\mathbf{r} \mid \forall \mathbf{s} \in [\![A^c]\!], \mathbf{rs} \in [\![S]\!]\}$$

$$\left[\!\!\left[\bigwedge_{i=1}^{n} S_i\right]\!\!\right] = \{\mathbf{r} : A \mid \forall i, \pi_{S_i}(\mathbf{r}) \in [\![S_i]\!]\} \qquad \text{with } n > 1$$

Normalisation ✓

Ongoing work

1. Add the missing isomorphism: currification

$$\boxed{(A \wedge B) \Rightarrow C \equiv A \Rightarrow B \Rightarrow C}$$

New rule:

$$\lambda x^A.\lambda y^B.\mathbf{r} \quad \leftrightarrows \quad \lambda z^{A\wedge B}.\mathbf{r}[\pi_A(z)/x][\pi_B(z)/y]$$

Modified beta rule:

$$\text{If } \mathbf{s} : A, \text{ then } (\lambda x^A.\mathbf{r})\mathbf{s} \to \mathbf{r}[\mathbf{s}/x]$$

Future work

2. Move to System F

Not trivial: our interpretation is not stable under substitution

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Room for improvement:

To find an interpretation of $\wedge$ stable under substitution

Part 2: Relation with probabilistic calculi

Non-deterministic vs. Probabilistic λ-calculus

Non-determinism	**Probabilities**
$\mathbf{r} + \mathbf{s}$ non-deterministic superposition (run **r** or **s**, non-deterministically)	$p.\mathbf{r} + q.\mathbf{s}$ probabilistic superposition (run **r** with probability p or **s** with probability q)

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- Non-deterministic projector
- Logical characterisation
- Quantitative characterisation in LL
- Etc.

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Goal: To move from ND to Prob. without loosing the connections with logic

Outline

Goal: To move from Non-determinism to Probilities

- General technique
- Application to our particular case

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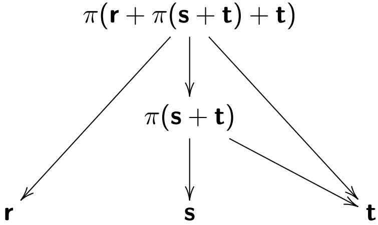
Intuition

From non-determinism to probabilities



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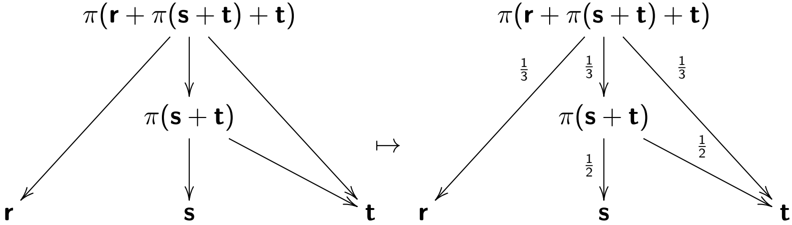
From non-determinism to probabilities



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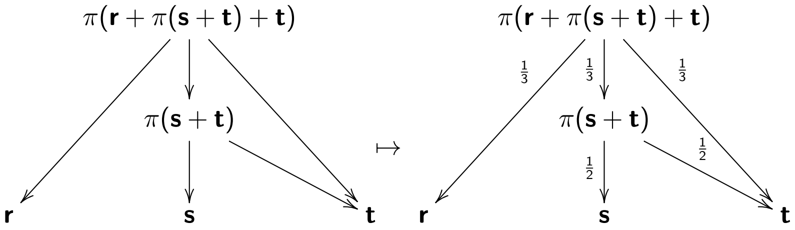

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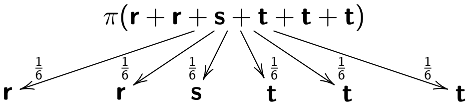
An easier way...



Intuition

Generalising the problem to abstract rewrite systems

Idea: to define a variant of a Lebesgue measure for sets of real numbers, on the space of traces

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1st Define an intuitive measure on single rewrites

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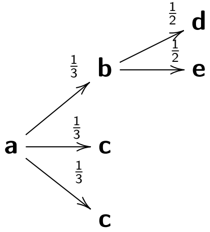

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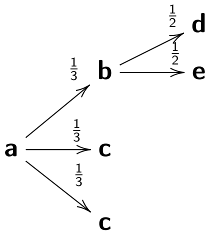

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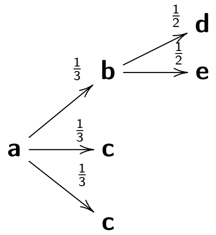

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Formalisation

Strategies

Λ: set of objects $\quad \rightarrow: \Lambda \times \Lambda \rightarrow \mathbb{N} \quad \mathbf{a} \rightarrow \mathbf{b}$ notation for $\rightarrow(\mathbf{a}, \mathbf{b}) \neq 0$.

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Degree

$$\rho(\mathbf{a}) = \sum_{\mathbf{b}} \to (\mathbf{a}, \mathbf{b})$$

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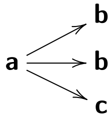

$\rho(\mathbf{a}) = 3$

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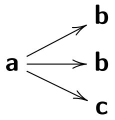
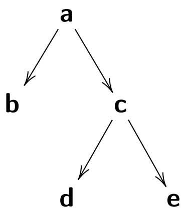


$\rho(\mathbf{a}) = 3$

Strategy

$f(\mathbf{a}) = \mathbf{b}$ implies $\mathbf{a} \to \mathbf{b}$ $\quad$ Ω = set of all the strategies

e.g. Rewrite system



$\Omega = \{f, g, h, i\}$, with

$$f(\mathbf{a}) = \mathbf{b} \qquad g(\mathbf{a}) = \mathbf{b}$$
$$f(\mathbf{c}) = \mathbf{d} \qquad g(\mathbf{c}) = \mathbf{e}$$

$$h(\mathbf{a}) = \mathbf{c} \qquad i(\mathbf{a}) = \mathbf{c}$$
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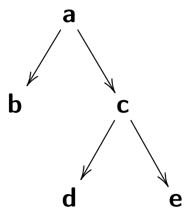
Formalisation

Boxes

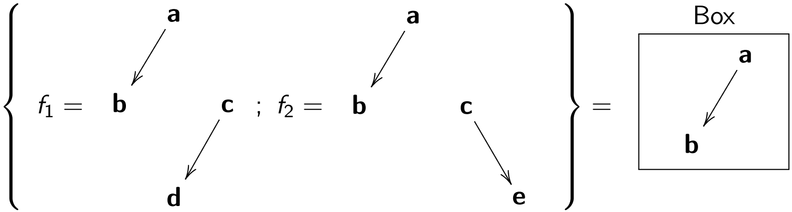
e.g. Rewrite system:



Box

$B \subseteq \Omega$ of the form

$$B = \{f \mid f(\mathbf{a}_1) = \mathbf{b}_1, \dots, f(\mathbf{a}_n) = \mathbf{b}_n\}$$



$$\{f_1; f_2\} = \{f \mid f(\mathbf{a}) = \mathbf{b}\}$$

Formalisation

Measure on boxes

Measure on boxes

If $B = \{f \mid f(\mathbf{a}_1) = \mathbf{b}_1, \ldots, f(\mathbf{a}_n) = \mathbf{b}_n\}$ then

$$\mathrm{p}(B) = \prod_{i=1}^{n} \frac{\to(\mathbf{a}_i, \mathbf{b}_i)}{\rho(\mathbf{a}_i)} \quad \left(\begin{array}{c} \to(\mathbf{a}_i, \mathbf{b}_i) \\ \text{ways to arrive to } \mathbf{b}_i \text{ from } \mathbf{a}_i \\ \rho(\mathbf{a}_i) \\ \text{nb. of rewrites from } \mathbf{a}_i \end{array}\right)$$

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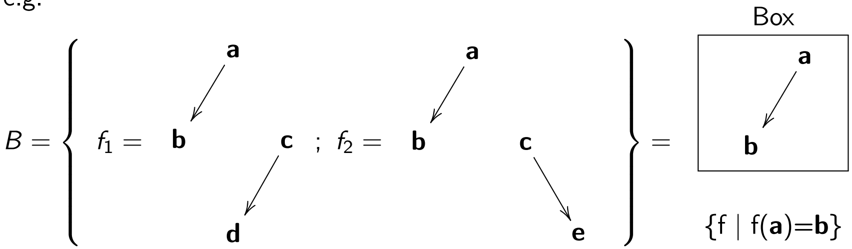

$$\mathrm{p}(B) = \frac{\rightarrow(\mathbf{a}, \mathbf{b})}{\rho(\mathbf{a})} = \frac{1}{2}$$

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Generalising the problem to abstract rewrite systems

Idea: to define a variant of a Lebesgue measure for sets of real numbers, on the space of traces

1st Define an intuitive measure on boxes

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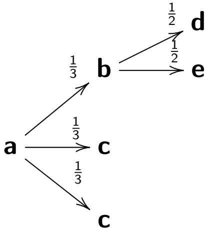

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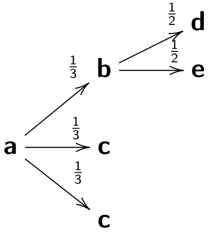

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Formalisation

Probability function

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Let $S \in \mathcal{P}(\Omega)$, $S \neq \emptyset$

$$\mathrm{P}(\emptyset) = 0$$

$$\mathrm{P}(S) = \inf \left\{ \sum_{B \in \mathcal{C}} \mathrm{p}(B) \mid \mathcal{C} \text{ is a countable family of boxes s.t. } S \subseteq \bigcup_{B \in \mathcal{C}} B \right\}$$

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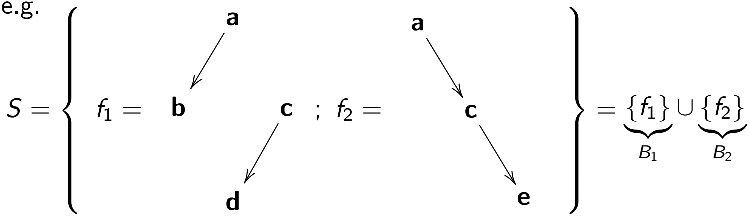

$$\boxed{\mathrm{P}(S) = \mathrm{p}(B_1) + \mathrm{p}(B_2) = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}}$$

Formalisation

Lebesgue measure and probability space

Lebesgue measurable

A is Lebesgue measurable if $\forall S \in \mathcal{P}(\Omega)$

$$\mathtt{P}(S) = \mathtt{P}(S \cap A) + \mathtt{P}(S \cap A^{\sim})$$

$\mathcal{A} = \{A \subseteq \Omega \mid A \text{ is Lebesgue measurable}\}$

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Theorem

$(\Omega, \mathcal{A}, \mathrm{P})$ *is a probability space*

- Ω is the set of all possible strategies
- $\mathcal{A}$ is the set of events
- P is the probability function

Proof.

We show that it satisfies the Kolmogorov axioms. □

Outline

Goal: To move from Non-determinism to Probilities

- General technique
- Application to our particular case

From non-determinism to probabilities

The calculus λ_+ (Polymorphic version)

$$A, B, C ::= X \mid A \Rightarrow B \mid A \wedge B \mid \forall X.A$$

$$\mathbf{r}, \mathbf{s}, \mathbf{t} ::= x^A \mid \lambda x^A.\mathbf{r} \mid \mathbf{rs} \mid \mathbf{r} + \mathbf{s} \mid \pi_A(\mathbf{r}) \mid \Lambda X.\mathbf{r} \mid \mathbf{r}\{A\}$$

$$\mathbf{r} : A \qquad \pi_A(\mathbf{r} + \mathbf{s}) \to \mathbf{r}$$

Non-determinism:

If $\mathbf{r} : A \quad \mathbf{s} : A$

$$\pi_A(\mathbf{r} + \mathbf{s})$$

$$\mathbf{r} \swarrow \qquad \searrow \mathbf{s}$$

From non-determinism to probabilities

The calculus λ_+^p

ARS $\lambda_+^{\downarrow}$

- Closed normal terms of λ_+ are objects of $\lambda_+^{\downarrow}$
- If $\mathbf{r}_1, \ldots, \mathbf{r}_n$ are objects, then $\pi_A(\mathbf{r}_1 + \cdots + \mathbf{r}_n)$ is an object

The rewrite rules have multiplicities: e.g.$\pi_A(\mathbf{r} + \mathbf{r}) \to \mathbf{r}$ with multiplicity 2

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Theorem

$(\Omega, \mathcal{A}, \mathrm{P})$: *probability space over* $\lambda_+^{\downarrow}$
$B_{\mathbf{r}_i} = \{f \mid f(\pi_A(\sum_{j=1}^n m_j.\mathbf{r}_j)) = \mathbf{r}_i\}$: *a box*

$$\mathrm{P}(B_{\mathbf{r}_i}) = \frac{m_i}{\sum_{j=1}^n m_j}$$

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$$\mathrm{P}(B_{\mathbf{r}_i}) = \frac{m_i}{\sum_{j=1}^n m_j}$$

Probabilistic calculus λ_+^p

Replace rule "If $\mathbf{r} : A$, then $\pi_A(\mathbf{r} + \mathbf{s}) \to \mathbf{r}$" by
$\pi_A(\sum_{i=1}^n m_i.\mathbf{r}_i + \mathbf{s}) \to \mathbf{r}_i$ with probability $\frac{m_i}{\sum_{i=1}^n m_j}$

From non-determinism to probabilities

$\lambda_+^p \leftarrow$ **Alg**

Algebraic calculi (Probabilistic version)

$$\mathbf{r},\mathbf{s},\mathbf{t} ::= \quad x^A \mid \lambda x^A.\mathbf{r} \mid \mathbf{rs} \mid \Lambda X.\mathbf{r} \mid \mathbf{r}\{A\} \mid \sum_{i=1}^{n} p_i.\mathbf{r}_i \qquad \text{with} \begin{cases} n > 0, \\ p_i \in \mathbb{Q}(0,1] \\ \sum_{i=1}^{n} p_i = 1 \end{cases}$$

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$$\mathbf{r},\mathbf{s},\mathbf{t} \ ::= \quad x^A \mid \lambda x^A.\mathbf{r} \mid \mathbf{rs} \mid \Lambda X.\mathbf{r} \mid \mathbf{r}\{A\} \mid \sum_{i=1}^{n} p_i.\mathbf{r}_i \qquad \text{with} \begin{cases} n > 0, \\ p_i \in \mathbb{Q}(0,1] \\ \sum_{i=1}^{n} p_i = 1 \end{cases}$$

From Alg to λ_+^p

$$[\![\sum_{i=1}^{n} \frac{n_i}{d_i}.\mathbf{r}_i]\!] = \pi_A(\sum_{i=1}^{n} m_i.[\![\mathbf{r}_i]\!]) \quad \text{where} \begin{cases} \mathbf{r}_i : A \\ n_i, d_i \in \mathbb{N}^* \\ m_i = n_i(\prod_{\substack{k=1 \\ k\neq i}}^{n} d_k) \end{cases} \text{for } i = 1, \dots, n$$

Theorem (Alg to λ_+^p)

If $\boldsymbol{r} \to^* \sum_{i=1}^{n} p_i.\boldsymbol{s}_i$ in Alg and $[\![\boldsymbol{s}_i]\!] \to^* \boldsymbol{t}_i$,
then $[\![\boldsymbol{r}]\!] \to^* \boldsymbol{t}_i$ with probability p_i in λ_+^p.

From non-determinism to probabilities

$\lambda_+^p \to$ **Alg**

Algebraic calculi (Probabilistic version)

$$\mathbf{r}, \mathbf{s}, \mathbf{t} \ ::= \quad x^A \mid \lambda x^A.\mathbf{r} \mid \mathbf{rs} \mid \Lambda X.\mathbf{r} \mid \mathbf{r}\{A\} \mid \sum_{i=1}^{n} p_i.\mathbf{r}_i \qquad \text{with} \begin{cases} n > 0, \\ p_i \in \mathbb{Q}(0,1] \\ \sum_{i=1}^{n} p_i = 1 \end{cases}$$

From λ_+^p to Alg

If $\pi_A(\mathbf{r}) \to \mathbf{s}_i$ with probability p_i, for $i = 1, \ldots, n$, $\quad (\!|\pi_A(\mathbf{r})|\!) = \sum_{i=1}^{n} p_i.(\!|\mathbf{s}_i|\!)$

Remark: if $\pi_A(\mathbf{r})$ normal, there is no translation

Theorem (λ_+^p to Alg)

- *If* $\mathbf{r} \to \mathbf{s}$*, with probability* 1*, then* $(\!|\mathbf{r}|\!) \to (\!|\mathbf{s}|\!)$
- *If* $\mathbf{r} \to \mathbf{s}_i$ *with probability* p_i*, for* $i = 1, \ldots, n$*, then* $(\!|\mathbf{r}|\!) = \sum_{i=1}^{n} p_i.(\!|\mathbf{s}_i|\!)$. □

Sumarising

Part 1: Isomorphisms

- We introduced a new calculus where isomorphic propositions have the same proofs
- We provided a proof of strong normalisation for simply types

Part 2: From non-determinism to probabilities

- We provide a general technique to transform a non-deterministic calculus into a probabilistic one
- We have a way to transform λ_+ into λ_+^p
- We get a simpler calculus, encoding an algebraic calculus, without losing the connections with logic