Automated method in proving graph properties

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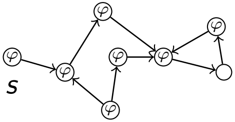
Outline

Motivation



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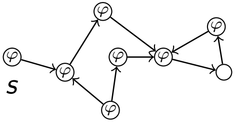

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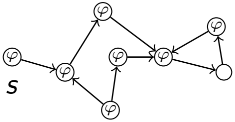

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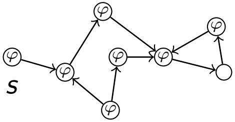

is there an infinite path such that all the vertices of the path are in φ*?*

- by graph traversal algorithm
- model checking of $G, s \models EG\varphi$
- It can also be expressed as

$$\exists Y\Big(s \in Y \wedge \forall x\Big(x \in Y \Rightarrow \varphi(x) \wedge \exists x'(\textbf{edge}(x, x') \wedge x' \in Y)\Big)\Big)$$

Ordered Polarized Resolution Modulo

$$\text{Resolution}\ \frac{P \vee C \qquad \neg Q \vee D}{\sigma(C \vee D)}\ a, b, c \qquad \text{Factoring}\ \frac{L \vee K \vee C}{\sigma(L \vee C)}\ d$$

$$\text{Ext.Narr.}\ \frac{P \vee C}{\sigma(D \vee C)}\ a, b, \text{ and } \underline{\neg Q} \vee D \text{ is a one-way clause of } \mathcal{R}$$

$$\text{Ext.Narr.}\ \frac{\neg Q \vee D}{\sigma(C \vee D)}\ a, c, \text{ and } \underline{P} \vee C \text{ is a one-way clause of } \mathcal{R}$$

[a] $\sigma = mgu(P, Q)$ [b] $P \in \delta(P \vee C)$ [c] $\neg Q \in \delta(\neg Q \vee D)$
[d] L and K maximal in $L \vee K \vee C, \sigma = mgu(L, K)$ [e] L maximal in $L \vee C$

Figure: Inference rule of the $\mathrm{OPRM}^{\succ}_{\mathcal{R}}$

Language

Definition

The two-sorted language $\mathcal{L}$ with one sort for vertices and one sort for classes of vertices contains

- constants $s_1, \ldots, s_n$ for the vertices of the graph
- a binary predicate symbol **edge**
- a binary predicate symbol $\in$
- a constant $\varnothing$
- a binary function symbol *add*

The theory of a graph

Definition (The theory $\mathcal{T}_G$)

The theory $\mathcal{T}_G$ for a graph G contains

- $\mathbf{edge}(s_i, s_j)$;
- $\varphi(s_i)$ or $\neg varphi(s_i)$;
- $s_i = s_i$ and $\neg s_i = s_j$;
- $\forall x. \neg x \in \varnothing$,
  $\forall x \forall y \forall Z (x \in add(y, Z) \Leftrightarrow (x = y \vee x \in Z))$.

Clausal form of the theory

$Q(s_1, \ldots, s_k)$	$\underline{Q(s_1, \ldots, s_k)}$
$\neg Q(s_1, \ldots, s_k)$	$\underline{\neg Q(s_1, \ldots, s_k)}$
$\forall x. \neg x \in \varnothing$	$\underline{\neg(x \in \varnothing)}$
$\forall x \forall y \forall Z (x \in add(y, Z) \Leftrightarrow (x = y \vee x \in Z))$	$\underline{\neg x \in add(y, Z)} \vee x = y \vee x \in Z$
	$\underline{x \in add(x, Z)}$
	$\underline{x \in add(y, Z)} \vee \neg x \in Z$
$s_i = s_i$	$\underline{s_i = s_i}$
$\neg s_i = s_j$	$\underline{\neg s_i = s_j}$

Outline

Which kind of problem can be expressed by $\mathcal{L}$

- Reachability or connectivity,
- k-colorable problem,
- bisimulation problem,
- problems can be expressed by Temporal logic, ...

Reachability

Example

For two vertices s_1, s_2 in a graph G, the property that starting from s_1, s_2 is reachable can be expressed as

$$\forall Y\Big(s_1 \in Y \wedge \forall x\Big(x \in X \Rightarrow \forall y(\mathbf{edge}(x,y) \Rightarrow y \in Y)\Big) \Rightarrow s_2 \in Y\Big).$$

and s_2 is non-reachable can be expressed as

$$\exists Y\Big(s_1 \in Y \wedge \forall x\Big(x \in Y \Rightarrow (\neg x = s_2 \wedge \forall y(\mathbf{edge}(x,y) \Rightarrow y \in Y))\Big)\Big).$$

3-colorable

Example (3-colorable)

The property that graph G is 3-colorable can be expressed as:

$$\exists X, Y, Z \Big(Part(X, Y, Z) \wedge \forall x \forall y \Big(edge(x, y) \wedge \neg x = y \Rightarrow$$
$$\neg(x \in X \wedge y \in X) \wedge \neg(x \in Y \wedge y \in Y) \wedge \neg(x \in Y \wedge y \in Y) \Big) \Big)$$

where $Part(X, Y, Z)$ expresses that (X, Y, Z) is a partition of the domain. The formula $Part(X, Y, Z)$ is written as follows:

$$\forall x \Big((x \in X \vee x \in Y \vee x \in Z) \wedge$$
$$(\neg(x \in X \wedge x \in Y) \wedge \neg(x \in Y \wedge x \in Z) \wedge \neg(x \in X \wedge x \in Z)) \Big)$$

Bisimulation

Example

Given a graph G, the property that two vertices s_1 and s_2 in G are bisimilar can be expressed as:

$$\exists B\Big((s_1, s_2) \in B \wedge \forall x \forall y \Big((x, y) \in B \Rightarrow \\ (\forall x'(\mathbf{edge}(x, x') \Rightarrow \exists y''(\mathbf{edge}(y, y'') \wedge (x', y'') \in B)) \\ \wedge \forall y'(\mathbf{edge}(y, y') \Rightarrow \exists x''(\mathbf{edge}(x, x'') \wedge (x'', y') \in B)))\Big)\Big).$$

B is a class of binary tuple of vertices.

Which kind of problem do we consider?

Properties for single vertices in a graph which can be defined inductively or coinductively.

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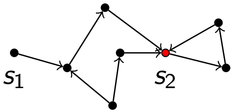
For example the reachability between two vertices can be defined inductively,



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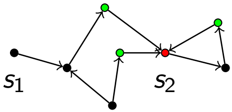

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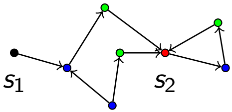

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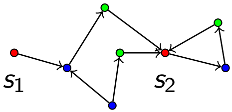

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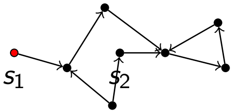

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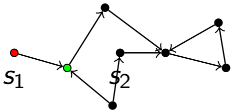

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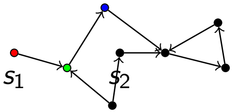

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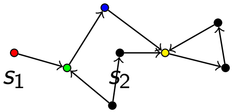

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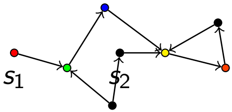

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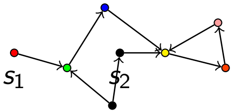

The necessity of coinductive definition

For the two examples before, as we can prove that whether two vertices are reachable by the inductive definition, why should we care about the coinductive definition?

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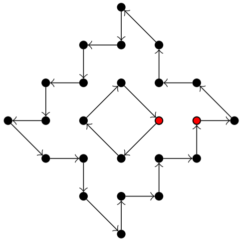

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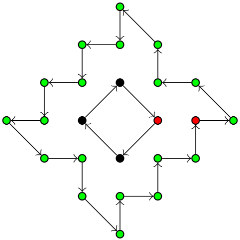

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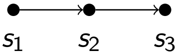

Example

Given a graph as above. We prove that starting from s_1, s_3 is reachable. The property can be expressed by

$$\forall Y \Big(s_1 \in Y \wedge \forall x \Big(x \in X \Rightarrow \forall y \big(\mathbf{edge}(x, y) \Rightarrow y \in Y \big) \Big) \Rightarrow s_3 \in Y \Big).$$

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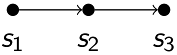

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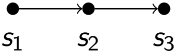
The negation of it is

$$s_1 \in Y_c \tag{1a}$$

$$\neg x \in Y_c \vee \neg edge(x, y) \vee y \in Y_c \tag{1b}$$

$$\neg s_3 \in Y_c \tag{1c}$$



$$\dfrac{\dfrac{\dfrac{\dfrac{\neg s_3 \in Y_c \qquad \neg x \in Y_c \vee \neg \mathbf{edge}(x,y) \vee y \in Y_c}{\neg x \in Y_c \vee \neg \mathbf{edge}(x, s_3) \qquad \underline{\mathbf{edge}(s_2, s_3)}}}{\neg s_2 \in Y_c \qquad \neg x \in Y_c \vee \neg \mathbf{edge}(x,y) \vee y \in Y_c}}{\neg x \in Y_c \vee \neg \mathbf{edge}(x, s_2) \qquad \underline{\mathbf{edge}(s_1, s_2)}}}{\dfrac{\neg s_1 \in Y_c \qquad s_1 \in Y_c}{\square}}$$

Coinductive formulas

Definition (coinductive definition)

For any unary predicate P, if the set T of vertices can be expressed by the formula

$$\forall v(v \in T \Rightarrow (P(v) \wedge \exists v'(\mathbf{edge}(v, v') \wedge v' \in T))), \text{ or}$$

$$\forall v(v \in T \Rightarrow (P(v) \wedge \forall v'(\mathbf{edge}(v, v') \Rightarrow v' \in T))),$$

we say that T is defined coinductively. And we denote these two kinds of formulas by co-rule(v, T).

Coinductive formulas

Definition (coinductive formula)

Formulas of the form

$$\exists X(s \in X \wedge \forall x.\text{co-rule}(x, X))$$

are called coinductive formulas.

Coinductive formulas

Definition (coinductive clause, traversal clause)

- The clausal form of the negation of coinductive formulas are called coinductive clauses (co-clause).
- The set of traversal clauses is inductively defined as the set containing all the co-clauses and closed by the inference rules of $\mathsf{OPRM}^{\succ}_{\mathcal{R}}$.

Coinductive formulas

Definition ((non-)traversed literal, rule-literal, v-literal, ex-literal)

- Literals of the form $\neg s \in Y / s \in Y$ are called traversed/non-traversed literals.
- Literals of the form $\neg$co-rule(s, Y) are called rule-literals.
- The literals with only vertex terms are called v-literals.
- The literals that are renamings of some other formulas by not rule-literals are called ex-literals.

Coinductive formulas

Definition (next-to-traverse vertex (literal))

Given a clause C, if both traversed literal $\neg s \in Y$ and $\neg$co-rule(s, Y) are in C, we denote s as the next-to-traverse vertex of C. The function ntv is defined as

$$\text{ntv}(C) = \{s \mid \text{both } \neg s \in Y \text{ and } \neg\text{co-rule}(s, Y) \text{ are literals of C}\}.$$

And we name the rule-literal in C with a next-to-traverse vertex as next-to-traverse-literal.

Selection function

Definition (selection function)

$$\delta(C) = \begin{cases} a = \{l \mid l \text{ is a v-literal}\} & a \neq \emptyset, \\ b = \{l \mid l \text{ is an ex-literal}\} & a = \emptyset \text{ and } b \neq \emptyset, \\ c = \{l \mid l \text{ is a next-to-traverse-literal}\} & a, b = \emptyset \text{ and } c \neq \emptyset, \\ d = \{l \mid l \text{ is a rule-literal}\} & a, b, c = \emptyset \text{ and } d \neq \emptyset, \\ e = \{l \mid l \text{ is a traversed literal}\} & a, b, c, d = \emptyset \text{ and } e \neq \emptyset, \\ f = \{l \mid l \text{ is a non-traversed literal}\}. & \textit{otherwise} \end{cases}$$

Example

Example

Given a graph as above. We prove that starting from s_1, s_3 is unreachable.

$$\exists Y\Big(s_1 \in Y \wedge \forall x\Big(x \in Y \Rightarrow (\neg x = s_3 \wedge \forall y(\mathbf{edge}(x,y) \Rightarrow y \in Y))\Big)\Big).$$

After renaming and removing object quantifier $\forall$, the formula is

$$\exists Y\Big(s_1 \in Y \wedge \bigwedge_{1\leqslant i\leqslant 3} \text{co-rule}(s_i, Y)\Big) \tag{2a}$$

$$\wedge\ \forall x \forall Y\Big(\text{co-rule}(x, Y) \Leftrightarrow (x \in Y \Rightarrow (\neg x = s_3 \wedge \bigwedge_{1\leqslant i\leqslant 3} \psi(x, s_i, Y)))\Big) \tag{2b}$$

$$\wedge\ \forall x \forall y \forall Y\Big(\psi(x, y, Y) \Leftrightarrow (\mathbf{edge}(x, y) \Rightarrow y \in Y)\Big) \tag{2c}$$

The clausal form of the negation of (2a) is

$$\neg s_1 \in Y \vee \bigvee_{1 \leqslant i \leqslant 3} \neg \text{co-rule}(s_i, Y) \tag{3}$$

we take (2b) and (2c) as axioms. The $\Leftarrow$ part of (2b) can be transformed into one-way clauses

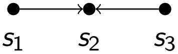
$$\underline{\text{co-rule}(x, Y)} \vee x \in Y \tag{4a}$$

$$\underline{\text{co-rule}(x, Y)} \vee x = s_3 \vee \bigvee_{1 \leqslant i \leqslant 3} \neg \psi(x, s_i, Y) \tag{4b}$$

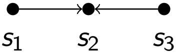
The $\Leftarrow$ part of (2c) can be transformed into one-way clauses

$$\underline{\psi(x, y, Y)} \vee \mathbf{edge}(x, y) \tag{5a}$$

$$\underline{\psi(x, y, Y)} \vee \neg y \in Y \tag{5b}$$



$$\dfrac{\neg s_1 \in Y \vee \neg\text{co-rule}(s_1, Y)^* \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \qquad \underline{\text{co-rule}(x, Y)} \vee x = s_3 \vee \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(x, s_i, Y}{\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee s_1 = s_3^* \vee \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(s_1, s_i, Y) \qquad \underline{\neg s_1 = s_3}}$$



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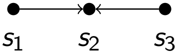

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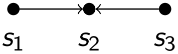

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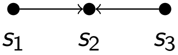
$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \neg\psi(s_1, s_2, Y) \vee \neg\psi(s_1, s_3, Y) \vee \textbf{edge}(s_1, s_1)^* \qquad \underline{\neg\textbf{edge}(s_1, s_1)}$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \neg\psi(s_1, s_2, Y)^* \vee \neg\psi(s_1, s_3, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \textbf{edge}(x, y)$$



$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_1, Y)^* \vee \neg \text{co-rule}(s_2, Y) \vee \neg \text{co-rule}(s_3, Y) \qquad \underline{\text{co-rule}(x, Y)} \vee x = s_3 \vee \bigvee_{1 \leqslant i \leqslant 3} \neg \psi(x, s_i, Y$$

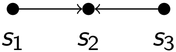
$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_2, Y) \vee \neg \text{co-rule}(s_3, Y) \vee s_1 = s_3^* \vee \bigvee_{1 \leqslant i \leqslant 3} \neg \psi(s_1, s_i, Y) \qquad \underline{\neg s_1 = s_3}$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_2, Y) \vee \neg \text{co-rule}(s_3, Y) \vee \bigvee_{1 \leqslant i \leqslant 3} \neg \psi(s_1, s_i, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \mathbf{edge}(x, y)$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_2, Y) \vee \neg \text{co-rule}(s_3, Y) \vee \neg \psi(s_1, s_2, Y) \vee \neg \psi(s_1, s_3, Y) \vee \mathbf{edge}(s_1, s_1)^* \qquad \underline{\neg \mathbf{edge}(s_1, s_1)}$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_2, Y) \vee \neg \text{co-rule}(s_3, Y) \vee \neg \psi(s_1, s_2, Y)^* \vee \neg \psi(s_1, s_3, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \mathbf{edge}(x, y)$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_2, Y) \vee \neg \text{co-rule}(s_3, Y) \vee \neg \psi(s_1, s_2, Y) \vee \mathbf{edge}(s_1, s_3)^* \qquad \underline{\neg \mathbf{edge}(s_1, s_3)}$$



$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_1, Y)^* \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \qquad \underline{\text{co-rule}(x, Y)} \vee x = s_3 \vee \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(x, s_i, Y$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee s_1 = s_3^* \vee \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(s_1, s_i, Y) \qquad \underline{\neg s_1 = s_3}$$

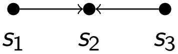
$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(s_1, s_i, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \mathbf{edge}(x, y)$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \neg\psi(s_1, s_2, Y) \vee \neg\psi(s_1, s_3, Y) \vee \mathbf{edge}(s_1, s_1)^* \qquad \underline{\neg\mathbf{edge}(s_1, s_1)}$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \neg\psi(s_1, s_2, Y)^* \vee \neg\psi(s_1, s_3, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \mathbf{edge}(x, y)$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \neg\psi(s_1, s_2, Y) \vee \mathbf{edge}(s_1, s_3)^* \qquad \underline{\neg\mathbf{edge}(s_1, s_3)}$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \neg\psi(s_1, s_2, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \neg y \in Y$$



$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_1, Y)^* \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \qquad \underline{\text{co-rule}(x, Y)} \vee x = s_3 \vee \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(x, s_i, Y$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee s_1 = s_3^* \vee \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(s_1, s_i, Y) \qquad \underline{\neg s_1 = s_3}$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(s_1, s_i, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \mathbf{edge}(x, y)$$

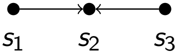
$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \neg\psi(s_1, s_2, Y) \vee \neg\psi(s_1, s_3, Y) \vee \mathbf{edge}(s_1, s_1)^* \qquad \underline{\neg\mathbf{edge}(s_1, s_1)}$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \neg\psi(s_1, s_2, Y)^* \vee \neg\psi(s_1, s_3, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \mathbf{edge}(x, y)$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \neg\psi(s_1, s_2, Y) \vee \mathbf{edge}(s_1, s_3)^* \qquad \underline{\neg\mathbf{edge}(s_1, s_3)}$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \neg\psi(s_1, s_2, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \neg y \in Y$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y)^* \vee \neg\text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \qquad \underline{\text{co-rule}(x, Y)} \vee x = s_3 \vee \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(x, s_i, Y)$$



$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_1, Y)^* \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \qquad \underline{\text{co-rule}(x, Y)} \vee x = s_3 \vee \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(x, s_i, Y$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee s_1 = s_3^* \vee \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(s_1, s_i, Y) \qquad \underline{\neg s_1 = s_3}$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(s_1, s_i, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \mathbf{edge}(x, y)$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \neg\psi(s_1, s_2, Y) \vee \neg\psi(s_1, s_3, Y) \vee \mathbf{edge}(s_1, s_1)^* \qquad \underline{\neg\mathbf{edge}(s_1, s_1)}$$

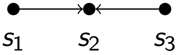
$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \neg\psi(s_1, s_2, Y)^* \vee \neg\psi(s_1, s_3, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \mathbf{edge}(x, y)$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \neg\psi(s_1, s_2, Y) \vee \mathbf{edge}(s_1, s_3)^* \qquad \underline{\neg\mathbf{edge}(s_1, s_3)}$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \neg\psi(s_1, s_2, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \neg y \in Y$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y)^* \vee \neg\text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \qquad \underline{\text{co-rule}(x, Y)} \vee x = s_3 \vee \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(x, s_i, Y)$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \vee s_2 = s_3^* \vee \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(s_2, s_i, Y) \qquad \underline{\neg s_2 = s_3}$$



$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_1, Y)^* \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \qquad \underline{\text{co-rule}(x, Y)} \vee x = s_3 \vee \textstyle\bigvee_{1 \leqslant i \leqslant 3} \neg\psi(x, s_i, Y$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee s_1 = s_3^* \vee \textstyle\bigvee_{1 \leqslant i \leqslant 3} \neg\psi(s_1, s_i, Y) \qquad \underline{\neg s_1 = s_3}$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \textstyle\bigvee_{1 \leqslant i \leqslant 3} \neg\psi(s_1, s_i, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \mathbf{edge}(x, y)$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \neg\psi(s_1, s_2, Y) \vee \neg\psi(s_1, s_3, Y) \vee \mathbf{edge}(s_1, s_1)^* \qquad \underline{\neg\mathbf{edge}(s_1, s_1)}$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \neg\psi(s_1, s_2, Y)^* \vee \neg\psi(s_1, s_3, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \mathbf{edge}(x, y)$$

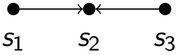
$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \neg\psi(s_1, s_2, Y) \vee \mathbf{edge}(s_1, s_3)^* \qquad \underline{\neg\mathbf{edge}(s_1, s_3)}$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \neg\psi(s_1, s_2, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \neg y \in Y$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y)^* \vee \neg\text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \qquad \underline{\text{co-rule}(x, Y)} \vee x = s_3 \vee \textstyle\bigvee_{1 \leqslant i \leqslant 3} \neg\psi(x, s_i, Y)$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \vee s_2 = s_3^* \vee \textstyle\bigvee_{1 \leqslant i \leqslant 3} \neg\psi(s_2, s_i, Y) \qquad \underline{\neg s_2 = s_3}$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \vee \textstyle\bigvee_{1 \leqslant i \leqslant 3} \neg\psi(s_2, s_i, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \mathbf{edge}(x, y)$$



$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_1, Y)^* \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \qquad \underline{\text{co-rule}(x, Y)} \vee x = s_3 \vee \bigvee_{1\leqslant i\leqslant 3} \neg\psi(x, s_i, Y$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee s_1 = s_3^* \vee \bigvee_{1\leqslant i\leqslant 3} \neg\psi(s_1, s_i, Y) \qquad \underline{\neg s_1 = s_3}$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \bigvee_{1\leqslant i\leqslant 3} \neg\psi(s_1, s_i, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \mathbf{edge}(x, y)$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \neg\psi(s_1, s_2, Y) \vee \neg\psi(s_1, s_3, Y) \vee \mathbf{edge}(s_1, s_1)^* \qquad \underline{\neg\mathbf{edge}(s_1, s_1)}$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \neg\psi(s_1, s_2, Y)^* \vee \neg\psi(s_1, s_3, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \mathbf{edge}(x, y)$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \neg\psi(s_1, s_2, Y) \vee \mathbf{edge}(s_1, s_3)^* \qquad \underline{\neg\mathbf{edge}(s_1, s_3)}$$

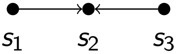
$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \neg\psi(s_1, s_2, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \neg y \in Y$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y)^* \vee \neg\text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \qquad \underline{\text{co-rule}(x, Y)} \vee x = s_3 \vee \bigvee_{1\leqslant i\leqslant 3} \neg\psi(x, s_i, Y)$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \vee s_2 = s_3^* \vee \bigvee_{1\leqslant i\leqslant 3} \neg\psi(s_2, s_i, Y) \qquad \underline{\neg s_2 = s_3}$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \vee \bigvee_{1\leqslant i\leqslant 3} \neg\psi(s_2, s_i, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \mathbf{edge}(x, y)$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \vee \neg\psi(s_2, s_2, Y) \vee \neg\psi(s_2, s_3, Y) \vee \mathbf{edge}(s_2, s_1)^* \qquad \underline{\neg\mathbf{edge}(s_2, s_1)}$$



$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_1, Y)^* \vee \neg \text{co-rule}(s_2, Y) \vee \neg \text{co-rule}(s_3, Y) \qquad \underline{\text{co-rule}(x, Y)} \vee x = s_3 \vee \bigvee_{1 \leqslant i \leqslant 3} \neg \psi(x, s_i, Y$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_2, Y) \vee \neg \text{co-rule}(s_3, Y) \vee s_1 = s_3^* \vee \bigvee_{1 \leqslant i \leqslant 3} \neg \psi(s_1, s_i, Y) \qquad \underline{\neg s_1 = s_3}$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_2, Y) \vee \neg \text{co-rule}(s_3, Y) \vee \bigvee_{1 \leqslant i \leqslant 3} \neg \psi(s_1, s_i, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \mathbf{edge}(x, y)$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_2, Y) \vee \neg \text{co-rule}(s_3, Y) \vee \neg \psi(s_1, s_2, Y) \vee \neg \psi(s_1, s_3, Y) \vee \mathbf{edge}(s_1, s_1)^* \qquad \underline{\neg \mathbf{edge}(s_1, s_1)}$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_2, Y) \vee \neg \text{co-rule}(s_3, Y) \vee \neg \psi(s_1, s_2, Y)^* \vee \neg \psi(s_1, s_3, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \mathbf{edge}(x, y)$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_2, Y) \vee \neg \text{co-rule}(s_3, Y) \vee \neg \psi(s_1, s_2, Y) \vee \mathbf{edge}(s_1, s_3)^* \qquad \underline{\neg \mathbf{edge}(s_1, s_3)}$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_2, Y) \vee \neg \text{co-rule}(s_3, Y) \vee \neg \psi(s_1, s_2, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \neg y \in Y$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_2, Y)^* \vee \neg \text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \qquad \underline{\text{co-rule}(x, Y)} \vee x = s_3 \vee \bigvee_{1 \leqslant i \leqslant 3} \neg \psi(x, s_i, Y)$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \vee s_2 = s_3^* \vee \bigvee_{1 \leqslant i \leqslant 3} \neg \psi(s_2, s_i, Y) \qquad \underline{\neg s_2 = s_3}$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \vee \bigvee_{1 \leqslant i \leqslant 3} \neg \psi(s_2, s_i, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \mathbf{edge}(x, y)$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \vee \neg \psi(s_2, s_2, Y) \vee \neg \psi(s_2, s_3, Y) \vee \mathbf{edge}(s_2, s_1)^* \qquad \underline{\neg \mathbf{edge}(s_2, s_1)}$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \vee \neg \psi(s_2, s_2, Y)^* \vee \neg \psi(s_2, s_3, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \mathbf{edge}(x, y)$$

$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_1, Y)^* \lor \neg\text{co-rule}(s_2, Y) \lor \neg\text{co-rule}(s_3, Y) \qquad \underline{\text{co-rule}(x, Y)} \lor x = s_3 \lor \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(x, s_i, Y$$

$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_2, Y) \lor \neg\text{co-rule}(s_3, Y) \lor s_1 = s_3^* \lor \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(s_1, s_i, Y) \qquad \underline{\neg s_1 = s_3}$$

$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_2, Y) \lor \neg\text{co-rule}(s_3, Y) \lor \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(s_1, s_i, Y)^* \qquad \underline{\psi(x, y, Y)} \lor \mathbf{edge}(x, y)$$

$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_2, Y) \lor \neg\text{co-rule}(s_3, Y) \lor \neg\psi(s_1, s_2, Y) \lor \neg\psi(s_1, s_3, Y) \lor \mathbf{edge}(s_1, s_1)^* \qquad \underline{\neg\mathbf{edge}(s_1, s_1)}$$

$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_2, Y) \lor \neg\text{co-rule}(s_3, Y) \lor \neg\psi(s_1, s_2, Y)^* \lor \neg\psi(s_1, s_3, Y)^* \qquad \underline{\psi(x, y, Y)} \lor \mathbf{edge}(x, y)$$

$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_2, Y) \lor \neg\text{co-rule}(s_3, Y) \lor \neg\psi(s_1, s_2, Y) \lor \mathbf{edge}(s_1, s_3)^* \qquad \underline{\neg\mathbf{edge}(s_1, s_3)}$$

$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_2, Y) \lor \neg\text{co-rule}(s_3, Y) \lor \neg\psi(s_1, s_2, Y)^* \qquad \underline{\psi(x, y, Y)} \lor \neg y \in Y$$

$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_2, Y)^* \lor \neg\text{co-rule}(s_3, Y) \lor \neg s_2 \in Y \qquad \underline{\text{co-rule}(x, Y)} \lor x = s_3 \lor \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(x, s_i, Y)$$

$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_3, Y) \lor \neg s_2 \in Y \lor s_2 = s_3^* \lor \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(s_2, s_i, Y) \qquad \underline{\neg s_2 = s_3}$$

$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_3, Y) \lor \neg s_2 \in Y \lor \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(s_2, s_i, Y)^* \qquad \underline{\psi(x, y, Y)} \lor \mathbf{edge}(x, y)$$

$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_3, Y) \lor \neg s_2 \in Y \lor \neg\psi(s_2, s_2, Y) \lor \neg\psi(s_2, s_3, Y) \lor \mathbf{edge}(s_2, s_1)^* \qquad \underline{\neg\mathbf{edge}(s_2, s_1)}$$

$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_3, Y) \lor \neg s_2 \in Y \lor \neg\psi(s_2, s_2, Y)^* \lor \neg\psi(s_2, s_3, Y)^* \qquad \underline{\psi(x, y, Y)} \lor \mathbf{edge}(x, y)$$

$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_3, Y) \lor \neg s_2 \in Y \lor \mathbf{edge}(s_2, s_2)^* \lor \neg\psi(s_2, s_3, Y) \qquad \underline{\neg s_2 = s_2}$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_1, Y)^* \vee \neg \text{co-rule}(s_2, Y) \vee \neg \text{co-rule}(s_3, Y) \qquad \underline{\text{co-rule}(x, Y)} \vee x = s_3 \vee \bigvee_{1 \leqslant i \leqslant 3} \neg \psi(x, s_i, Y$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_2, Y) \vee \neg \text{co-rule}(s_3, Y) \vee s_1 = s_3^* \vee \bigvee_{1 \leqslant i \leqslant 3} \neg \psi(s_1, s_i, Y) \qquad \underline{\neg s_1 = s_3}$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_2, Y) \vee \neg \text{co-rule}(s_3, Y) \vee \bigvee_{1 \leqslant i \leqslant 3} \neg \psi(s_1, s_i, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \textbf{edge}(x, y)$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_2, Y) \vee \neg \text{co-rule}(s_3, Y) \vee \neg \psi(s_1, s_2, Y) \vee \neg \psi(s_1, s_3, Y) \vee \textbf{edge}(s_1, s_1)^* \qquad \underline{\neg \textbf{edge}(s_1, s_1)}$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_2, Y) \vee \neg \text{co-rule}(s_3, Y) \vee \neg \psi(s_1, s_2, Y)^* \vee \neg \psi(s_1, s_3, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \textbf{edge}(x, y)$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_2, Y) \vee \neg \text{co-rule}(s_3, Y) \vee \neg \psi(s_1, s_2, Y) \vee \textbf{edge}(s_1, s_3)^* \qquad \underline{\neg \textbf{edge}(s_1, s_3)}$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_2, Y) \vee \neg \text{co-rule}(s_3, Y) \vee \neg \psi(s_1, s_2, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \neg y \in Y$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_2, Y)^* \vee \neg \text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \qquad \underline{\text{co-rule}(x, Y)} \vee x = s_3 \vee \bigvee_{1 \leqslant i \leqslant 3} \neg \psi(x, s_i, Y)$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \vee s_2 = s_3^* \vee \bigvee_{1 \leqslant i \leqslant 3} \neg \psi(s_2, s_i, Y) \qquad \underline{\neg s_2 = s_3}$$

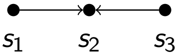
$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \vee \bigvee_{1 \leqslant i \leqslant 3} \neg \psi(s_2, s_i, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \textbf{edge}(x, y)$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \vee \neg \psi(s_2, s_2, Y) \vee \neg \psi(s_2, s_3, Y) \vee \textbf{edge}(s_2, s_1)^* \qquad \underline{\neg \textbf{edge}(s_2, s_1)}$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \vee \neg \psi(s_2, s_2, Y)^* \vee \neg \psi(s_2, s_3, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \textbf{edge}(x, y)$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \vee \textbf{edge}(s_2, s_2)^* \vee \neg \psi(s_2, s_3, Y) \qquad \underline{\neg s_2 = s_2}$$

$$\neg s_1 \in Y \vee \neg \text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \vee \neg \psi(s_2, s_3, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \textbf{edge}(x, y)$$



$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_1, Y)^* \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \qquad \underline{\text{co-rule}(x, Y)} \vee x = s_3 \vee \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(x, s_i, Y$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee s_1 = s_3^* \vee \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(s_1, s_i, Y) \qquad \underline{\neg s_1 = s_3}$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(s_1, s_i, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \mathbf{edge}(x, y)$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \neg\psi(s_1, s_2, Y) \vee \neg\psi(s_1, s_3, Y) \vee \mathbf{edge}(s_1, s_1)^* \qquad \underline{\neg\mathbf{edge}(s_1, s_1)}$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \neg\psi(s_1, s_2, Y)^* \vee \neg\psi(s_1, s_3, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \mathbf{edge}(x, y)$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \neg\psi(s_1, s_2, Y) \vee \mathbf{edge}(s_1, s_3)^* \qquad \underline{\neg\mathbf{edge}(s_1, s_3)}$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y) \vee \neg\text{co-rule}(s_3, Y) \vee \neg\psi(s_1, s_2, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \neg y \in Y$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_2, Y)^* \vee \neg\text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \qquad \underline{\text{co-rule}(x, Y)} \vee x = s_3 \vee \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(x, s_i, Y)$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \vee s_2 = s_3^* \vee \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(s_2, s_i, Y) \qquad \underline{\neg s_2 = s_3}$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \vee \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(s_2, s_i, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \mathbf{edge}(x, y)$$

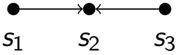
$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \vee \neg\psi(s_2, s_2, Y) \vee \neg\psi(s_2, s_3, Y) \vee \mathbf{edge}(s_2, s_1)^* \qquad \underline{\neg\mathbf{edge}(s_2, s_1)}$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \vee \neg\psi(s_2, s_2, Y)^* \vee \neg\psi(s_2, s_3, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \mathbf{edge}(x, y)$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \vee \mathbf{edge}(s_2, s_2)^* \vee \neg\psi(s_2, s_3, Y) \qquad \underline{\neg s_2 = s_2}$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \vee \neg\psi(s_2, s_3, Y)^* \qquad \underline{\psi(x, y, Y)} \vee \mathbf{edge}(x, y)$$

$$\neg s_1 \in Y \vee \neg\text{co-rule}(s_3, Y) \vee \neg s_2 \in Y \vee \mathbf{edge}(s_2, s_3)^* \qquad \underline{\neg\mathbf{edge}(s_2, s_3)}$$



$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_1, Y)^* \lor \neg\text{co-rule}(s_2, Y) \lor \neg\text{co-rule}(s_3, Y) \qquad \underline{\text{co-rule}(x, Y)} \lor x = s_3 \lor \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(x, s_i, Y$$

$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_2, Y) \lor \neg\text{co-rule}(s_3, Y) \lor s_1 = s_3^* \lor \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(s_1, s_i, Y) \qquad \underline{\neg s_1 = s_3}$$

$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_2, Y) \lor \neg\text{co-rule}(s_3, Y) \lor \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(s_1, s_i, Y)^* \qquad \underline{\psi(x, y, Y)} \lor \textbf{edge}(x, y)$$

$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_2, Y) \lor \neg\text{co-rule}(s_3, Y) \lor \neg\psi(s_1, s_2, Y) \lor \neg\psi(s_1, s_3, Y) \lor \textbf{edge}(s_1, s_1)^* \qquad \underline{\neg\textbf{edge}(s_1, s_1)}$$

$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_2, Y) \lor \neg\text{co-rule}(s_3, Y) \lor \neg\psi(s_1, s_2, Y)^* \lor \neg\psi(s_1, s_3, Y)^* \qquad \underline{\psi(x, y, Y)} \lor \textbf{edge}(x, y)$$

$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_2, Y) \lor \neg\text{co-rule}(s_3, Y) \lor \neg\psi(s_1, s_2, Y) \lor \textbf{edge}(s_1, s_3)^* \qquad \underline{\neg\textbf{edge}(s_1, s_3)}$$

$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_2, Y) \lor \neg\text{co-rule}(s_3, Y) \lor \neg\psi(s_1, s_2, Y)^* \qquad \underline{\psi(x, y, Y)} \lor \neg y \in Y$$

$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_2, Y)^* \lor \neg\text{co-rule}(s_3, Y) \lor \neg s_2 \in Y \qquad \underline{\text{co-rule}(x, Y)} \lor x = s_3 \lor \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(x, s_i, Y)$$

$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_3, Y) \lor \neg s_2 \in Y \lor s_2 = s_3^* \lor \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(s_2, s_i, Y) \qquad \underline{\neg s_2 = s_3}$$

$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_3, Y) \lor \neg s_2 \in Y \lor \bigvee_{1 \leqslant i \leqslant 3} \neg\psi(s_2, s_i, Y)^* \qquad \underline{\psi(x, y, Y)} \lor \textbf{edge}(x, y)$$

$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_3, Y) \lor \neg s_2 \in Y \lor \neg\psi(s_2, s_2, Y) \lor \neg\psi(s_2, s_3, Y) \lor \textbf{edge}(s_2, s_1)^* \qquad \underline{\neg\textbf{edge}(s_2, s_1)}$$

$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_3, Y) \lor \neg s_2 \in Y \lor \neg\psi(s_2, s_2, Y)^* \lor \neg\psi(s_2, s_3, Y)^* \qquad \underline{\psi(x, y, Y)} \lor \textbf{edge}(x, y)$$

$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_3, Y) \lor \neg s_2 \in Y \lor \textbf{edge}(s_2, s_2)^* \lor \neg\psi(s_2, s_3, Y) \qquad \underline{\neg s_2 = s_2}$$

$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_3, Y) \lor \neg s_2 \in Y \lor \neg\psi(s_2, s_3, Y)^* \qquad \underline{\psi(x, y, Y)} \lor \textbf{edge}(x, y)$$

$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_3, Y) \lor \neg s_2 \in Y \lor \textbf{edge}(s_2, s_3)^* \qquad \underline{\neg\textbf{edge}(s_2, s_3)}$$

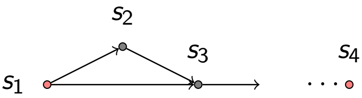
$$\neg s_1 \in Y \lor \neg\text{co-rule}(s_3, Y)^* \lor \neg s_2 \in Y \qquad \underline{\text{co-rule}(x, Y)} \lor x \in Y$$

$$\neg s_1 \in Y^* \lor \neg s_2 \in Y^* \lor s_3 \in Y$$

$$
\begin{array}{c}
\neg s_1 \in Y^* \vee \neg s_2 \in Y^* \vee s_3 \in Y \qquad \underline{x \in add(x, Z)} \\
\hline
\neg s_2 \in add(s_1, Y')^* \vee s_3 \in add(s_1, Y') \quad \underline{x \in add(y, Z)} \vee \neg x \in Z \\
\hline
\neg s_2 \in Y'^* \vee s_3 \in add(s_1, Y') \quad \underline{x \in add(x, Z)} \\
\hline
s_3 \in add(s_1, add(s_2, Y'')) \quad \underline{\neg x \in add(y, Z)} \vee x = y \vee x \in Z \\
\hline
s_3 = s_1^* \vee s_3 \in add(s_2, Y'') \quad \underline{\neg s_3 = s_1} \\
\hline
s_3 \in add(s_2, Y'')^* \quad \underline{\neg x \in add(y, Z)} \vee x = y \vee x \in Z \\
\hline
s_3 = s_2^* \vee s_3 \in Y'' \quad \underline{\neg s_3 = s_2} \\
\hline
s_3 \in Y''^* \quad \underline{\neg x \in \varnothing} \\
\hline
\square
\end{array}
$$

Simplification

While traversing a graph, for some vertices have been visited already, we do not need to visit them the second time.



Simplification

Definition (Path subsumption elimination rule(PSER))

For a set of traversal clauses $C_1, \ldots, C_n$, we have the rule

$$\frac{C_1 \cdots C_n}{C_i (1 \leqslant i \leqslant n)} \, \mathsf{ntv}(C_1) = \cdots = \mathsf{ntv}(C_n)$$

meaning that if there are clauses of the form $C_1, \ldots, C_n$, such that all the clauses have the same next-to-traverse vertices, they can be replaced by only one of them.

Simplification

Definition (Class-tautology clause)

Clauses of the form $\neg s \in Y \vee s \in Y \vee C$ are called class-tautology clauses.

Definition (Class Tautology elimination rule (TER_c))

Class-tautology clauses can be removed.

Completeness

Theorem

$\mathrm{OPRM}^{\succ}_{\mathcal{R}}$ *with Path subsumption elimination rule and Class tautology elimination rule is complete.*

Outline

Efficiency Comparison

Table: Efficiency Comparison

Vertices	time/s & result															
	EU				EF				AF				EG			
	new		old		new		old		new		old		new		old	
10	0.021	p	0.019	p	0.020	p	0.022	p	0.019	n	0.021	n	0.027	p	25.648	p
20	0.024	p	0.022	p	0.024	p	0.033	p	0.028	n	0.033	n	0.098	p	- -	-
100	0.264	n	- -	-	0.301	p	1.097	p	0.241	n	0.543	n	1.098	p	- -	-
200	1.044	p	1.603	p	1.142	p	8.295	p	0.919	n	3.067	n	8.350	p	- -	-
300	2.321	p	3.640	p	2.727	p	28.082	p	2.104	n	9.204	n	49.627	p	- -	-
350	3.209	p	5.241	p	3.834	p	45.351	p	2.981	n	14.269	n	50.892	p	- -	-
400	- -	-	- -	-	- -	-	- -	-	- -	-	- -	-	- -	-	- -	-

^{new} $OPRM^{\succ}_{\mathcal{R}}$ with PSER and TER_c; ^{old} $OPRM^{\succ}_{\mathcal{R}}$; ^{p} provable; ^{n} not provable; $^{--}$ time out.

Bibliography

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  Axiomatizing truth in a finite model

Thank you!