Equivalence of CSG and NCG

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1 The problem

1. First transform the following NCG into a CSG,

$$L = \{a^n b^n c^n \mid n \geq 1\} : S \to aSBc \mid abc,\ cB \to Bc,\ bB \to bb.$$

2. Prove that each NCG can be transformed into an equivalent CSG.

2 The solution

1. Assign each rule an identifier.

$$\begin{array}{ll} R_1 : S \to aSBc & R_2 : S \to abc, \\ R_3 : cB \to Bc & R_4 : bB \to bb \end{array}$$

Only R_3 is not of the desired form.

Step 1 Introduce new nonterminals C for the terminal c in order to remove terminals from the left sides of production rules.

$R_3 : cB \to Bc$ is replaced by $R_{31} : CB \to BC$, $R_{32} : C \to c$.
R_1 is replaced by $R_1' : S \to aSBC$, R_2 is replaced by $R_2' : S \to abC$.

Step 2 $R_{31} : CB \to BC$ is replaced by the following rules,

$$R_{311} : CB \to D_1 B,\ R_{312} : D_1 B \to D_1 D_2,\ R_{313} : D_1 D_2 \to BD_2, R_{314} : BD_2 \to BC.$$

Let G' be the resulting CSG.
Intuitively, one step $cB \to Bc$ in G is simulated by several derivation steps of the CSG G',

- to start the simulation, CB is replaced by $D_1 D_2$ using the rule $CB \to D_1 B$ and $D_1 B \to D_1 D_2$,
- then $D_1 D_2$ is removed and BC is derived left-to-right by the rule $D_1 D_2 \to BD_2$, $BD_2 \to BC$.

Note that in G', the execution of the rules $R_{311}, R_{312}, R_{313}, R_{314}$ may not be consecutive. But we have the following observation:

If there is a derivation $S \Rightarrow^* \alpha$ in G' such that $R_{311}, R_{312}, R_{313}, R_{314}$ are used, then there must be another derivation in which $R_{311}, R_{312}, R_{313}, R_{314}$ are applied consecutively.

Let's illustrate this with an example: The derivation

$$S \xrightarrow{R_1'} aSBC \xrightarrow{R_1'} aaSBCBC \xrightarrow{R_{311}} aaSBD_1BC \xrightarrow{R_1'} aaabCBD_1BC \xrightarrow{R_{312}} aaabCBD_1D_2C \to \dots$$

can be replaced by

$$S \xrightarrow{R_1'} aSBC \xrightarrow{R_1'} aaSBCBC \xrightarrow{R_1'} aaabCBCBC \xrightarrow{R_{311}} aaabCBD_1BC \xrightarrow{R_{312}} aaabCBD_1D_2C \to \dots$$

2. Suppose $G = (\mathcal{N}, \Sigma, \mathcal{P}, S)$ is an NCG.

Similar to the example above, by introducing a nonterminal A_a for each terminal $a \in \Sigma$, we can get a NCG such that the left sides of all rules only contain nonterminals. Let G' be the resulting grammar.

Assign an identifier for each production rule G', say $R_1, \dots, R_n$.

For each rule R_i of the form $A_1 \dots A_m \to B_1 \dots B_n$ (where $2 \le m \le n$) such that for every i, $B_i \in \mathcal{N}$, do the following: Introduce m new (distinct) nonterminals $C_1^i, \dots, C_m^i$ and replace R_i by the following rules,

$$\begin{gathered} A_1 \dots A_m \to C_1^i A_2 \dots A_m \to C_1^i C_2^i A_3 \dots A_m \to \dots \to \\ C_1^i C_2^i \dots C_{m-1}^i A_m \to C_1^i C_2^i \dots C_{m-1}^i C_m^i \to B_1 C_2^i \dots C_m^i \to \\ B_1 B_2 C_3^i \dots C_m^i \to B_1 \dots B_{m-1} C_m^i \to B_1 \dots B_{m-1} B_m \dots B_n. \end{gathered}$$

Note that the nonterminals $C_1^i, \dots, C_m^i$ are distinct, thus the position information of the left-side of the rule R_i is encoded by these nondeterminals; in addition, for distinct i, j, $C_r^i \neq C_s^j$ for any r, s, in other words, no newly introduced nonterminals can be reused for different rules.

To prove the correctness of the transformation, we should prove that

Claim. for every $\alpha, \beta \in (\mathcal{N} \cup \Sigma)^*$, $\alpha \Rightarrow_G \beta$ iff $\alpha \Rightarrow_{G'} \beta$.

In the following, we will prove the claim for the situation that *exactly one rule* in G, say $R_1 : A_1 \dots A_m \to B_1 \dots B_n$, is replaced by the a set of new rules (denoted as $\overline{R_1}$)

$$\begin{gathered} A_1 \dots A_m \to C_1 A_2 \dots A_m \to C_1 C_2 A_3 \dots A_m \to \dots \to \\ C_1 C_2 \dots C_{m-1} A_m \to C_1 C_2 \dots C_{m-1} C_m \to B_1 C_2 \dots C_m \to \\ B_1 B_2 C_3 \dots C_m \to B_1 \dots B_{m-1} C_m \to B_1 \dots B_{m-1} B_m \dots B_n. \end{gathered}$$

The proof for the general case can be obtained easily by an induction on the number of replaced rules in G.

We show the following: In the derivation $\alpha \Rightarrow_{G'} \beta$, suppose

$$\begin{gathered} \alpha_1 A_1 \dots A_m \beta_1 \to_{G'} \alpha_1 C_1 A_2 \dots A_m \beta_1 \\ \Rightarrow_{G'} \alpha_2 C_1 A_2 \dots A_m \beta_2 \to_{G'} \alpha_2 C_1 C_2 A_3 \dots A_m \beta_2 \to_{G'} \dots \\ \to_{G'} \alpha_{m-1} C_1 C_2 \dots C_{m-1} A_m \beta_{m-1} \Rightarrow_{G'} \alpha_m C_1 C_2 \dots C_{m-1} A_m \beta_m \\ \to_{G'} \alpha_m C_1 C_2 \dots C_{m-1} C_m \beta_m \Rightarrow_{G'} \alpha_{m+1} C_1 C_2 \dots C_{m-1} C_m \beta_{m+1} \\ \to_{G'} \alpha_{m+1} B_1 C_2 \dots C_m \beta_{m+1} \Rightarrow_{G'} \alpha_{m+2} B_1 C_2 \dots C_m \beta_{m+2} \\ \to_{G'} \alpha_{m+2} B_1 B_2 C_3 \dots C_m \beta_{m+2} \Rightarrow_{G'} \dots \\ \to_{G'} \alpha_{2m-1} B_1 \dots B_{m-1} C_m \beta_{2m-1} \Rightarrow_{G'} \alpha_{2m} B_1 \dots B_{m-1} C_m \beta_{2m} \\ \to_{G'} \alpha_{2m} B_1 \dots B_{m-1} B_m \dots B_n \beta_{2m} \end{gathered}$$

such that

- $\forall i : 1 \le i < m$, $\alpha_i \Rightarrow_{G'} \alpha_{i+1}$ and $A_{i+1} \dots A_m \beta_i \Rightarrow_{G'} A_{i+1} \dots A_m \beta_{i+1}$.
- $\forall i : 0 \le i < m$, $\alpha_{m+i} B_1 \dots B_i \Rightarrow_{G'} \alpha_{m+i+1} B_1 \dots B_i$ and $\beta_{m+i} \Rightarrow_{G'} \beta_{m+i+1}$.

The derivation can be reordered to gather the derivation steps corresponding to the rules in $\overline{R_1}$ (the red color below) together as follows.

$$
\begin{gathered}
\alpha_1 A_1 A_2 \dots A_m \beta_1 \Rightarrow_{G'} \alpha_2 A_1 A_2 \dots A_m \beta_2 \\
\Rightarrow_{G'} \alpha_3 A_1 A_2 \dots A_m \beta_3 \Rightarrow_{G'} \alpha_4 A_1 A_2 \dots A_m \beta_4 \\
\Rightarrow_{G'} \cdots \Rightarrow_{G'} \alpha_m A_1 A_2 \dots A_m \beta_m \\
\rightarrow_{G'} \alpha_m C_1 A_2 \dots A_m \beta_m \rightarrow_{G'} \alpha_m C_1 C_2 \dots A_m \beta_m \\
\rightarrow_{G'} \cdots \rightarrow_{G'} \alpha_m C_1 \dots C_m \beta_m \\
\rightarrow_{G'} \alpha_m B_1 C_2 \dots C_m \beta_m \rightarrow_{G'} \alpha_m B_1 B_2 C_3 \dots C_m \beta_m \\
\rightarrow_{G'} \cdots \rightarrow_{G'} \alpha_m B_1 \dots B_{m-1} C_m \beta_m \rightarrow_{G'} \alpha_m B_1 \dots B_{m-1} B_m \dots B_n \beta_m \\
\Rightarrow_{G'} \alpha_{m+1} B_1 \dots B_n \beta_{m+1} \Rightarrow_{G'} \alpha_{m+2} B_1 \dots B_n \beta_{m+2} \\
\Rightarrow_{G'} \cdots \Rightarrow_{G'} \alpha_{2m-1} B_1 \dots B_n \beta_{2m-1} \Rightarrow_{G'} \alpha_{2m} B_1 \dots B_n \beta_{2m}
\end{gathered}
$$

From the new derivation, we can continue reordering the derivation steps, and gather other derivation steps corresponding to the rules in $\overline{R_1}$ together, without separating again those derivation steps that have been gathered together.

Therefore, finally we get a derivation such that

> for each execution of the rules in $\overline{R_1}$ in the derivation, all its derivation steps are gathered together (called a $\overline{R_1}$ block).

Finally we can replace the $\overline{R_1}$ blocks into one derivation step using the rule $A_1 \dots A_m \rightarrow B_1 \dots B_n$ and get a derivation in the original grammar G.