Automata theory and its applications

Lecture 7-8: Visibly pushdown languages

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Outline

1. Visibly pushdown automata (VPA)
2. Closure properties
3. Visibly pushdown grammar (VPG)
4. Logical characterization
   - Equivalence of NFA and MSO
   - Equivalence of VPA and MSO_μ
5. Decision problems

Motivation

Parenthesises in arithmetic expressions

$$(\; (\; (\; 5 + x \;) * y + z \;) * (\; u - v \;) \;) \; / \; w$$

Motivation

Curly brackets in C Programs

```
for(...)
{
  if (...)
  {
    for (...)
    {
      ...
    }
  }
  else
  {
    ...
  }
}
```

Motivation

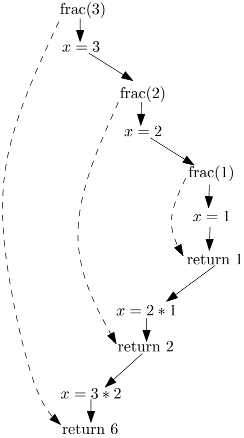
XML documents

⟨library⟩
⟨catalog⟩
⟨book⟩
⟨title⟩
Computational Complexity
⟨/title⟩
⟨author⟩
Christos H. Papadimitriou
⟨/author⟩
⟨book⟩
⟨catalog⟩
⟨library⟩

Motivation

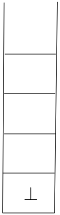
Recursive function calls and returns

```
frac(int y)
{
 int x = y;
 if (y >= 2)
 {
  x = y * frac(y − 1);
  return x;
 }
 else
 {
  return x;
 }
}
```



Motivation

$$(\ (\ (\ 5\ +\ x\ )\ *\ y\ +\ z\ )\ *\ (\ u\ -\ v\ )\ )\ /\ w$$



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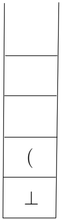
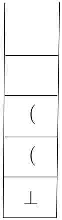
$$(\; (\; (\; 5 \; + \; x \;) \; * \; y \; + \; z \;) \; * \; (\; u \; - \; v \;) \;) \; / \; w$$



Motivation

$(\ (\ (\ 5 \ + \ x \ ) \ * \ y \ + \ z \ ) \ * \ (\ u \ - \ v \ ) \ ) \ / \ w$



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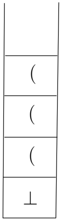

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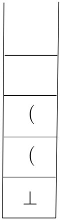

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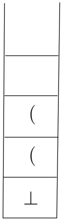

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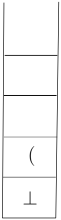


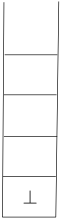
Motivation

Stack operations determined by the input symbol

$$(\; (\; (\; 5 \; + \; x \;) \; * \; y \; + \; z \;) \; * \; (\; u \; - \; v \;) \;) \; / \; w$$

(: Push

): Pop



Motivation

Stack operations determined by the input symbol

```
for(...)
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  if (...)
  {
    for (...)
    {
      ...
    }
  }
  else
  {
    ...
  }
}
```
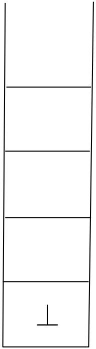

{: Push

}: Pop



Motivation

Stack operations determined by the input symbol

$\langle$library$\rangle$
$\langle$catalog$\rangle$
$\langle$book$\rangle$
$\langle$title$\rangle$
Computational Complexity
$\langle$/title$\rangle$
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$\langle$/author$\rangle$
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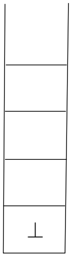
$\langle$catalog$\rangle$
$\langle$library$\rangle$

$\langle tag \rangle$: Push

$\langle /tag \rangle$: Pop



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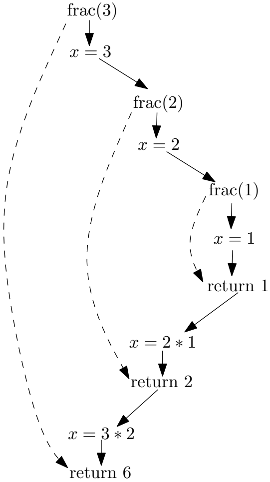
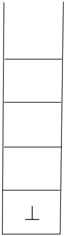


frac: Push
return: Pop



Visibly pushdown automata (VPA)

The alphabet Σ is partitioned into $\widetilde{\Sigma} = \langle \Sigma_c, \Sigma_r, \Sigma_l \rangle$

- Σ_c: finite set of calls,
- Σ_r: finite set of returns,
- Σ_l: finite set of local actions.

A (nondeterministic) VPA $\mathcal{A}$ is a tuple $(Q, \widetilde{\Sigma}, \Gamma, \delta, q_0, \bot, F)$, where

- Q is a finite set of states,
- $\widetilde{\Sigma}$ is the input alphabet,
- Γ is the stack alphabet,
- $\delta \subseteq Q \times \Sigma_c \times Q \times (\Gamma \backslash \{\bot\}) \ \cup \ Q \times \Sigma_r \times \Gamma \times Q \ \cup \ Q \times \Sigma_l \times Q$,
- q_0 is the initial state,
- $\bot$ is the bottom symbol of the stack,
- $F \subseteq Q$ is the set of final states.

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- q_0 is the initial state,
- $\bot$ is the bottom symbol of the stack,
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Remark:

- No ε-transitions,
- Exactly one symbol is pushed in each call transition.

Visibly pushdown automata (VPA)

A (nondeterministic) VPA $\mathcal{A}$ is a tuple $(Q, \widetilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$, where

- Q is a finite set of states,
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- $\delta \subseteq Q \times \Sigma_c \times Q \times (\Gamma \backslash \{\perp\}) \ \cup \ Q \times \Sigma_r \times \Gamma \times Q \ \cup \ Q \times \Sigma_l \times Q$,
- q_0 is the initial state,
- $\perp$ is the bottom symbol of the stack,
- $F \subseteq Q$ is the set of final states.

A *deterministic* VPA is a VPA $\mathcal{A} = (Q, \widetilde{\Sigma}, \Gamma, \delta, q_0, F)$ such that

- for every $(q, a) \in Q \times \Sigma_c$, there is at most one pair $(q', \gamma) \in Q \times (\Gamma \backslash \{\perp\})$ such that $(q, a, q', \gamma) \in \delta$,
- for every $(q, a, \gamma) \in Q \times \Sigma_r \times \Gamma$, there is at most one $q' \in Q$ such that $(q, a, \gamma, q') \in \delta$,
- for every $(q, a) \in Q \times \Sigma_l$, there is at most one $q' \in Q$ such that $(q, a, q') \in \delta$.

A deterministic VPA is *complete* if "at most" is replaced by "exactly".

Visibly pushdown automata (VPA): continued

A run of a VPA $\mathcal{A}$ over a word $w = a_1 \dots a_n$ is

a sequence $(q_0, \alpha_0)(q_1, \alpha_1) \dots (q_n, \alpha_n)$ s.t.

- $\forall i. q_i \in Q$,
- $\alpha_0 = \perp$,
- $\forall i : 1 \leqslant i < n$, one of the following holds,

Call $a_i \in \Sigma_c$, $\exists \gamma \in \Gamma \backslash \{\perp\}.(q_i, a_i, q_{i+1}, \gamma) \in \delta$, $\alpha_{i+1} = \gamma \alpha_i$,

Return $a_i \in \Sigma_r$,

- $\exists \gamma \in \Gamma \backslash \{\perp\}.(q_i, a_i, \gamma, q_{i+1}) \in \delta$, $\alpha_i = \gamma \alpha_{i+1}$,
- or $(q_i, a_i, \perp, q_{i+1}) \in \delta$ and $\alpha_i = \alpha_{i+1} = \perp$.

Local $a_i \in \Sigma_l$, $(q_i, a_i, q_{i+1}) \in \delta$ and $\alpha_{i+1} = \alpha_i$.

Visibly pushdown automata (VPA): continued

A run of a VPA $\mathcal{A}$ over a word $w = a_1 \ldots a_n$ is

a sequence $(q_0, \alpha_0)(q_1, \alpha_1) \ldots (q_n, \alpha_n)$ s.t.

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Call $a_i \in \Sigma_c$, $\exists \gamma \in \Gamma \backslash \{\perp\}.(q_i, a_i, q_{i+1}, \gamma) \in \delta$, $\alpha_{i+1} = \gamma\alpha_i$,

Return $a_i \in \Sigma_r$,

- $\exists \gamma \in \Gamma \backslash \{\perp\}.(q_i, a_i, \gamma, q_{i+1}) \in \delta$, $\alpha_i = \gamma\alpha_{i+1}$,
- or $(q_i, a_i, \perp, q_{i+1}) \in \delta$ and $\alpha_i = \alpha_{i+1} = \perp$.

Local $a_i \in \Sigma_l$, $(q_i, a_i, q_{i+1}) \in \delta$ and $\alpha_{i+1} = \alpha_i$.

A run $(q_0, \alpha_0) \ldots (q_n, \alpha_n)$ is *accepting* if $q_n \in F$.

A word w is accepted by a VPA $\mathcal{A}$ if $\exists$ an accepting run of $\mathcal{A}$ over w.

The set of words accepted by $\mathcal{A}$ is denoted by $\mathcal{L}(\mathcal{A})$.

Remark: Acceptance of VPAs are defined by final states, not by empty stack.

Well-matched words

Let $\widetilde{\Sigma} = \langle \Sigma_c, \Sigma_r, \Sigma_l \rangle$.
The set of *well-matched* words $w \in \Sigma^*$ is defined inductively as follows,

- ε is well-matched,
- if w' is well matched, then
  $$w = aw' \textit{ or } w = w'a \textit{ such that } a \in \Sigma_l \textit{ is well-matched,}$$
- if w' is well-matched, then
  $$w = aw'b \textit{ such that } a \in \Sigma_c, b \in \Sigma_r \textit{ is well-matched.}$$
- if w' and w'' are well-matched, then
  $$w = w'w'' \textit{ is well-matched.}$$

Example: (())() is well-matched, while neither ()()) nor (() is.

Remark. As a result of the acceptance by final states,

VPAs over $\widetilde{\Sigma}$ may accept non-well-matched words.

Visibly pushdown languages (VPL)

A language $L \subseteq \Sigma^*$ is a *visibly pushdown language* with respect to $\widetilde{\Sigma}$ if *there is a VPA $\mathcal{A}$ over $\widetilde{\Sigma}$, satisfying that $\mathcal{L}(\mathcal{A}) = L$.*

Example:

The language $\{a^n b^n \mid n \geqslant 1\}$ is a VPL with respect to $\widetilde{\Sigma} = \langle \{a\}, \{b\}, \varnothing \rangle$.

Homework

Let $L \subseteq \{a, b\}^*$ be the set of words with "equal number of a's and b's".

Prove that L is not a VPL with respect to any partition of $\Sigma = \{a, b\}$.

Embedding of CFL as VPLs

Proposition. For every CFL $L \subseteq \Sigma^*$, there are a VPL $L' \subseteq (\Sigma')^*$ with respect to some $\tilde{\Sigma}'$ and a homomorphism $h : (\Sigma')^* \to \Sigma^*$ such that $L = h(L')$.

Let L be a CFL defined by a PDA $\mathcal{A} = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ (accept. by final states).

W.l.o.g, suppose that each $(q, a, X, \alpha) \in \delta$ satisfies that

$\alpha = \varepsilon$ (pop) or $\alpha = X$ (stable) or $\alpha = YX$ (push).

Let $\Sigma' = (\Sigma \cup \{\sigma_\varepsilon\}) \times \{c, r, l\}$ and

$\tilde{\Sigma}' = \langle(\Sigma \cup \{\sigma_\varepsilon\}) \times \{c\}, (\Sigma \cup \{\sigma_\varepsilon\}) \times \{r\}, (\Sigma \cup \{\sigma_\varepsilon\}) \times \{l\}\rangle$

From $\mathcal{A}$, define a VPA $\mathcal{A}' = (Q, \tilde{\Sigma}', \Gamma, \delta', q_0, Z_0, F)$ over $\tilde{\Sigma}'$, where δ' is defined by the following rules,

- if $(q, a, X, q', \varepsilon) \in \delta$, then $(q, (a, r), X, q') \in \delta'$,
- if $(q, a, X, q', X) \in \delta$, then add a new state q_1,
  $(q, (a, r), X, q_1), (q_1, (\sigma_\varepsilon, c), q_2, X) \in \delta'$.
- if $(q, a, X, q', YX) \in \delta$, then add two new states q_1, q_2, and
  $(q, (a, r), X, q_1), (q_1, (\sigma_\varepsilon, c), q_2, X), (q_2, (\sigma_\varepsilon, c), q', Y) \in \delta'$.

Let $h : (\Sigma')^* \to \Sigma^*$ be a homomorphism defined by

$\forall a \in \Sigma, s \in \{c, r, l\}.\ h((a, s)) = a, h(\sigma_\varepsilon, s) = \varepsilon$.

Then $L = h(\mathcal{L}(\mathcal{A}'))$.

Outline

Union and intersection

Proposition. VPLs with respect to $\widetilde{\Sigma}$ are closed under union and intersection.
Let $\mathcal{A}_1 = (Q_1, \widetilde{\Sigma}, \Gamma_1, \delta_1, q_0^1, \bot_1, F_1)$ and $\mathcal{A}_2 = (Q_2, \widetilde{\Sigma}, \Gamma_2, \delta_2, q_0^2, \bot_2, F_2)$ be two VPAs.

Union.

Without loss of generality, suppose $\bot_1 = \bot_2 = \bot$.

The VPA $\mathcal{A} = (Q_1 \cup Q_2 \cup \{q_0\}, \widetilde{\Sigma}, \Gamma_1 \cup \Gamma_2, \delta, q_0, \bot, F_1 \cup F_2)$ such that

$$\delta = \begin{array}{l} \qquad\qquad\qquad \delta_1 \cup \delta_2 \cup \\ \{(q_0, a, q', \gamma) \mid (q_0^1, a, q', \gamma) \in \delta_1 \text{ or } (q_0^2, a, q', \gamma) \in \delta_2\} \cup \\ \quad \{(q_0, a, \gamma, q') \mid (q_0^1, a, \gamma, q') \in \delta_1 \text{ or } (q_0^2, a, \gamma, q') \in \delta_2\} \end{array}$$

defines $\mathcal{L}(\mathcal{A}_1) \cup \mathcal{L}(\mathcal{A}_2)$.

Intersection.

The VPA $\mathcal{A} = (Q_1 \times Q_2, \widetilde{\Sigma}, \Gamma_1 \times \Gamma_2, \delta, (q_0^1, q_0^2), (\bot_1, \bot_2), F_1 \times F_2)$ such that

$$\delta = \begin{array}{l} \{((q_1, q_2), a, (q_1', q_2'), (\gamma_1, \gamma_2)) \mid (q_1, a, q_1', \gamma_1) \in \delta_1, (q_2, a, q_2', \gamma_2) \in \delta_2\} \cup \\ \quad \{((q_1, q_2), a, (\gamma_1, \gamma_2), (q_1', q_2')) \mid (q_1, a, \gamma_1, q_1') \in \delta_1, (q_2, a, \gamma_2, q_2') \in \delta_2\} \end{array}$$

defines $\mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{A}_2)$.

Complementation

Theorem. For every VPA $\mathcal{A}$, a deterministic VPA $\mathcal{A}'$ can be constructed such that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$.

Corollary. VPLs with respect to $\widetilde{\Sigma}$ are closed under complementation.

Proof.

Suppose L is defined by a complete deterministic VPA $\mathcal{A} = (Q, \widetilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$.

Then $\mathcal{A}' = (Q, \widetilde{\Sigma}, \Gamma, \delta, q_0, \perp, Q \backslash F)$ defines $\Sigma^* \backslash \mathcal{L}(\mathcal{A})$. $\square$

Complementation

Illustration of the intuition of the proof of the Theorem.

In an obviously way, we can define $(q, \alpha) \xrightarrow{w} (q', \alpha')$:

the reachability of the config. (q', α') *from* (q, α) *by reading* w.

Observation. Suppose $(q, \alpha) \xrightarrow{w} (q', \alpha')$ and w is well-matched, then $\alpha = \alpha'$.

Point I.

A well-matched word w *can be seen as a relation* $S_w \subseteq Q \times Q$,

without changing the content of the stack.

Complementation

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Point I.

A well-matched word w can be seen as a relation $S_w \subseteq Q \times Q$, without changing the content of the stack.

Point II.

Suppose w is well-matched.

- $S_\varepsilon = \mathrm{Id}_Q$.
- If $w = aw'$ with $a \in \Sigma_l$, then
$$S_w = \{(q, q') \mid \exists q''.(q, a, q'') \in \delta, (q'', q') \in S_{w'}\}.$$
Similarly for $w = w'a$.
- If $w = aw'b$ with $a \in \Sigma_c$ and $b \in \Sigma_r$, then
$S_w = \{(q, q') \mid \exists q_1, q_2, \gamma.\ (q, a, q_1, \gamma) \in \delta, (q_1, q_2) \in S_{w'}, (q_2, b, \gamma, q') \in \delta\}.$

Complementation

Illustration of the intuition of the proof of the Theorem.

Point III. Get inspirations from the subset construction for NFAs.

Question:

What info. should be remembered after reading a word w in a NFA?

Complementation

Illustration of the intuition of the proof of the Theorem.

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Question:

What info. should be remembered after reading a word w in a NFA?

Answer:

The set of states reachable from q_0 after reading w.

Complementation

Illustration of the intuition of the proof of the Theorem.

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Question:

What info. should be remembered
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Illustration of the intuition of the proof of the Theorem.

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Answer:

Let me think for a while . . .

Complementation

Illustration of the intuition of the proof of the Theorem.

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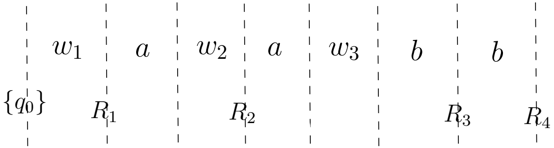
Consider $w_1 a w_2 a w_3 bb$ s.t. $a \in \Sigma_c, b \in \Sigma_r$ and w_1, w_2, w_3 are well-matched.



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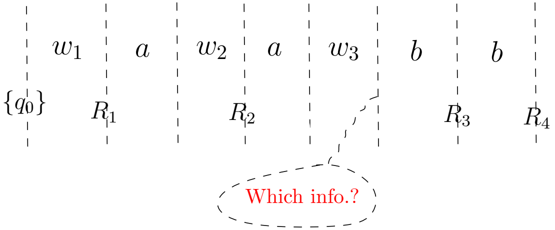

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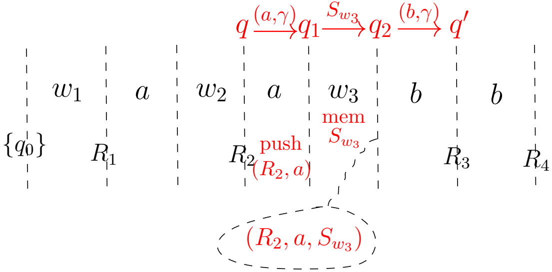

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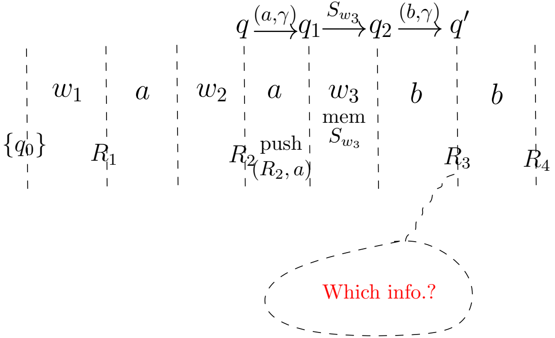

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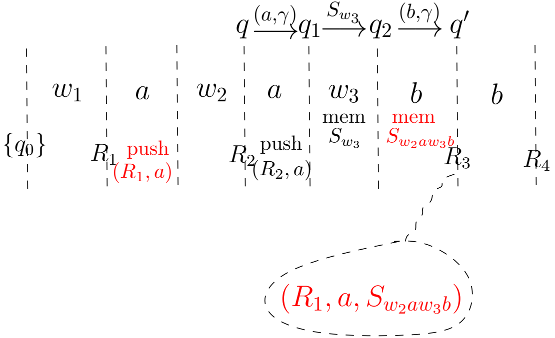

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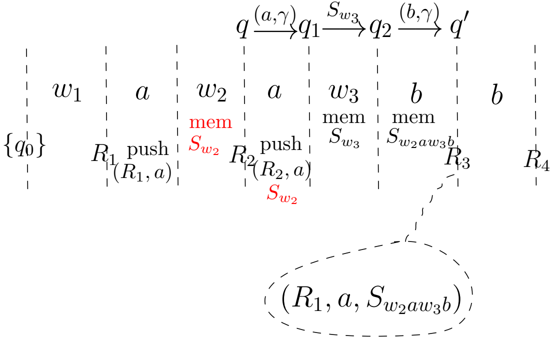
after reading a word w in a nondeterministic VPA?

Consider $w_1 a w_2 a w_3 bb$ s.t. $a \in \Sigma_c, b \in \Sigma_r$ and w_1, w_2, w_3 are well-matched.



Complementation: continued

The construction for determinization $\mathcal{A}' = (Q', \tilde{\Sigma}, \Gamma', \delta', (\mathrm{Id}_Q, \{q_0\}), F')$:

- Q': (S, R) such that $S \subseteq Q \times Q, R \subseteq Q$,
- Γ': letters (S, R, a) such that $S \subseteq Q \times Q, R \subseteq Q, a \in \Sigma_c$,
- $F' = \{(S, R) \mid R \cap F \neq \emptyset\}$,
- δ':

Local if $a \in \Sigma_l$, then $((S, R), a, (S', R')) \in \delta'$, where
$R' = \{q' \mid \exists q \in R.(q, a, q') \in \delta\}$, $S' = \{(q, q') \mid \exists q_1.(q, q_1) \in S, (q_1, a, q') \in \delta\}$.

Call if $a \in \Sigma_c$, then $((S, R), a, (\mathrm{Id}_Q, R'), (S, R, a)) \in \delta'$, where
$R' = \{q' \mid \exists q \in R, \gamma \in \Gamma.(q, a, q', \gamma) \in \delta\}$.

Return if $a \in \Sigma_r$, then $((S, R), a, (S'', R'', a'), (S', R')) \in \delta'$, where

$$S' = \left\{ (q, q') \;\middle|\; \begin{array}{c} \exists q_1, q_2, q_3, \gamma \in \Gamma. \\ (q, q_1) \in S'', (q_1, a', q_2, \gamma) \in \delta, (q_2, q_3) \in S, (q_3, a, \gamma, q') \in \delta \end{array} \right\},$$

$$R' = \left\{ q' \;\middle|\; \begin{array}{c} \exists q_1, q_2, q_3, \gamma \in \Gamma. \\ q_1 \in R'', (q_1, a', q_2, \gamma) \in \delta, (q_2, q_3) \in S, (q_3, a, \gamma, q') \in \delta \end{array} \right\},$$

or $((S, R), a, \perp, (S', R')) \in \delta'$, where
$S' = \{(q, q') \mid \exists q''.(q, q'') \in S, (q'', a, \perp, q') \in \delta\}$,
$R' = \{q' \mid \exists q \in R.(q, a, \perp, q') \in \delta\}$.

Outline

1. Visibly pushdown automata (VPA)
2. Closure properties
3. Visibly pushdown grammar (VPG)
4. Logical characterization
   - Equivalence of NFA and MSO
   - Equivalence of VPA and MSO_{μ}
5. Decision problems

Visibly pushdown grammar (VPG)

A CFG $G = (\mathcal{N}, \Sigma, \mathcal{P}, S)$ is a VPG over $\widetilde{\Sigma}$ if $\mathcal{N}$ can be partitioned into $\mathcal{N}_0$ and $\mathcal{N}_1$, and each rule in $\mathcal{P}$ is of the following forms,

- $X \to \varepsilon$,
- $X \to aY$ such that if $X \in \mathcal{N}_0$, then $a \in \Sigma_l, Y \in \mathcal{N}_0$,
- $X \to aYbZ$ such that $a \in \Sigma_c$, $b \in \Sigma_r$, $Y \in \mathcal{N}_0$, and if $X \in \mathcal{N}_0$, then $Z \in \mathcal{N}_0$.

Example: Let $\widetilde{\Sigma} = (\{a\}, \{b\}, \emptyset)$. Then the VPG

$$S \to aSbC \mid aTbC, T \to \varepsilon, C \to \varepsilon,$$

such that $\mathcal{N}_0 = \{S, T, C\}$ defines $\{a^n b^n \mid n \geqslant 1\}$.

Equivalence of VPA and VPG

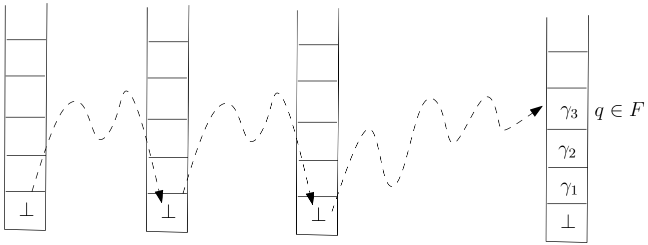
Theorem. VPA $\equiv$ VPG.

From VPA to VPG.

Let $\mathcal{A} = (Q, \widetilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$ be a VPA.

The intuition: Utilizing the nonterminals $[q, \gamma, p]$ with the meaning

the top symbol of the stack is γ,
and from state q,
by reading a well-matched word,
state p can be reached.



Equivalence of VPA and VPG

Theorem. VPA $\equiv$ VPG.

From VPA to VPG.

Let $\mathcal{A} = (Q, \widetilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$ be a VPA.

Construct a VPG $(\mathcal{N}_0, \mathcal{N}_1, \widetilde{\Sigma}, \mathcal{P}, S)$ as follows.

- $Q = \{(q, \perp) \mid q \in Q\} \cup \{q \mid q \in Q\} \cup \{[q, \gamma, p] \mid q, p \in Q, \gamma \in \Gamma \backslash \{\perp\}\}$,
  - $(q, \perp)$: the state is q and the stack is empty,
  - q: the state is q and the stack is nonempty.
- $\mathcal{N}_0 = \{[q, \gamma, p] \mid q, p \in Q, \gamma \in \Gamma \backslash \{\perp\}\}$, $S = (q_0, \perp)$,
- $\mathcal{P}$ is defined by the following rules,
  - if $(q, a, q') \in \delta$ s.t. $a \in \Sigma_l$, then
    $(q, \perp) \to a(q', \perp)$, $q \to aq'$, $[q, \gamma, p] \to a[q', \gamma, p]$.
  - if $(q, a, q', \gamma), (p', b, \gamma, p) \in \delta$ s.t. $a \in \Sigma_c, b \in \Sigma_r$, then
    $[q, \gamma_1, r] \to a[q', \gamma, p']b[p, \gamma_1, r]$, $(q, \perp) \to a(q', \gamma, p')b(p, \perp)$,
    $q \to a(q', \gamma, p')bp$.
  - if $(q, a, q', \gamma) \in \delta$ s.t. $a \in \Sigma_c$, then
    $(q, \perp) \to aq'$, $q \to aq'$, $(q, \perp) \to a[q', \gamma, p]$, $q \to a[q', \gamma, p]$.
  - if $(q, a, \perp, q') \in \delta$ s.t. $a \in \Sigma_r$, then $(q, \perp) \to a(q', \perp)$.
  - $\forall q \in Q. [q, \gamma, q] \to \varepsilon$,
  - $\forall q \in F. q \to \varepsilon, (q, \perp) \to \varepsilon$.

Equivalence of VPA and VPG: continued

From VPG to VPA.
Let $G = (\mathcal{N}_0, \mathcal{N}_1, \widetilde{\Sigma}, \mathcal{P}, S)$ be a VPG.
Construct VPA $\mathcal{A} = (\mathcal{N}, \widetilde{\Sigma}, \Sigma_r \times \mathcal{N} \cup \{\bot, \$\}, \delta, S, F)$ as follows.

- δ is defined by the following rules,
  - if $X \to aY$ s.t. $a \in \Sigma_l$, then $(X, a, Y) \in \delta$,
  - if $X \to aY$ s,t. $a \in \Sigma_c$, then $(X, a, Y, \$) \in \delta$,
  - if $X \to aY$ s.t. $a \in \Sigma_r$, then $(X, a, \$, Y) \in \delta$ and $(X, a, \bot, Y) \in \delta$,
  - if $X \to aYbZ$, then $(X, a, Y, (b, Z)) \in \delta$,
  - if $X \to \varepsilon$ and $X \in \mathcal{N}_0$, then $(X, b, (b, Y), Y) \in \delta$.
- $\mathcal{A}$ accepts if the state is in X s.t. $X \to \varepsilon$ and the top symbol is \$ or $\bot$.

Equivalence of VPA and VPG: continued

From VPG to VPA.
Let $G = (\mathcal{N}_0, \mathcal{N}_1, \widetilde{\Sigma}, \mathcal{P}, S)$ be a VPG.
Construct VPA $\mathcal{A} = (\mathcal{N}, \widetilde{\Sigma}, \Sigma_r \times \mathcal{N} \cup \{\perp, \$\}, \delta, S, F)$ as follows.

- δ is defined by the following rules,
  - if $X \to aY$ s.t. $a \in \Sigma_l$, then $(X, a, Y) \in \delta$,
  - if $X \to aY$ s,t. $a \in \Sigma_c$, then $(X, a, Y, \$) \in \delta$,
  - if $X \to aY$ s.t. $a \in \Sigma_r$, then $(X, a, \$, Y) \in \delta$ and $(X, a, \perp, Y) \in \delta$,
  - if $X \to aYbZ$, then $(X, a, Y, (b, Z)) \in \delta$,
  - if $X \to \varepsilon$ and $X \in \mathcal{N}_0$, then $(X, b, (b, Y), Y) \in \delta$.
- $\mathcal{A}$ accepts if the state is in X s.t. $X \to \varepsilon$ and the top symbol is \$ or $\perp$.

  Adapt $\mathcal{A}$ *into* $\mathcal{A}' = (\mathcal{N} \times \Gamma, \widetilde{\Sigma}, \Gamma, \delta', (S, \perp), \{(X, \gamma) \mid X \to \varepsilon, \gamma = \$, \perp\})$
  by adding the top symbol of the stack into the states.

  - if $X \to aY$ s.t. $a \in \Sigma_l$, then $\forall \gamma.((X, \gamma), a, (Y, \gamma)) \in \delta'$,
  - if $X \to aY$ s,t. $a \in \Sigma_c$, then $\forall \gamma.((X, \gamma), a, (Y, \$), (\$, \gamma)) \in \delta'$,
  - if $X \to aY$ s.t. $a \in \Sigma_r$, then
    $\forall \gamma.((X, \gamma), a, \perp, (Y, \perp)) \in \delta$ and $\forall \gamma.((X, \$), a, (\$, \gamma), (Y, \gamma)) \in \delta'$,
  - if $X \to aYbZ$, then $\forall \gamma.((X, \gamma), a, (Y, (b, Z)), ((b, Z), \gamma)) \in \delta'$,
  - if $X \to \varepsilon$ and $X \in \mathcal{N}_0$, then $\forall \gamma.((X, (b, Z)), b, ((b, Z), \gamma), (Z, \gamma)) \in \delta'$.

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Outline

Monadic Second-Order Logic (MSO)

Syntax.

$\varphi := P_\sigma(x) \mid x = y \mid \mathrm{suc}(x, y) \mid X(x) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \exists x \varphi_1 \mid \exists X \varphi_1$,

where $\sigma \in \Sigma$.

An MSO *sentence* is a MSO formula without free variables.

Semantics.

A *structure* $\mathcal{S}$ over Σ is

- a domain $S = \{1, \ldots, n\}$,
- an interpretation of all the unary predicates $P_\sigma \in \Sigma$ over S, denoted by $(P_\sigma)^{\mathcal{S}}$.

Example. Let $\Sigma = \{a, b\}$. Then $\mathcal{S} = (\{1, 2, 3\}, (P_a)^{\mathcal{S}} = \{1\}, (P_b)^{\mathcal{S}} = \{2, 3\})$ is a structure over Σ.

A word $w = a_1 \ldots a_n$ can be seen as a structure $\mathcal{S}_w$ over Σ,

- the domain of $\mathcal{S}_w$, denoted by S_w, is $\{1, \ldots, n\}$,
- the interpretation of every $P_\sigma \in \Sigma$ is the set of positions with the letter σ in w.

Monadic Second-Order Logic (MSO)

Syntax.

$\varphi := P_\sigma(x) \mid x = y \mid \mathrm{suc}(x, y) \mid X(x) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \exists x \varphi_1 \mid \exists X \varphi_1$,

where $\sigma \in \Sigma$.

An MSO *sentence* is a MSO formula without free variables.

Semantics.

Given a MSO formula φ, a *valuation* of $\mathrm{free}(\varphi)$ over a structure $\mathcal{S}$ is a mapping $\mathcal{I}$ such that

- for every $x \in \mathrm{free}(\varphi)$, $\mathcal{I}(x) \in S$,
- for every $X \in \mathrm{free}(\varphi)$, $\mathcal{I}(X) \subseteq S$.

Monadic Second-Order Logic (MSO)

Syntax.

$\varphi := P_\sigma(x) \mid x = y \mid \mathrm{suc}(x, y) \mid X(x) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \exists x \varphi_1 \mid \exists X \varphi_1$,

where $\sigma \in \Sigma$.

An MSO *sentence* is a MSO formula without free variables.

Semantics.

A MSO formula φ is satisfied over a word $w = a_1 \dots a_n$, with a valuation $\mathcal{I}$ of free(φ) over $\mathcal{S}_w$, denoted by $(w, \mathcal{I}) \models \varphi$, is defined as follows,

- $(w, \mathcal{I}) \models P_\sigma(x)$ iff $a_{\mathcal{I}(x)} = \sigma$,
- $(w, \mathcal{I}) \models x = y$ iff $\mathcal{I}(x) = \mathcal{I}(y)$,
- $(w, \mathcal{I}) \models \mathrm{suc}(x, y)$ iff $\mathcal{I}(x) + 1 = \mathcal{I}(y)$,
- $(w, \mathcal{I}) \models X(x)$ iff $\mathcal{I}(x) \in \mathcal{I}(X)$,
- $(w, \mathcal{I}) \models \varphi_1 \vee \varphi_2$ iff $(w, \mathcal{I}) \models \varphi_1$ or $(w, \mathcal{I}) \models \varphi_2$,
- $(w, \mathcal{I}) \models \neg\varphi_1$ iff not $(w, \mathcal{I}) \models \varphi_1$,
- $(w, \mathcal{I}) \models \exists x \varphi_1$ iff there is $j \in S_w$ such that $(w, \mathcal{I}[x \to j]) \models \varphi_1$,
- $(w, \mathcal{I}) \models \exists X \varphi_1$ iff there is $J \subseteq S_w$ such that $(w, \mathcal{I}[X \to J]) \models \varphi_1$.

Monadic Second-Order Logic (MSO)

Syntax.

$\varphi := P_\sigma(x) \mid x = y \mid \mathrm{suc}(x,y) \mid X(x) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \exists x \varphi_1 \mid \exists X \varphi_1$,

where $\sigma \in \Sigma$.

An MSO *sentence* is a MSO formula without free variables.

Semantics.

Let φ be a MSO sentence.
The language defined by φ, denoted $\mathcal{L}(\varphi)$: The set of words satisfying φ.

A language $L \subseteq \Sigma^*$ is *MSO-definable*
if
there is a MSO sentence φ such that $\mathcal{L}(\varphi) = L$.

Monadic Second-Order Logic (continued)

Abbreviations.

- $\varphi_1 \wedge \varphi_2 = \neg(\neg\varphi_1 \vee \neg\varphi_2)$,
- $\varphi_1 \rightarrow \varphi_2 = \neg\varphi_1 \vee \varphi_2$,
- $\forall x \varphi_1 = \neg\exists x(\neg\varphi_1)$,
- $x < y = \begin{array}{c} \neg x = y \wedge \\ \forall X \left(\left(X(x) \wedge \forall z_1 \forall z_2 (X(z_1) \wedge \mathrm{suc}(z_1, z_2) \rightarrow X(z_2))\right) \rightarrow X(y) \right) \end{array}$,
- $\mathrm{first}(x) = \forall y(x = y \vee x < y)$,
- $\mathrm{last}(x) = \forall y(x = y \vee y < x)$.

Example.

$$\neg\exists x\ \mathrm{first}(x),$$

$$\exists x \exists y (P_a(x) \wedge P_b(y) \wedge x < y),$$

$$\exists X \left(\begin{array}{c} \exists x(\mathrm{first}(x) \wedge X(x)) \wedge \\ \forall x \forall y \forall z(\mathrm{suc}(x, y) \wedge \mathrm{suc}(y, z) \wedge X(x) \rightarrow X(z)) \\ \wedge \forall x(X(x) \rightarrow P_a(x)) \end{array} \right).$$

NFA≡MSO

From NFA to MSO

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ be a NFA. Let $Q = \{q_0, q_1, \ldots, q_n\}$. Construct the MSO formula φ as follows,

$$\exists q_0 \ldots q_n (\varphi_{init} \wedge \varphi_{trans} \wedge \varphi_{final}),$$

where

- $\varphi_{init} = \exists x(\text{first}(x) \wedge \bigvee_{(q_0,a,q)\in\delta} (P_a(x) \wedge q(x))),$
- $\varphi_{trans} = \forall x \forall y(\text{suc}(x, y) \to \bigvee_{(q,a,q')\in\delta} q(x) \wedge P_a(y) \wedge q'(y)),$
- $\varphi_{final} = \exists x(\text{last}(x) \wedge \bigvee_{q\in F} q(x)).$

Then $\mathcal{L}(\varphi) = \mathcal{L}(\mathcal{A})$.

NFA≡MSO

From MSO to NFA.

A normal form for MSO formulas

New modalities,

$$X \subseteq Y,\ \mathrm{Sing}(X),\ suc(X, Y).$$

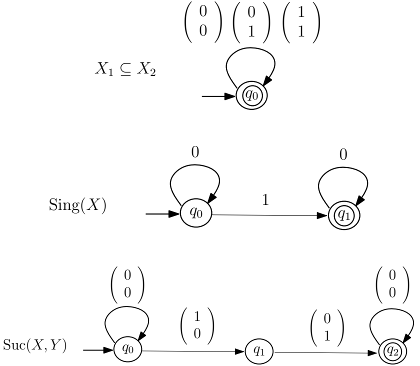
Then a MSO formula φ can be transformed into a normal form φ' by the following rules,

- if $\varphi = P_\sigma(x)$, then $\varphi' = \mathrm{Sing}(X) \wedge X \subseteq P_\sigma$,
- if $\varphi = x = y$, then $\varphi' = \mathrm{Sing}(X) \wedge \mathrm{Sing}(Y) \wedge X \subseteq Y \wedge Y \subseteq X$,
- if $\varphi = \mathrm{suc}(x, y)$, then $\varphi' = \mathrm{suc}(X, Y)$,
- if $\varphi = Z(x)$, then $\varphi' = \mathrm{Sing}(X) \wedge X \subseteq Z$,
- if $\varphi = \varphi_1 \vee \varphi_2$, then $\varphi' = \varphi_1' \vee \varphi_2'$,
- if $\varphi = \neg\varphi_1$, then $\varphi' = \neg\varphi_1'$,
- if $\varphi = \exists x \varphi_1$, then $\varphi' = \exists X(\mathrm{Sing}(X) \wedge \varphi_1')$,
- if $\varphi = \exists X \varphi_1$, then $\varphi' = \exists X \varphi_1'$.

From MSO to NFA.

$$\varphi := X \subseteq P_\sigma \mid P_\sigma \subseteq X \mid X \subseteq Y \mid \mathrm{Sing}(X) \mid \mathrm{suc}(X,Y) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \exists X \varphi_1.$$

Let $\varphi(X_1, \ldots, X_k)$ be a MSO formula in the normal form.
We construct a NFA $\mathcal{A} = (Q, \Sigma \times \{0,1\}^k, \delta, q_0, F)$ as follows.



NFA≡MSO

From MSO to NFA.

$$\varphi := X \subseteq P_\sigma \mid P_\sigma \subseteq X \mid X \subseteq Y \mid \mathrm{Sing}(X) \mid \mathrm{suc}(X,Y) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \exists X \varphi_1.$$

Let $\varphi(X_1, \dots, X_k)$ be a MSO formula in the normal form.
We construct a NFA $\mathcal{A} = (Q, \Sigma \times \{0,1\}^k, \delta, q_0, F)$ as follows.

- $\varphi = \varphi_1 \vee \varphi_2$

  NFAs are closed under union,

- $\varphi = \neg\varphi_1$

  NFAs are closed under complementation,

- $\varphi = \exists X \varphi_1$

  NFAs are closed under projection (a special case of homomorphisms), e.g. $(b_1, \dots, b_k) \to (b_2, \dots, b_k)$.

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MSO_μ

Fix $\tilde{\Sigma}$.

Given a word $w = a_1 \dots a_n \in \Sigma^*$, a binary relation $\mu(x, y)$ can be defined such that

$\mu(i,j)$ *iff* a_i *is a call and* a_j *is a matching return.*

Example. In the word "(( )) ( ) ((", $\mu(1,4), \mu(2,3), \mu(5,6)$ hold.

Syntax of MSO_μ over $\tilde{\Sigma}$.

$$\varphi := P_\sigma(x) \mid x = y \mid \mathrm{suc}(x,y) \mid X(x) \mid \mu(x,y) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \exists x \varphi_1 \mid \exists X \varphi_1,$$

where $\sigma \in \Sigma$.

Semantics of MSO_μ over $\tilde{\Sigma}$.

- $(w, \mathcal{I}) \models \mu(x,y)$ iff $\mu(\mathcal{I}(x), \mathcal{I}(y))$ holds on w.

Example. Let $\tilde{\Sigma} = (\{a\}, \{b\}, \{c\})$

$$\forall x (P_a(x) \to \exists y \exists z (P_b(y) \wedge P_c(z) \wedge x < z \wedge z < y \wedge \mu(x,y)))$$

VPA $\equiv$ MSO$_\mu$

From VPA to MSO$_\mu$.

Let $\mathcal{A} = (Q, \widetilde{\Sigma}, \Gamma, \delta, q_0, \perp, F)$ be a VPA, $Q = \{q_0, \ldots, q_n\}$, $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$.
Define $\varphi := \exists q_0 \ldots q_n P_{\gamma_1} \ldots P_{\gamma_k} (\varphi_{init} \wedge \varphi_{trans} \wedge \varphi_{final})$ as follows,

- $$\varphi_{init} = \exists x \left(\text{first}(x) \wedge \left(\begin{array}{c} \bigvee\limits_{(q_0,a,q)\in\delta} (P_a(x) \wedge q(x)) \bigvee \\ \bigvee\limits_{(q_0,a,q,\gamma)\in\delta} (P_a(x) \wedge q(x) \wedge P_\gamma(x)) \bigvee \\ \bigvee\limits_{(q_0,a,\perp,q)\in\delta} (P_a(x) \wedge q(x) \wedge P_\perp(x)) \end{array} \right)\right),$$
- $\varphi_{trans} = \forall x \forall y \, (\text{suc}(x, y) \rightarrow \psi_{call} \vee \psi_{return} \vee \psi_{local})$, where
  - $\psi_{call} = \bigvee\limits_{(q,a,q',\gamma)\in\delta} (q(x) \wedge P_a(y) \wedge q'(y) \wedge P_\gamma(y))$,
  - $$\psi_{return} = \begin{array}{c} \bigvee\limits_{(q,a,\gamma,q')\in\delta} (q(x) \wedge P_a(y) \wedge q'(y) \wedge P_\gamma(y) \wedge \exists z(\mu(z, y) \wedge P_\gamma(z))) \bigvee \\ \bigvee\limits_{(q,a,\perp,q')\in\delta} (q(x) \wedge P_a(y) \wedge q'(y) \wedge P_\perp(y) \wedge \neg\exists z(\mu(z, y))) \end{array},$$
  - $\psi_{local} = \bigvee\limits_{(q,a,q')\in\delta} (q(x) \wedge P_a(y) \wedge q'(y))$.
- $$\varphi_{final} = \exists x \left(\text{last}(x) \wedge \bigvee_{q\in F} q(x) \right).$$

From MSO$_\mu$ to VPA.

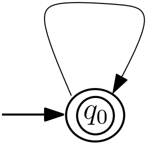
$$\varphi := \begin{array}{c} X \subseteq P_\sigma \mid P_\sigma \subseteq X \mid X \subseteq Y \mid \mathrm{Sing}(X) \mid \\ \mathrm{suc}(X,Y) \mid \mu(X,Y) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \exists X \varphi_1 \end{array},$$

$$\sigma', \sigma'' \neq \sigma : \quad (\sigma',0), \downarrow (\sigma',0) \;\; (\sigma',0), \uparrow (\sigma'',0), (\sigma,1), \perp$$

$$(\sigma',0)$$

$$(\sigma,1) \downarrow (\sigma,1) \;\; (\sigma,0), \downarrow (\sigma,0)$$

$X \subseteq P_\sigma : \sigma \in \Sigma_c$



VPA $\equiv$ MSO$_\mu$: continued

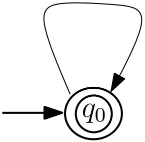
From MSO$_\mu$ to VPA.

$$\varphi := \begin{array}{c} X \subseteq P_\sigma \mid P_\sigma \subseteq X \mid X \subseteq Y \mid \mathrm{Sing}(X) \mid \\ \mathrm{suc}(X,Y) \mid \mu(X,Y) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \exists X \varphi_1 \end{array},$$

$\sigma', \sigma'' \neq \sigma :$

$$(\sigma',0), \downarrow (\sigma',0) \quad (\sigma',0), \uparrow (\sigma'',0), \perp$$
$$(\sigma',0)$$
$$(\sigma,1) \uparrow (\sigma',0), \perp \quad (\sigma,0) \uparrow (\sigma',0), \perp$$

$X \subseteq P_\sigma : \sigma \in \Sigma_r$

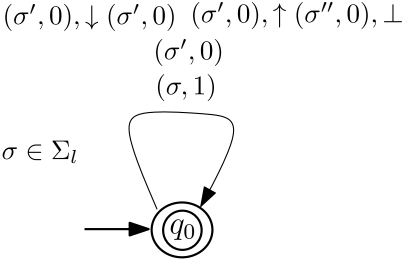


From MSO$_\mu$ to VPA.

$$\varphi := \begin{array}{c} X \subseteq P_\sigma \mid P_\sigma \subseteq X \mid X \subseteq Y \mid \mathrm{Sing}(X) \mid \\ \mathrm{suc}(X,Y) \mid \mu(X,Y) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \exists X \varphi_1 \end{array},$$

$\sigma', \sigma'' \neq \sigma$:

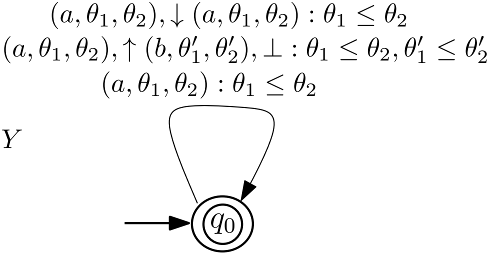
$X \subseteq P_\sigma : \sigma \in \Sigma_l$



From MSO$_\mu$ to VPA.

$$\varphi := \begin{array}{c} X \subseteq P_\sigma \mid P_\sigma \subseteq X \mid X \subseteq Y \mid \mathrm{Sing}(X) \mid \\ \mathrm{suc}(X,Y) \mid \mu(X,Y) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \exists X \varphi_1 \end{array},$$

$X \subseteq Y$



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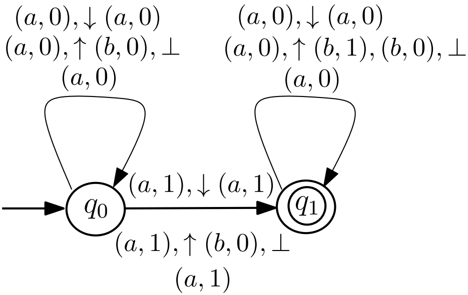
Sing(X)



VPA $\equiv$ MSO$_\mu$: continued

From MSO$_\mu$ to VPA.

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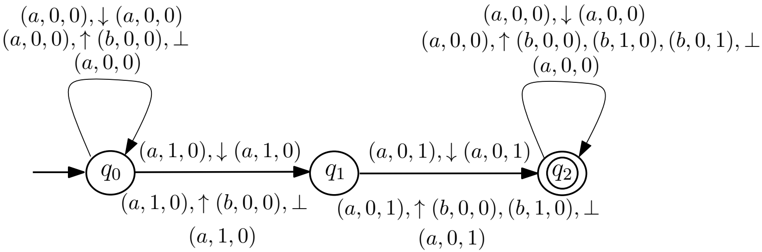
suc(X,Y)



VPA $\equiv$ MSO$_\mu$: continued

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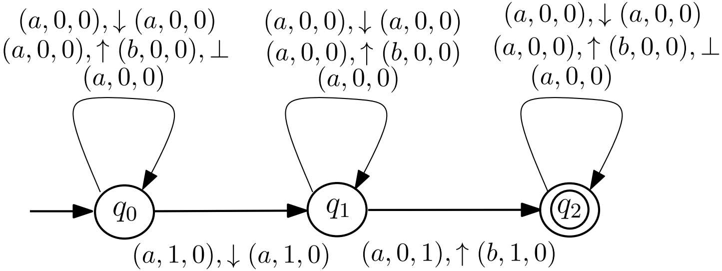
$\mu(X,Y)$



Outline

1. Visibly pushdown automata (VPA)
2. Closure properties
3. Visibly pushdown grammar (VPG)
4. Logical characterization
   - Equivalence of NFA and MSO
   - Equivalence of VPA and MSO_μ
5. Decision problems

Nonemptiness

Theorem. The nonemptiness of VPA can be solved in $O(n^3)$ time.

A VPA can be transformed into an equivalent VPG in $O(n^3)$ *time.*

The emptiness of a CFG can be solved in linear time.

Language inclusion

Theorem. The language inclusion of VPA is EXPTIME-complete.

Upper bound.

Given two VPAs $\mathcal{A}_1$ and $\mathcal{A}_2$,

- determinize $\mathcal{A}_2$ into $\mathcal{A}_2'$,
- complement $\mathcal{A}_2'$ into $\mathcal{B}$,
- test whether $\mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{B}) = \emptyset$.

The determinization procedure can be fulfilled in EXPTIME.

Language inclusion

Theorem. The language inclusion of VPA is EXPTIME-complete.

Lower bound.

The universality of VPA is EXPTIME-hard.

Language inclusion

Theorem. The language inclusion of VPA is EXPTIME-complete.

Lower bound.

The universality of VPA is EXPTIME-hard.

Result from complexity theory: APSPACE = EXPTIME.

An *alternating* TM (ATM) is a TM $M = (Q_\vee, Q_\wedge, \Sigma, \Gamma, \delta, q_0, B, F)$ such that

- the state set is divided into two disjoint subsets, $Q_\vee$ ("or" state), $Q_\wedge$ ("and" state),
- for every $q \in Q$ and $a \in \Gamma$, $|\delta(q, a)| = 2$.

A *run* of an ATM M over an input $w \in \Sigma^*$ is a configuration tree s.t.

- the root of the tree is the initial configuration,
- for every node (configuration) $\alpha q \beta$ in the tree, if $q \in Q_\vee$, then $\alpha q \beta$ has one of its successor config. as its unique child in the tree,
- for every node (configuration) $\alpha q \beta$ in the tree, if $q \in Q_\wedge$, then the two successor config. of $\alpha q \beta$ are both its children in the tree.

APSPACE: The class of languages accepted by ATMs using polynomial space.

Language inclusion

Theorem. The language inclusion of VPA is EXPTIME-complete.

Lower bound.

The universality of VPA is EXPTIME-hard.

Reduction from

the membership problem of alternating TMs using polynomial space.

Let $M = (Q_\vee, Q_\wedge, \Sigma, \Gamma, \delta, q_0, B, F)$ be a ATM using linear space, say cn.
Let t be a accepting run of M over an input w.
Use C_x*'s (where* $x \in \{0,1\}^*$*) to denote the nodes of* t*,*
e.g. the root is C_ε, while the left child of the root is C_0, and so on.
Encode t by a word θ which is generated by a DFS traversal of t.

Language inclusion

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Encode t by a word θ which is generated by a DFS traversal of t.
Initially set $\theta = \varepsilon$.

1. The traversal starts from the root C_ε.
2. When a node C_x is visited for the first time, then $\theta = \theta(fC_x)$,
3. When a node C_x is visited again by backtracking from its right-child, then $\theta = \theta(b\overline{C_x})^r$.

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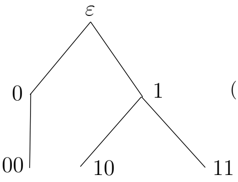
Encode t by a word θ which is generated by a DFS traversal of t.

Such a word θ *is called a successful computation of* M*.*



$$(fC_\varepsilon)(fC_0)(fC_{00})(b\overline{C_{00}})^r(b\overline{C_0})^r(fC_1)(fC_{10})(b\overline{C_{10}})^r(fC_{11})(b\overline{C_{11}})^r(b\overline{C_1})^r(b\overline{C_\varepsilon})^r$$

Language inclusion

Theorem. The language inclusion of VPA is EXPTIME-complete.

Lower bound.

The universality of VPA is EXPTIME-hard.

Reduction from

the membership problem of alternating TMs using polynomial space.

Let $M = (Q_\vee, Q_\wedge, \Sigma, \Gamma, \delta, q_0, B, F)$ be a ATM using linear space, say cn.
Let $\Gamma' = \Gamma \cup Q \cup \bar{\Gamma} \cup \bar{Q} \cup \{f, b\}$, $\tilde{\Gamma}' = \langle \Gamma \cup Q \cup \{f\}, \bar{\Gamma} \cup \bar{Q} \cup \{b\} \rangle$.
The format of a successful computation θ,
e.g. well-matched call-returns, except consistencies of consecutive config.
can be checked by a deterministic VPA $\mathcal{A}$.

A word $w \in (\Gamma')^*$ is a *unsuccessful* computation of M if one of the following conditions holds,

- w is not accepted by $\mathcal{A}$,
- there is a subword $fC_x fC_{x0}$ or $\overline{C^r_{x0}} b \overline{C^r_x} b$ or $\overline{C^r_{x1}} b \overline{C^r_x} b$, such that $C_x \not\vdash C_{x0}$, or $C_x \not\vdash C_{x1}$: Guess an index $i : 1 < i < cn + 1$, and check the relationship of the $(i-1, i, i+1)$-th symbol of C_x and the i-th symbol of C_{x0}, ...

Language inclusion

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Lower bound.

The universality of VPA is EXPTIME-hard.

Reduction from

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The set of unsuccessful computations of M

can be accepted by a nondeterministic VPA $\mathcal{B}$ of polynomial size.

$$M \textit{ does not accept } w \textit{ iff } \mathcal{L}(\mathcal{B}) = (\Gamma')^*.$$

Automata over infinite words