Automata theory and its applications

Lecture 9-11: Automata over infinite words

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December 20, 2012

Outline

1. Motivation
2. Büchi automata
3. Closure properties
4. Equivalence with MSO
5. Decision problem
6. Muller, Rabin, Strett, and Parity automata
7. Determinization
8. Equivalence with WMSO

Why infinite words ?

Reactive systems: reacting continuously with the environment

- Operating systems,
- Communicating protocols,
- Control programs,
- Vending machines,
- ...

Salient feature of reactive systems:

Nonterminating

The behavior of reactive systems:

A set of infinite words.

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Büchi automata (BA)

A Büchi automata $\mathcal{B}$ is a tuple $(Q, \Sigma, \delta, q_0, F)$ where

- Q: finite set of states, Σ: alphabet,
- q_0: initial state, $F \subseteq Q$: set of final states,
- $\delta \subseteq Q \times \Sigma \times Q$.

A run ρ of a Büchi automata $\mathcal{B}$ over an ω-word $w = a_1 a_2 \cdots \in \Sigma^\omega$ is

$$\textit{a state sequence } q_0 q_1 \ldots \textit{ such that } \forall i \geqslant 0.(q_i, a_{i+1}, q_{i+1}) \in \delta.$$

$\text{Inf}(\rho)$: the set of states occurring infinitely often in ρ.

$$\textit{A run is accepting iff } \text{Inf}(\rho) \cap F \neq \emptyset.$$

An ω-word w is accepted by $\mathcal{B}$ if there is an accepting run of $\mathcal{B}$ over w.

Let $\mathcal{L}(\mathcal{B})$ denote the set of ω-words accepted by $\mathcal{B}$.

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$$\textit{a state sequence } q_0 q_1 \ldots \textit{ such that } \forall i \geqslant 0.(q_i, a_{i+1}, q_{i+1}) \in \delta.$$

$\mathrm{Inf}(\rho)$: the set of states occurring infinitely often in ρ.

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An ω-word w is accepted by $\mathcal{B}$ if there is an accepting run of $\mathcal{B}$ over w.

Let $\mathcal{L}(\mathcal{B})$ denote the set of ω-words accepted by $\mathcal{B}$.

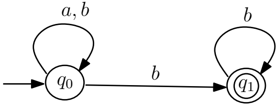
A deterministic Büchi automaton (DBA) $\mathcal{B}$ is a BA $(Q, \Sigma, \delta, q_0, F)$ s.t.

$$\forall q \in Q, a \in \Sigma, \exists \textit{ at most one } q' \in Q \textit{ such that } (q, a, q') \in \delta.$$

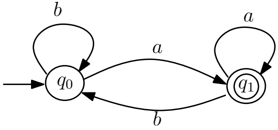
Then δ in a DBA can be seen as a partial function $\delta : Q \times \Sigma \to Q$.

Büchi automata: Example

"The letter a occurs only finitely often"



"The letter a occurs infinitely often"



Büchi automata: Several notations

Let $\mathcal{B} = (Q, \Sigma, \delta, q_0, F)$ be a BA, $q, q' \in Q$, and $w = a_1 \dots a_n \in \Sigma^*$.

A *partial run* of $\mathcal{B}$ over w from q to q' is a state sequence $q_1 q_2 \dots q_{n+1}$ such that

- $\forall i \leqslant n.(q_i, a_i, q_{i+1}) \in \delta$,
- $q_1 = q$, $q_{n+1} = q'$.

$q \xrightarrow{w} q'$:

there is a partial run of $\mathcal{B}$ over w from q to q'.

$q \xrightarrow[F]{w} q'$:

there is a partial run of $\mathcal{B}$ over w from q to q' which contains an accepting state.

Theorem. Let $L \subseteq \Sigma^\omega$. Then

L can be defined by a BA iff $L = \bigcup_{1 \leqslant i \leqslant n} U_i V_i^\omega$,

where $\forall i : 1 \leqslant i \leqslant n$. $U_i, V_i \subseteq \Sigma^*$ are regular and $\varepsilon \notin V_i$.

Proof.

Only if direction:

Suppose that L is defined by a BA $\mathcal{B} = (Q, \Sigma, \delta, q_0, F)$.

Let $L_{qq'} = \{w \in \Sigma^* \mid q \xrightarrow{w} q'\}$. Then $L = \bigcup_{q \in F} L_{q_0 q}(L_{qq} \backslash \{\varepsilon\})^\omega$.

ω-regular languages

Theorem. Let $L \subseteq \Sigma^\omega$. Then

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Proof.

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Suppose that L is defined by a BA $\mathcal{B} = (Q, \Sigma, \delta, q_0, F)$.

Let $L_{qq'} = \{w \in \Sigma^* \mid q \xrightarrow{w} q'\}$. Then $L = \bigcup_{q \in F} L_{q_0 q}(L_{qq} \backslash \{\varepsilon\})^\omega$.

If direction: Suppose $L = \bigcup_{1 \leqslant i \leqslant n} U_i V_i^\omega$.

Since Büchi automata are closed under union (which will be shown later), it is sufficient to prove that $U_i V_i^\omega$ can be defined by a BA.

Let $\mathcal{A}_1 = (Q_1, \Sigma, \delta_1, q_0^1, F_1)$ (resp. $\mathcal{A}_2 = (Q_2, \Sigma, \delta_2, q_0^2, F_2)$) define U_i (resp. V_i).

W.l.o.g. assume that there are no transitions (q, a, q_0^2) with $q \in Q_2$.

Then $\mathcal{B} = (Q_1 \cup Q_2, \Sigma, \delta, q_0^1, \{q_0^2\})$ defines L, where

$$\delta = \begin{array}{c} \delta_1 \cup \delta_2 \cup \big\{(q, a, q') \mid q \in F_1, (q_0^2, a, q') \in \delta_2\big\} \\ \cup\big\{(q, a, q_0^2) \mid \exists q' \in F_2, (q, a, q') \in \delta_2\big\} \end{array}.$$

□

Expressibility of DBA

Let $L \subseteq \Sigma^*$. Define $\overrightarrow{L} = \{w \in \Sigma^\omega \mid \exists^\omega n.\ w_1 \dots w_n \in L\}$.

Proposition. Let $L \subseteq \Sigma^\omega$. Then

L can be defined by a DBA iff $L = \overrightarrow{L'}$ for some regular language $L' \subseteq \Sigma^*$.

Proof.

Only if direction:

Suppose L is defined by the DBA $\mathcal{B} = (Q, \Sigma, \delta, q_0, F)$.

Let L' be defined by the DFA $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$, then $L = \overrightarrow{L'}$.

It is trivial that $L \subseteq \overrightarrow{L'}$.

Suppose $w \in \overrightarrow{L'}$. Then there exist infinitely many $n \in \mathbb{N}$ s.t. $w_1 \dots w_n \in L'$.

For each such n, let $q_0 \dots q_n$ be the accepting run of $\mathcal{A}$ over $w_1 \dots w_n$.

Then $q_0 \dots q_n \dots$ is an accepting run of $\mathcal{B}$ over w. Therefore, $w \in L$.

If direction:

Let $L = \overrightarrow{L'}$ and $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ be a DFA defining L'.

Then the DFA $\mathcal{B} = (Q, \Sigma, \delta, q_0, F)$ defines L. □

Expressibility of DBA

Let $L \subseteq \Sigma^*$. Define $\overrightarrow{L} = \{w \in \Sigma^\omega \mid \exists^\omega n.\ w_1 \ldots w_n \in L\}$.

Proposition. Let $L \subseteq \Sigma^\omega$. Then
L can be defined by a DBA iff $L = \overrightarrow{L'}$ for some regular language $L' \subseteq \Sigma^*$.

Proposition. BA is strictly more expressive than DBA.

Proof.

The language L "The letter a occurs only finitely often"
is not expressible in DBA.

For contradiction, assume that L is defined by a DBA $\mathcal{B}$.

Consider ab^ω. The run of $\mathcal{B}$ over ab^ω is accepting. Let $n_1 \in \mathbb{N}$ s.t. $q_0 \xrightarrow[F]{ab^{n_1}} q_1$.

Consider $ab^{n_1}ab^\omega$. Let $n_2 \in \mathbb{N}$ s.t. $q_1 \xrightarrow[F]{ab^{n_2}} q_2$.

Continue like this, we can get an ω-word $ab^{n_1}ab^{n_2}\ldots$ which is accepted by $\mathcal{B}$, while on the other hand contains infinitely many a's, a contradiction. □

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Union and intersection

Proposition. The class of ω-regular languages is closed under union and intersection.

Proof.

Let $\mathcal{A}_1 = (Q_1, \Sigma, \delta_1, q_0^1, F_1), \mathcal{A}_2 = (Q_2, \Sigma, \delta_2, q_0^2, F_2)$ define resp. L_1, L_2.

Union:

The BA $\mathcal{A} = (Q_1 \cup Q_2 \cup \{q_0\}, \Sigma, \delta, q_0, F_1 \cup F_2)$ defines $L_1 \cup L_2$, where

$$\delta = \delta_1 \cup \delta_2 \cup \{(q_0, a, q) \mid (q_0^1, a, q) \in \delta_1\} \cup \{(q_0, a, q) \mid (q_0^2, a, q) \in \delta_2\}.$$

Intersection:

The BA $\mathcal{A} = (Q_1 \times Q_2 \times \{0, 1, 2\}, \Sigma, \delta, (q_0^1, q_0^2, 0), Q_1 \times Q_2 \times \{2\})$ defines $L_1 \cap L_2$, where δ is defined as follows,

Suppose $(q_1, a, q_1') \in \delta_1$ and $(q_2, a, q_2') \in \delta_2$.

- If $q_1' \notin F_1$, then $((q_1, q_2, 0), a, (q_1', q_2', 0)) \in \delta$,
  otherwise, $((q_1, q_2, 0), a, (q_1', q_2', 1)) \in \delta$.
- If $q_2' \notin F_2$, then $((q_1, q_2, 1), a, (q_1', q_2', 1)) \in \delta$,
  otherwise, $((q_1, q_2, 1), a, (q_1', q_2', 2)) \in \delta$.
- $((q_1, q_2, 2), a, (q_1', q_2', 0)) \in \delta$.

□

Complementation

Theorem. The class of ω-regular languages is closed under complementation.

Let $L \subseteq \Sigma^\omega$ defined by a BA $\mathcal{B} = (Q, \Sigma, \delta, q_0, F)$.
Define a congruence $\sim_{\mathcal{B}}$ over Σ^* as follows:

$$u \sim_{\mathcal{B}} v \text{ iff } \forall q, q' \in Q.(q \xrightarrow{u} q' \Leftrightarrow q \xrightarrow{v} q') \text{ and } (q \xrightarrow[F]{u} q' \Leftrightarrow q \xrightarrow[F]{v} q').$$

Let $[u]$ denote the equivalence class of u under $\sim_{\mathcal{B}}$.

Complementation

Theorem. The class of ω-regular languages is closed under complementation.

Lemma. $\sim_{\mathcal{B}}$ is of finite index.

Lemma. $\sim_{\mathcal{B}}$ saturates L, namely,

for every $u, v \in \Sigma^*$, $[u][v]^\omega \cap L \neq \emptyset$ *implies that* $[u][v]^\omega \subseteq L$.

Proof.

Suppose $u_1 v_1 v_2 \cdots \in L$ s.t. $u_1 \in [u]$ and $v_1, v_2, \cdots \in [v]$.
We prove that $u'_1 v'_1 v'_2 \cdots \in L$ for every $u'_1 \in [u]$ and $v'_1, v'_2, \cdots \in [v]$.
There exists an accepting run ρ of $\mathcal{B}$ over $u_1 v_1 v_2 \ldots$.
Let $q_1, q_2, \ldots$ be the states in ρ such that $q_0 \xrightarrow{u_1} q_1$, $\forall i \geqslant 1. q_i \xrightarrow{v_i} q_{i+1}$.
Then there are $i_1 < i_2 < \ldots$ s.t.

$$q_1 \xrightarrow[F]{v_1 \ldots v_{i_1}} q_{i_1+1}, \ \forall j \geqslant 1. q_{i_j+1} \xrightarrow[F]{v_{i_j+1} \ldots v_{i_{j+1}}} q_{i_{j+1}+1}.$$

So $q_0 \xrightarrow{u'_1} q_1$, $q_1 \xrightarrow[F]{v'_1 \ldots v'_{i_1}} q_{i_1+1}$, and $\forall j \geqslant 1. q_{i_j+1} \xrightarrow[F]{v'_{i_j+1} \ldots v'_{i_{j+1}}} q_{i_{j+1}+1}$.
Therefore, $u'_1 v'_1 v'_2 \ldots$ is accepted by $\mathcal{B}$, thus in L. □

Complementation

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Lemma. $\forall w \in \Sigma^\omega$, $\exists u, v \in \Sigma^*$ s.t. $w \in [u][v]^\omega$.

Proof.

For a pair (i, j) such that $i < j$, assign a *color* $[w_i \dots w_{j-1}]$.
From Ramsey theorem,

$\exists$ a color $[v]$ and an infinite sequence $1 \leqslant i_1 < i_2 < \dots$ s.t.

$\forall j < k$, the pair (i_j, i_k) is assigned the color $[v]$.

Let $u = w_1 \dots w_{i_1 - 1}$. Then
$w = (w_1 \dots w_{i_1-1})(w_{i_1} \dots w_{i_2-1})(w_{i_2} \dots w_{i_3-1}) \cdots \in [u][v]^\omega$. □

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for every $u, v \in \Sigma^*$, $[u][v]^\omega \cap L \neq \emptyset$ *implies that* $[u][v]^\omega \subseteq L$.

Lemma. $\forall w \in \Sigma^\omega, \exists u, v \in \Sigma^*$ s.t. $w \in [u][v]^\omega$.

Lemma. $\forall u \in \Sigma^*$ s.t. $[u]$ is regular.

Proof.

It is sufficient to prove that $L_{qq'} = \left\{ w \mid q \xrightarrow{w} q' \right\}$ and $L^F_{qq'} = \left\{ w \mid q \xrightarrow[F]{w} q' \right\}$ are regular for all q, q'.

$L_{qq'}$ is regular: Obvious.

$L^F_{qq'}$ is regular: Defined by the NFA $(Q \times \{0,1\}, \Sigma, \delta', (q,0), (q',1))$, where

$\forall p, p' \in Q$, *if* $(p, a, p') \in \delta$, *then* $((p,1), a, (p',1)) \in \delta'$, *and*
if $p' \notin F$, *then* $((p,0), a, (p',0)) \in \delta'$, *otherwise*, $((p,0), a, (p',1)) \in \delta'$.

□

Complementation

Theorem. The class of ω-regular languages is closed under complementation.

Lemma. $\sim_{\mathcal{B}}$ is of finite index.

Lemma. $\sim_{\mathcal{B}}$ saturates L, namely,

for every $u, v \in \Sigma^*$, $[u][v]^\omega \cap L \neq \emptyset$ *implies that* $[u][v]^\omega \subseteq L$.

Lemma. $\forall w \in \Sigma^\omega$, $\exists u, v \in \Sigma^*$ s.t. $w \in [u][v]^\omega$.

Lemma. $\forall u \in \Sigma^*$ s.t. $[u]$ is regular.

Proof of the theorem.

Let $S = \{([u],[v]) \mid [u][v]^\omega \cap L \neq \emptyset\}$. Then $\overline{L} = \bigcup\limits_{([u],[v]) \notin S} [u][v]^\omega$.

$\bigcup\limits_{([u],[v]) \notin S} [u][v]^\omega \subseteq \overline{L}$: If $([u],[v]) \notin S$, then $[u][v]^\omega \cap L = \emptyset$, so $[u][v]^\omega \subseteq \overline{L}$.

$\overline{L} \subseteq \bigcup\limits_{([u],[v]) \notin S} [u][v]^\omega$: For every $w \in \overline{L}$, there are $[u], [v]$ such that $w \in [u][v]^\omega$.
Because $([u],[v]) \in S$ implies $w \in [u][v]^\omega \subseteq L$, it follows $([u],[v]) \notin S$. □

Complementation

Theorem. The class of ω-regular languages is closed under complementation.

Lemma. $\sim_{\mathcal{B}}$ is of finite index.

Lemma. $\sim_{\mathcal{B}}$ saturates L, namely,

for every $u, v \in \Sigma^*$, $[u][v]^\omega \cap L \neq \emptyset$ *implies that* $[u][v]^\omega \subseteq L$.

Lemma. $\forall w \in \Sigma^\omega, \exists u, v \in \Sigma^*$ s.t. $w \in [u][v]^\omega$.

Lemma. $\forall u \in \Sigma^*$ s.t. $[u]$ is regular.

Complexity analysis

The automaton $\mathcal{B}'$ defining $\overline{L}$:

The union of the BAs for the languages $[u][v]^\omega$ *with* $([u],[v]) \notin S$.

The BA for $[u][v]^\omega$ can be easily obtained from the NFAs for resp. $[u]$ and $[v]$.

[u] is determined by $(\{(q,q') \mid q \xrightarrow{u} q'\}, \{(q,q') \mid q \xrightarrow[F]{u} q'\}) \Rightarrow$
$2^{2|Q|^2}$ *equivalence classes* $\Rightarrow 2^{2|Q|^2}$ *states in the NFA for* $[u]$ *and* $[v]$.

Conclusion: There are $2^{O(|Q|^2)}$ states in $\mathcal{B}'$.

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MSO over infinite words

Syntax.

$\varphi := P_\sigma(x) \mid x = y \mid \text{suc}(x,y) \mid X(x) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \exists x \varphi_1 \mid \exists X \varphi_1$,

where $\sigma \in \Sigma$.

A MSO formula φ is satisfied over an ω-word $w = a_1 \dots a_n \dots$, with a valuation $\mathcal{I}$ of Free(φ) over $\mathcal{S}_w$, denoted by $(w, \mathcal{I}) \models \varphi$, is defined as follows,

- $(w, \mathcal{I}) \models P_\sigma(x)$ iff $a_{\mathcal{I}(x)} = \sigma$,
- $(w, \mathcal{I}) \models x = y$ iff $\mathcal{I}(x) = \mathcal{I}(y)$,
- $(w, \mathcal{I}) \models \text{suc}(x,y)$ iff $\mathcal{I}(x) + 1 = \mathcal{I}(y)$,
- $(w, \mathcal{I}) \models X(x)$ iff $\mathcal{I}(x) \in \mathcal{I}(X)$,
- $(w, \mathcal{I}) \models \varphi_1 \vee \varphi_2$ iff $(w, \mathcal{I}) \models \varphi_1$ or $(w, \mathcal{I}) \models \varphi_2$,
- $(w, \mathcal{I}) \models \neg\varphi_1$ iff not $(w, \mathcal{I}) \models \varphi_1$,
- $(w, \mathcal{I}) \models \exists x \varphi_1$ iff there is $j \in S_w$ such that $(w, \mathcal{I}[x \to j]) \models \varphi_1$,
- $(w, \mathcal{I}) \models \exists X \varphi_1$ iff there is $J \subseteq S_w$ such that $(w, \mathcal{I}[X \to J]) \models \varphi_1$.

BA $\equiv$ MSO

From BA to MSO

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ be a BA. Let $Q = \{q_0, q_1, \ldots, q_n\}$. Construct the MSO formula φ as follows,

$$\exists q_0 \ldots q_n(\varphi_{init} \wedge \varphi_{trans} \wedge \varphi_{final}),$$

where

- $\varphi_{init} = \exists x(\text{First}(x) \wedge \bigvee_{(q_0,a,q)\in\delta} (P_a(x) \wedge q(x))),$
- $\varphi_{trans} = \forall x \forall y(\text{suc}(x,y) \rightarrow \bigvee_{(q,a,q')\in\delta} q(x) \wedge P_a(y) \wedge q'(y)),$
- $\varphi_{final} = \forall x \exists y \left(x < y \wedge \bigvee_{q\in F} q(y) \right).$

Then $\mathcal{L}(\varphi) = \mathcal{L}(\mathcal{A})$.

From MSO to BA

Similar to the construction of an NFA from a MSO formula.

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Nonemptiness

Input: Büchi automaton $\mathcal{B} = (Q, \Sigma, \delta, q_0, F)$.

Question: Is $\mathcal{L}(\mathcal{B}) \neq \varnothing$?

Find a SCC (strongly-connected-component) C satisfying the following conditions,

- C contains an accepting state,
- C is reachable from q_0.

Proposition. Nonemptiness of Büchi automata can be decided in linear time.

SCCs of a directed graph can be found in linear time by a DFS search.

Language inclusion

Input: Büchi automata $\mathcal{B}_1$ and $\mathcal{B}_2$.

Question: Is $\mathcal{L}(\mathcal{B}_1) \subseteq \mathcal{L}(\mathcal{B}_2)$?

Theorem. Language inclusion of Büchi automata is PSPACE-complete.

Upper bound.

Construct $\mathcal{B}_2'$ defining $\overline{\mathcal{L}(\mathcal{B}_2)}$ and test the emptiness of $\mathcal{L}(\mathcal{B}_1 \cap \mathcal{B}_2')$.

There are $|Q_1| 2^{O(|Q_2|^2)}$ *states in* $\mathcal{B}_1 \cap \mathcal{B}_2' \Rightarrow$
The nonemptiness of $\mathcal{B}_1 \cap \mathcal{B}_2'$ *can be decided in PSPACE*

- *Guess on the fly a path*
  from the initial state to a cycle containing an accepting state.
- *NPSPACE* $\equiv$ *PSPACE.*

Language inclusion

Input: Büchi automata $\mathcal{B}_1$ and $\mathcal{B}_2$.

Question: Is $\mathcal{L}(\mathcal{B}_1) \subseteq \mathcal{L}(\mathcal{B}_2)$?

Theorem. Language inclusion of Büchi automata is PSPACE-complete.

Lower bound.

Universality of Büchi automata ($\mathcal{L}(\mathcal{B}) = \Sigma^\omega$) is PSPACE-hard.

Reduction from the membership problem of PSPACE TMs.
Use BA to describe the unsuccessful computations of PSPACE TMs.

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$ be a linear space (say cn) TM.
In addition, let $\widehat{\Gamma} = \Gamma \cup Q \cup \{\$\}$.

A *successful computation* of M over w: $\$C_1\$C_2\$\ldots\$C_m\$\left(\widehat{\Gamma}\backslash\{\$\}\right)^\omega$ s.t.

- $\forall i$, $C_i \in \Gamma^j Q \Gamma^{cn-j}$ for some j,
- $\forall i < m$, $C_i \vdash_M C_{i+1}$,
- $C_1 = q_0 w B^{cn-n}$, $C_m \in \Gamma^* F \Gamma^*$.

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Various acceptance conditions

Acceptance conditions of ω-automata

- Muller condition: $(Q, \Sigma, \delta, q_0, \mathcal{F})$, where $\mathcal{F} \subseteq 2^Q$,

  A run ρ is accepting iff $\mathrm{Inf}(\rho) \in \mathcal{F}$.

- Rabin condition: $(Q, \Sigma, \delta, q_0, (U_i, V_i)_{1 \leqslant i \leqslant k})$, where $\forall i.\ U_i, V_i \subseteq Q$,

  A run ρ is accepting iff $\exists i.\ \mathrm{Inf}(\rho) \cap U_i = \emptyset \wedge \mathrm{Inf}(\rho) \cap V_i \neq \emptyset$.

- Strett condition: $(Q, \Sigma, \delta, q_0, (U_i, V_i)_{1 \leqslant i \leqslant k})$, where $\forall i.\ U_i, V_i \subseteq Q$,

  A run ρ is accepting iff $\forall i.\ \mathrm{Inf}(\rho) \cap V_i \neq \emptyset \rightarrow \mathrm{Inf}(\rho) \cap U_i \neq \emptyset$.

- Parity condition: $(Q, \Sigma, \delta, q_0, c)$, where $c : Q \rightarrow \{1, \ldots, k\}$,

  A run ρ is accepting iff $\min(\{c(q) \mid q \in \mathrm{Inf}(\rho)\})$ *is even.*

- Rabin chain condition: A Rabin condition $(U_i, V_i)_{1 \leqslant i \leqslant k}$ s.t.

  $U_1 \subseteq V_1 \subseteq U_2 \subseteq V_2 \subseteq \cdots \subseteq U_k \subseteq V_k$.

Various acceptance conditions

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  A run ρ is accepting iff $\mathrm{Inf}(\rho) \in \mathcal{F}$.

- Rabin condition: $(Q, \Sigma, \delta, q_0, (U_i, V_i)_{1 \leqslant i \leqslant k})$, where $\forall i.\ U_i, V_i \subseteq Q$,

  A run ρ is accepting iff $\exists i.\ \mathrm{Inf}(\rho) \cap U_i = \varnothing \wedge \mathrm{Inf}(\rho) \cap V_i \neq \varnothing$.

- Strett condition: $(Q, \Sigma, \delta, q_0, (U_i, V_i)_{1 \leqslant i \leqslant k})$, where $\forall i.\ U_i, V_i \subseteq Q$,

  A run ρ is accepting iff $\forall i.\ \mathrm{Inf}(\rho) \cap V_i \neq \varnothing \rightarrow \mathrm{Inf}(\rho) \cap U_i \neq \varnothing$.

- Parity condition: $(Q, \Sigma, \delta, q_0, c)$, where $c : Q \rightarrow \{1, \ldots, k\}$,

  A run ρ is accepting iff $\min(\{c(q) \mid q \in \mathrm{Inf}(\rho)\})$ *is even.*

- Rabin chain condition: A Rabin condition $(U_i, V_i)_{1 \leqslant i \leqslant k}$ s.t.

  $U_1 \subseteq V_1 \subseteq U_2 \subseteq V_2 \subseteq \cdots \subseteq U_k \subseteq V_k$.

Observation. Parity $\equiv$ Rabin chain.

Parity $\Rightarrow$ Rabin chain: $c : Q \rightarrow \{1, \ldots, 2k+1\}$

$\forall i : 1 \leqslant i \leqslant k.\ U_i = \{q \mid c(q) \leqslant 2i - 1\}, V_i = \{q \mid c(q) \leqslant 2i\}$.

Rabin chain $\Rightarrow$ Parity: $\forall i : 1 \leqslant i \leqslant k.\ c(U_i \backslash V_{i-1}) = 2i - 1,\ c(V_i \backslash U_i) = 2i$.

Equivalence of all the acceptance conditions

From Büchi to the other conditions:

Let $\mathcal{B} = (Q, \Sigma, \delta, q_0, F)$ be a BA.

- Muller: $(Q, \Sigma, \delta, q_0, \mathcal{F})$ with $\mathcal{F} = \{P \mid P \cap F \neq \emptyset\}$,
- Rabin: $(Q, \Sigma, \delta, q_0, (\emptyset, F))$,
- Strett: $(Q, \Sigma, \delta, q_0, (F, Q))$,
- Parity: $(Q, \Sigma, \delta, q_0, c)$ with $c(F) = 0$ and $c(Q \backslash F) = 1$.

From Parity to Strett:

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, c)$ be a Parity automaton and $c : Q \to \{1, \ldots, 2k+1\}$.

Then $\mathcal{A}$ is equivalent to the Strett automaton $(Q, \Sigma, \delta, q_0, (U_i, V_i)_{0 \leqslant i \leqslant k})$, where $U_i = \{q \mid c(q) \leqslant 2i\}$, $V_i = \{q \mid c(q) \leqslant 2i+1\}$.

From Rabin and Strett to Muller:

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, (U_i, V_i)_{1 \leqslant i \leqslant k})$ be a Rabin (resp. Strett) automaton. Then $\mathcal{A}$ is equivalent to the Muller automaton $(Q, \Sigma, \delta, q_0, \mathcal{F})$, where
$\mathcal{F} = \{F \mid \exists i. F \cap U_i = \emptyset \wedge F \cap V_i \neq \emptyset\}$
(resp. $\mathcal{F} = \{F \mid \forall i. F \cap V_i \neq \emptyset \to F \cap U_i \neq \emptyset\}$).

Equivalence of all the acceptance conditions

From Muller to Büchi

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, \mathcal{F})$ be a Muller automaton s.t.

$\mathcal{F} = \{F_1, \ldots, F_k\}$ and $\forall i : 1 \leqslant i \leqslant k$. $F_i = \{q_i^1, \ldots, q_i^{l_i}\}$.

Construct a Büchi automaton $\mathcal{B} = (Q', \Sigma, \delta', q_0', F')$ as follows.

- $Q' = Q \cup \{(q, i, j) \mid q \in Q, 1 \leqslant i \leqslant k, 0 \leqslant j \leqslant |F_i|\}$,
- $q_0' = q_0$,
- $F' = \{(q, i, |F_i|) \mid q \in Q, 1 \leqslant i \leqslant k\}$,
- δ' is defined as follows,
  - δ' contains all the transitions in δ,
  - for every transition $(q, a, q') \in \delta$ and every $i : 1 \leqslant i \leqslant k$ such that $q' \in F_i$, $(q, a, (q', i, 0)) \in \delta'$,
  - for every transition $(q, a, q') \in \delta$,
    - if $q, q' \in F_i$ and $q' = q_i^{j+1}$, then $((q, i, j), a, (q, i, j + 1)) \in \delta'$,
    - if $q, q' \in F_i$ and $q' \neq q_i^{j+1}$, then $((q, i, j), (q', i, j)) \in \delta'$,
  - for every transition $(q, a, q') \in \delta$, if $q, q' \in F_i$, then $((q, i, l_i), a, (q', i, 0)) \in \delta'$.

Transformation between deterministic automata

Theorem. Deterministic Muller, Rabin, Strett and Parity automata are expressively equivalent.

From Parity to Rabin and Strett, from Rabin and Strett to Muller:

Same as the nondeterministic automata.

Transformation between deterministic automata

Theorem. Deterministic Muller, Rabin, Strett and Parity automata are expressively equivalent.

From deterministic Muller to deterministic Parity (Rabin chain):

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, \mathcal{F})$ be a deterministic Muller automaton.
Suppose $Q = \{q_0, \ldots, q_n\}$.
The main idea.

Latest appearance record (LAR)

$$q_{i_0} q_{i_1} \ldots q_{i_r} \sharp q_{i_{r+1}} \ldots q_{i_n}$$

$$\Big\downarrow \quad \delta(q_{i_n}, a) = q_{i_s}$$

$$q_{i_0} q_{i_1} \ldots q_{i_{s-1}} \sharp q_{i_{s+1}} \ldots q_{i_n} q_{i_s}$$

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Theorem. Deterministic Muller, Rabin, Strett and Parity automata are expressively equivalent.

From deterministic Muller to deterministic Parity (Rabin chain):

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, \mathcal{F})$ be a deterministic Muller automaton.
Suppose $Q = \{q_0, \ldots, q_n\}$.
Construct a Parity automaton $\mathcal{A}' = (Q', \Sigma, \delta', q_0', (U_i, V_i)_{0 \leqslant i \leqslant n})$ as follows.

- Q' is the set of sequences $u \sharp v$ s.t. uv is a permutation of $q_0 \ldots q_n$.
- $q_0' = \sharp q_n q_{n-1} \ldots q_0$.
- if $\delta(q_{i_n}, a) = q_{i_s}$, then

$$\delta'(q_{i_0} \ldots q_{i_r} \sharp q_{i_{r+1}} \ldots q_{i_n}, a) = q_{i_0} \ldots q_{i_{s-1}} \sharp q_{i_{s+1}} \ldots q_{i_n} q_{i_s}.$$

  In particular, if $\delta(q_{i_n}, a) = q_{i_n}$, then

$$\delta'(q_{i_0} \ldots q_{i_r} \sharp q_{i_{r+1}} \ldots q_{i_n}, a) = q_{i_0} \ldots \sharp q_{i_n}.$$

- $U_i = \{u \sharp v \mid |u| < i\}$, $V_i = U_i \cup \{u \sharp v \mid |u| = i, \exists F \in \mathcal{F}.\ F = v\}$.

$$U_0 \subseteq V_0 \subseteq U_1 \subseteq V_1 \subseteq \cdots \subseteq U_n \subseteq V_n.$$

Transformation between deterministic automata

Theorem. Deterministic Muller, Rabin, Strett and Parity automata are expressively equivalent.

From deterministic Muller to deterministic Parity (Rabin chain):

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, \mathcal{F})$ be a deterministic Muller automaton.
Suppose $Q = \{q_0, \ldots, q_n\}$.
Construct a Parity automaton $\mathcal{A}' = (Q', \Sigma, \delta', q'_0, (U_i, V_i)_{0 \leqslant i \leqslant n})$ as follows.

Correctness of the construction.
Let $w \in \Sigma^\omega$ and ρ be the accepting run of $\mathcal{A}$ over w. Then $\mathrm{Inf}(\rho) = F \in \mathcal{F}$.
Consider the run ρ' of $\mathcal{A}'$ corresponding to ρ.

$\exists j$ *s.t. after the position* j *in* ρ, *only the states in* $\mathrm{Inf}(\rho)$ *appear* $\Longrightarrow$
$\exists j' \geqslant j$ *s.t. after the position* j' *in* ρ',
all the states in $\mathrm{Inf}(\rho)$ *are on the right side of LAR* $\Longrightarrow$
$\exists i$ *s.t. after the position* j' *in* ρ', *all the LARs* $u\sharp v$ *satisfy* $|u| \geqslant i$,
and $\exists^\omega u\sharp v$ *s.t.* $|u| = i$ *and* $v = \mathrm{Inf}(\rho) = F \Longrightarrow$
$\mathrm{Inf}(\rho') \cap U_i = \varnothing$ *and* $\mathrm{Inf}(\rho') \cap V_i \neq \varnothing$

Transformation between deterministic automata

Theorem. Deterministic Muller, Rabin, Strett and Parity automata are expressively equivalent.

From deterministic Muller to deterministic Parity (Rabin chain):

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, \mathcal{F})$ be a deterministic Muller automaton.

Suppose $Q = \{q_0, \ldots, q_n\}$.

Construct a Parity automaton $\mathcal{A}' = (Q', \Sigma, \delta', q'_0, (U_i, V_i)_{0 \leqslant i \leqslant n})$ as follows.

Correctness of the construction.

Let $w \in \Sigma^\omega$ and ρ' be the accepting run of $\mathcal{A}'$ over w.

$\exists i$ *s.t.* $\mathrm{Inf}(\rho') \cap U_i = \varnothing$ *and* $\mathrm{Inf}(\rho') \cap V_i \neq \varnothing \Longrightarrow$

$\exists F \in \mathcal{F}$ *and* j' *s.t.* $u \sharp v$ *in the position* j' *of* ρ' *satisfies* $|u| = i, v = F$,

and after the position j' *in* ρ',

all $u' \sharp v'$ *satisfy* $|u'| \geqslant i$, *and* $\exists^\omega u' \sharp v'$, $|u'| = i, v' = F \Longrightarrow$

Consider the run ρ *of* $\mathcal{A}$ *over* w*: After the position* j' *in* ρ,

only states in F *occur* (*o.w.* $u' \sharp v'$ *s.t.* $|u'| < i$ *occurs after* j' *in* ρ'),

and every state in F *occur infinitely often* (*o.w.* $\exists j'' > j'$, *all* $u' \sharp v'$ *after* j'' *satisfy* $|u'| > i$, *thus* $\mathrm{Inf}(\rho') \cap V_i = \varnothing$).

Therefore, ρ is accepting.

Outline

1 Motivation

2 Büchi automata

3 Closure properties

4 Equivalence with MSO

5 Decision problem

6 Muller, Rabin, Strett, and Parity automata

7 Determinization

8 Equivalence with WMSO

Deterministic Muller automata (DMA)

Proposition. The class of languages recognized by DMA is closed under all Boolean operations.

- Union: $\mathcal{A}_1 = (Q_1, \Sigma, \delta_1, q_0^1, \mathcal{F}_1)$ and $\mathcal{A}_2 = (Q_2, \Sigma, \delta_2, q_0^2, \mathcal{F}_2)$.
  $\mathcal{A} = (Q_1 \times Q_2, \Sigma, \delta, (q_0^1, q_0^2), \mathcal{F})$, where
  - $\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$,
  - $\mathcal{F} = \{S \subseteq Q_1 \times Q_2 \mid \text{proj}_2(S) \in \mathcal{F}_2\} \cup \{S \subseteq Q_1 \times Q_2 \mid \text{proj}_1(S) \in \mathcal{F}_1\}$.
- Intersection: $\mathcal{A}_1 = (Q_1, \Sigma, \delta_1, q_0^1, \mathcal{F}_1)$ and $\mathcal{A}_2 = (Q_2, \Sigma, \delta_2, q_0^2, \mathcal{F}_2)$.
  $\mathcal{A} = (Q_1 \times Q_2, \Sigma, \delta, (q_0^1, q_0^2), \mathcal{F})$, where
  - $\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$,
  - $\mathcal{F} = \{S \subseteq Q_1 \times Q_2 \mid \text{proj}_1(S) \in \mathcal{F}_1, \text{proj}_2(S) \in \mathcal{F}_2\}$.
- Complementation: $\mathcal{A} = (Q, \Sigma, \delta, q_0, \mathcal{F}) \Rightarrow \mathcal{B} = (Q, \Sigma, \delta, q_0, 2^Q \backslash \mathcal{F})$.

Expressibility of DMA

Theorem. An ω-language L is definable by a DMA iff
L is a Boolean combination of sets $\overrightarrow{W}$ for regular $W \subseteq \Sigma^*$.

Proof.

"If" direction:

- $\overrightarrow{W}$ is recognized by a deterministic Büchi automata,
- The class of languages recognized by DMAs is closed under all Boolean combinations.

"Only if" direction:
Suppose L is defined by a DMA $\mathcal{A} = (Q, \Sigma, \delta, q_0, \mathcal{F})$.
For every $q \in Q$, let W_q denote the language defined by DFA $(Q, \Sigma, \delta, q_0, \{q\})$.
Then

$$L = \bigcup_{F \in \mathcal{F}} \left(\bigcap_{q \in F} \overrightarrow{W_q} \cap \bigcap_{q \notin F} \overline{\overrightarrow{W_q}} \right).$$

□

Mcnaughton's theorem: NBA $\equiv$ DMA

Theorem. From every nondeterministic Büchi automaton, an equivalent DMA can be constructed.

NBA $\Rightarrow$ *Semi-deterministic Büchi automata (SDBA)* $\Rightarrow$ *DMA*

Using the slides and lecture notes by Bernd Finkbeiner.

NBA $\Rightarrow$ SDBA:

- Slides: `http://www.react.uni-saarland.de/teaching/automata-games-verification-12/downloads/intro6.pdf`
- Lecture notes: `http://www.react.uni-saarland.de/teaching/automata-games-verification-12/downloads/notes5.pdf`

SDBA $\Rightarrow$ DMA:

- Slides: `http://www.react.uni-saarland.de/teaching/automata-games-verification-12/downloads/intro7.pdf`
- Lecture notes: `http://www.react.uni-saarland.de/teaching/automata-games-verification-12/downloads/notes6.pdf`

Homework: Prove that the construction from SDBA to DMA is correct.

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1. Motivation
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8. Equivalence with WMSO

ω-regular $\equiv$ WMSO

WMSO:

The same syntax as MSO, with the interpretations of set variables restricted to finite sets.

WMSO to MSO: WMSO $\varphi \Rightarrow$ MSO φ'

$$(\exists X \eta)' = \exists X (\exists y \forall x (X(x) \rightarrow x \leqslant y) \wedge \eta').$$

From DMA to WMSO:

It is sufficient to show that $\overrightarrow{W}$ with W regular can be defined by a WMSO sentence φ.

W is regular $\Rightarrow \exists$ a MSO sentence ψ on finite words equivalent to W.

Then $\overrightarrow{W}$ is defined by $\forall x \exists y (x < y \wedge \psi_{\leqslant y})$, where $\psi_{\leqslant y}$ is obtained from ψ as follows:

- Replace every subformula $\exists X \eta$ with $\exists X (\forall x (X(x) \rightarrow x \leqslant y) \wedge \eta_{\leqslant y})$.
- Replace every subformula $\exists x \eta$ with $\exists x (x \leqslant y \wedge \eta_{\leqslant y})$.

Automata over finite trees