Automata theory and its applications

Lecture 17 -18: Automata over unranked trees

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Outline

1. Hedge automata
2. Closure properties
3. Minimization
4. Decision problems

Unranked trees: The motivation

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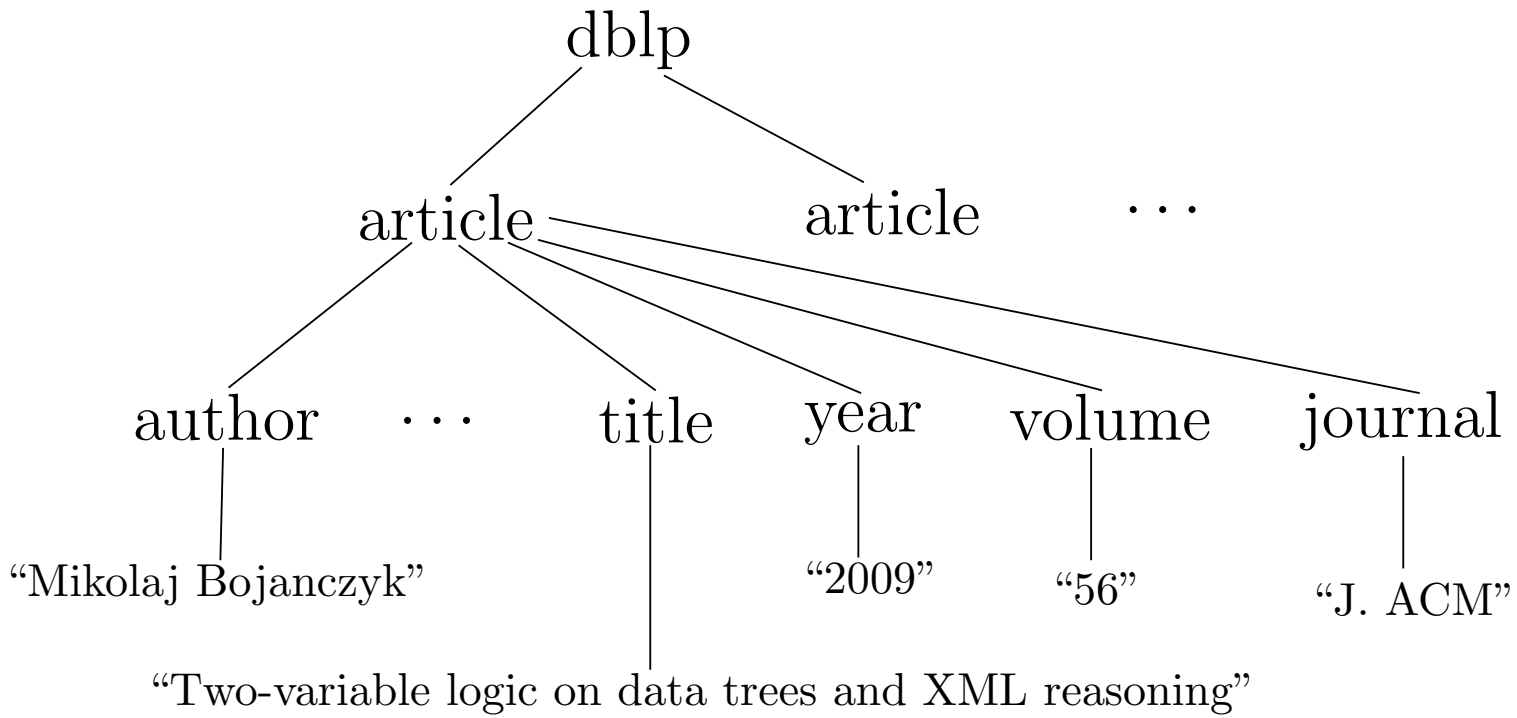
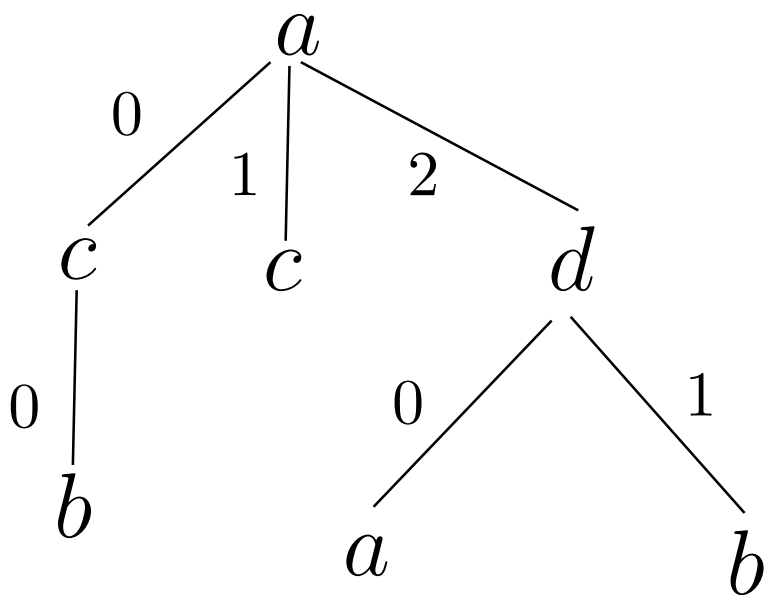
Unranked trees: The motivation



Unranked trees: The definition

An unranked tree t is a binary tuple (D, L), where

- $D \subseteq \mathbb{N}^*$ is a finite *tree domain*, that is,
  - D is prefix closed,
  - if $xi \in D$, then $x0, \ldots, x(i-1) \in D$.
- $L : D \to \Sigma$.



U_Σ: The set of unranked trees over Σ.

Nondeterministic finite hedge automata (NFHA)

A *hedge*: A sequence of unranked trees $t_1, \ldots, t_n$.

A NFHA $\mathcal{A}$ is a tuple (Q, Σ, δ, F), where

- $F \subseteq Q$,
- δ is a finite set of transition rules of the form $a(R) \to q$, where
  - $a \in \Sigma$, $q \in Q$,
  - $R \subseteq Q^*$ is a regular language over Q.

  The languages R are called the *horizontal languages*.

A *run* of $\mathcal{A}$ over a tree $t = (D, L)$ is a tree $r_{\mathcal{A},t} = (D, L')$ s.t.

$\forall x \in D$ *labeled by* a *and with children* $x0, \ldots, xi$,
$\exists$ *a rule* $a(R) \to q$ *s.t.* $r_{\mathcal{A},t}(x0) \ldots r_{\mathcal{A},t}(xi) \in R$ *and* $r_{\mathcal{A},t} = q$.

A run $r_{\mathcal{A},t}$ is *accepting* if $r_{\mathcal{A},t}(\varepsilon) \in F$.

Let $L \subseteq U_\Sigma$. Then L is regular if $\exists$ a NFHA $\mathcal{A}$ recognizing L.

Normalised and deterministic NFHA

A *normalised* NFHA is a NFHA $\mathcal{A} = (Q, \Sigma, \delta, F)$ s.t.

$\forall (a, q) \in \Sigma \times Q$, $\exists$ *at most one rule of the form* $a(R) \to q$ *in* δ.

Proposition. $\forall$ NFHA $\mathcal{A}$, $\exists$ an equivalent normalised NFHA $\mathcal{A}'$.

Proof.

$a(R_1) \to q, a(R_2) \to q \Longrightarrow a(R_1 \cup R_2) \to q$. □

Normalised and deterministic NFHA

A *deterministic* finite hedge automaton (DFHA) is a NFHA $\mathcal{A} = (Q, \Sigma, \delta, F)$ s.t.

$$\forall a \in \Sigma,\ \textit{if}\ a(R_1) \to q_1, a(R_2) \to q_2\ \textit{s.t.}\ q_1 \neq q_2,\ \textit{then}\ R_1 \cap R_2 = \varnothing.$$

Proposition. $\forall$ NFHA $\mathcal{A}$, $\exists$ an equivalent DFHA $\mathcal{A}'$ of exponential size.

Proof.

Let $\mathcal{A} = (Q, \Sigma, \delta, F)$ be a NFHA.
Assume $\mathcal{A}$ is normalised. For $(a, q) \in \Sigma \times Q$, let $R_{a,q}$ denote $R : a(R) \to q$.
Define $\mathcal{A}' = (2^Q, \Sigma, \delta', F')$ as follows.

- $F' = \{S \mid S \cap F \neq \varnothing\}$,
- $\forall a \in \Sigma, S \in 2^Q$, $a(R') \to S \in \delta'$, where
  $S_1 \dots S_n \in R'$ iff $S = \{q | \exists q_1 \in S_1, \dots, q_n \in S_n.\ q_1 \dots q_n \in R_{a,q}\}$.
  R' is regular:
  Define $L_{a,q}$ as $\{S_1 \dots S_n \mid \exists q_1 \in S_1, \dots, q_n \in S_n.\ q_1 \dots q_n \in R_{a,q}\}$.
  Then $R' = (\bigcup_{q \in S} L_{a,q}) \backslash (\bigcup_{q \notin S} L_{a,q})$.

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2. Closure properties
3. Minimization
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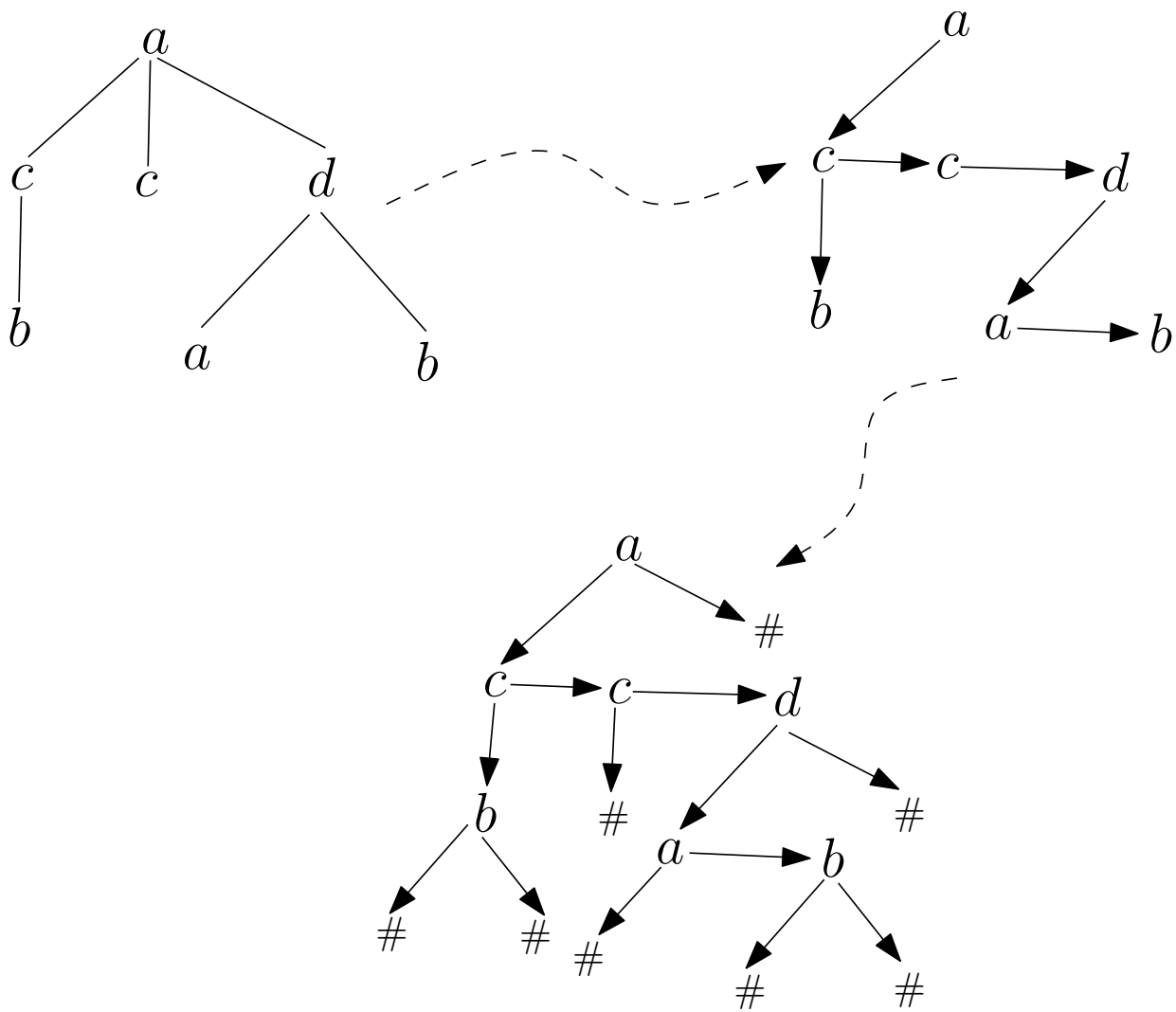
First-child-next-sibling (FCNS) encoding

An example:



First-child-next-sibling (FCNS) encoding

For every unranked tree t over the alphabet Σ, construct a ranked tree $\mathrm{fcns}(t)$ as follows.

- $\mathrm{fcns}(a) = a(\#, \#)$,
- $\mathrm{fcns}(a(t_1, \ldots, t_n)) = a(\mathrm{fcns}(t_1, \ldots, t_n), \#)$,
- for every hedge $t_1, \ldots, t_n$ with $n \geqslant 2$,
  $\mathrm{fcns}(t_1, \ldots, t_n) = \mathrm{fcns}(t_1) \mathbin{@} \mathrm{fcns}(t_2, \ldots, t_n)$, where $a(s, \#) \mathbin{@} s' = a(s, s')$.

For a tree language $L \subseteq U_\Sigma$, let $\mathrm{fcns}(L) = \{\mathrm{fcns}(t) \mid t \in L\}$.

Example:

$$\begin{aligned}\mathrm{fcns}(d(a,b)) &= d(\mathrm{fcns}(a,b), \#) = d(\mathrm{fcns}(a)@\mathrm{fcns}(b), \#) \\ &= d(a(\#,\#)@b(\#,\#), \#) = d(a(\#, b(\#,\#)), \#)\end{aligned}$$

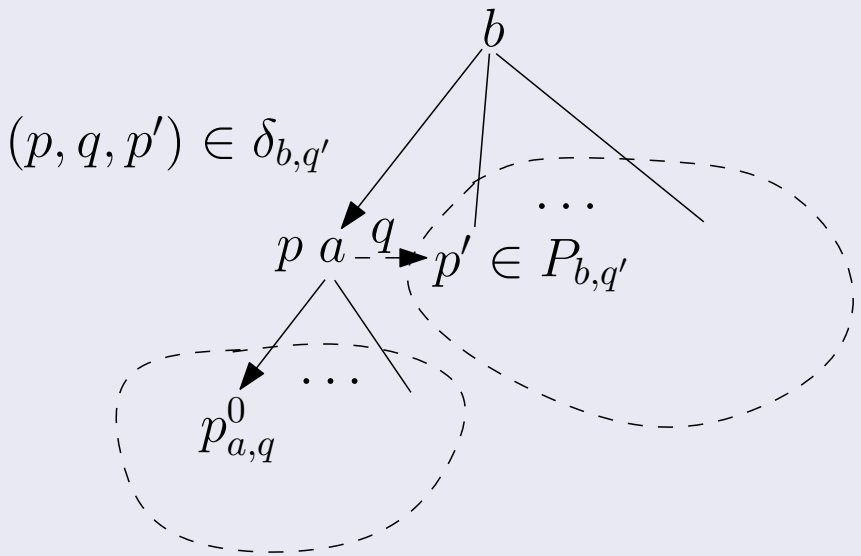
From unranked regular to ranked regular

Proposition. Let $L \subseteq U_\Sigma$. Then L is regular implies that fcns(L) is regular.

Proof.

Suppose L is recognized by a normalised NFHA $\mathcal{A} = (Q, \Sigma, \delta, F)$.
For every (a, q) s.t. $R_{a,q} \neq \emptyset$,
suppose that $R_{a,q}$ is recognized by a NFA $\mathcal{B}_{a,q} = (P_{a,q}, Q, \delta, p^0_{a,q}, F_{a,q})$.
W.l.o.g. assume that the state sets $P_{a,q}$ are disjoint from each other.

The intuition:



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W.l.o.g. assume that the state sets $P_{a,q}$ are disjoint from each other.

Define $\mathcal{A}' = (Q', \Sigma, \delta', F')$ as follows.

- $Q' = Q \cup \bigcup_{a,q} P_{a,q} \cup \{q_\#\}$,
- $F' = F$,
- δ' is defined by the following rules,
  - $(\#, q_\#) \in \delta'$,
  - $(p^0_{a,q}, q_\#, a, q) \in \delta'$,
  - if $\exists b, q, p', q'$ s.t. $p, p' \in P_{b,q'}$, $(p, q, p') \in \delta_{b,q'}$, then $(p^0_{a,q}, p', a, p) \in \delta'$,
  - if $\exists b, q, p', q'$ s.t. $\varepsilon \in R_{a,q}$, $p \in P_{b,q'}$, $p' \in F_{b,q'}$, $(p, q, p') \in \delta_{b,q'}$, then $(q_\#, q_\#, a, p) \in \delta'$,
  - if $\exists b, q, p', q'$ s.t. $\varepsilon \in R_{a,q}$, $p, p' \in P_{b,q'}$, $(p, q, p') \in \delta_{b,q'}$, then $(q_\#, p', a, p) \in \delta'$.

From ranked regular to unranked regular

Proposition. Let $L \subseteq T_{\Sigma \cup \{\#\}}$. Then L is regular implies $\mathrm{fcns}^{-1}(L)$ is regular.

A notation: $\mathrm{fcns}^{-1}(L) = \{t \in U_\Sigma \mid \mathrm{fcns}(t) \in L\}$.

Proof sketch.

Let $\mathcal{A} = (Q, \Sigma \cup \{\#\}, \delta, F)$ be a NBUT (over ranked trees).
We can define a NFHA $\mathcal{A}'$ such that the horizontal languages of $\mathcal{A}'$ are used to simulate the partial runs of $\mathcal{A}$ over the paths $x01^*$. □

Homework. Give the detailed construction for $\mathrm{fcns}^{-1}(L)$ from $\mathcal{A}$ in the above proof.

Closure properties

Corollary. The set of regular languages over unranked trees are closed under all Boolean operations.

Fact: The following facts hold for fcns and fcns^{-1}.

- $\text{fcns} : U_\Sigma \to T_\Sigma$ is an injective (non-surjective) function,
- $\text{fcns}^{-1} : T_\Sigma \to U_\Sigma$ is an injective and surjective partial function.

Proof.

Suppose $L_1, L_2 \subseteq U_\Sigma$ are regular.
The corollary follows from the closure property of regular languages over ranked trees and the following equations.

- $L_1 \cup L_2 = \text{fcns}^{-1}(\text{fcns}(L_1) \cup \text{fcns}(L_2))$, similarly for $\cap$,
- $U_\Sigma \backslash L_1 = \text{fcns}^{-1}(T_{\Sigma \cup \{\#\}} \backslash \text{fcns}(L_1))$.

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Representation of horizontal languages

We use DFA or NFA to represent the horizontal languages,

- $NFHA(NFA)$: NFHA with horizontal languages represented by NFA,
- $NFHA(DFA)$: NFHA with horizontal languages represented by DFA,
- $DFHA(NFA)$: DFHA with horizontal languages represented by NFA,
- $DFHA(DFA)$: DFHA with horizontal languages represented by DFA.

Nondeterminism of DFHA(DFA)

Let $\mathcal{A} = (Q, \Sigma, \delta, F)$ be a DFHA(DFA).
W.l.o.g, assume that $\mathcal{A}$ is normalised:

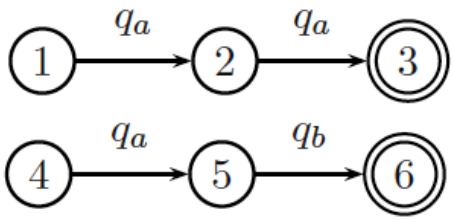
$\forall(a, q)$, $R_{a,q}$ is given by a DFA $\mathcal{B}_{a,q} = (Q_{a,q}, \Sigma, \delta_{a,q}, q^0_{a,q}, F_{a,q})$.

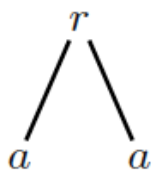
Size of a DFHA(DFA) $\mathcal{A}$: $|Q| + \sum_{a,q} |Q_{a,q}|$.

Nondet. choice of different DFAs:

$\mathcal{A} = (\{q_1, q_2, q_a, q_b\}, \{r, a, b\}, \delta, \{q_1, q_2\})$, where δ is defined as follows,

- $a(\{\varepsilon\}) \to q_a$, $b(\{\varepsilon\}) \to q_b$,
- $r(\{q_a q_a\}) \to q_1$, $r(\{q_a q_b\}) \to q_2$.





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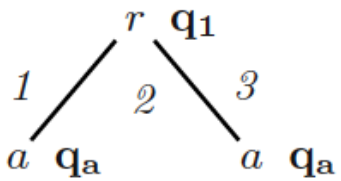
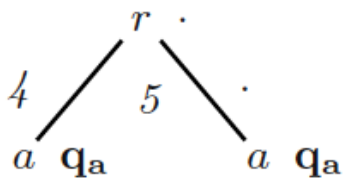


(b)

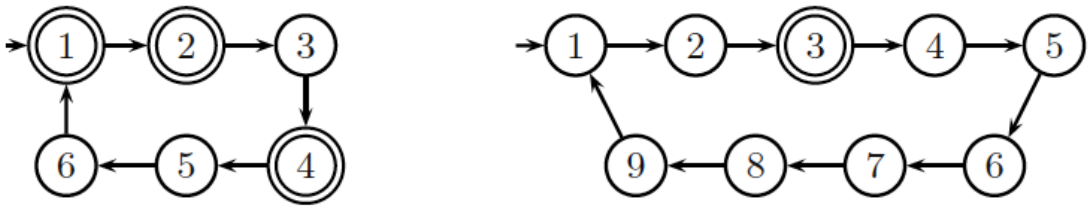


(c)

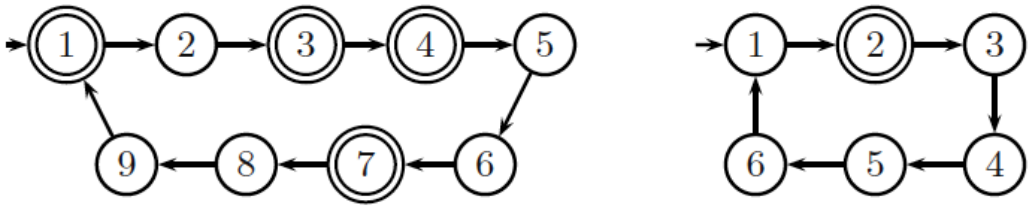
Minimization of DFHA(DFA) is dificult

No unique minimum DFHA(DFA) for a given language:

The language $\{r(a(w)) \mid w \in L\}$, where
$L = L_1 \cup L_2 \cup L_3$, $L_1 = (bbb)^*$, $L_2 = b(bbbbbb)^*$ and $L_3 = bb(bbbbbbbbb)^*$.



(a)



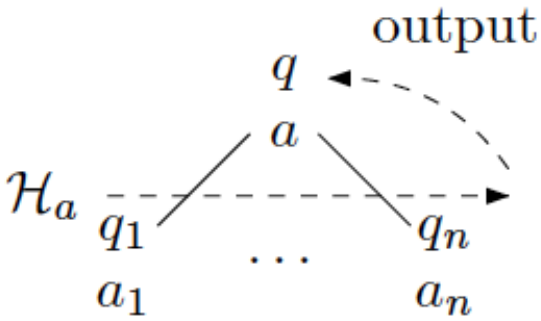
(b)

Theorem. DFHA(DFA) minimization is NP-complete.

Reference. W. Martens, and J. Niehren, Minimizing Tree Automata for Unranked Trees, DBPL 2005.

Deterministic stepwise hedge automata (DSHA)

The intuition (Deterministic horizontal automata with output):



A DSHA is a tuple $\mathcal{A} = (Q, \Sigma, \delta_0, F, \delta)$, where

- Q, Σ, F are as usual,
- $\delta_0 : \Sigma \to Q$ is the initial state assignment function,
- and $\delta : Q \times Q \to Q$ is the transition function.

For every $a \in \Sigma$, define δ_a^* as follows,

$$\delta_a^*(\varepsilon) = \delta_0(a),\ \delta_a^*(wq) = \delta(\delta_a^*(w), q).$$

A run of a DSHA $\mathcal{A} = (Q, \Sigma, \delta_0, F, \delta)$ over a tree $t = (D, L)$:

A tree $r_{\mathcal{A},t} = (D, L')$ s.t.

$\forall$ *node* x *with label* a *and children* $x0, \ldots, xk$,

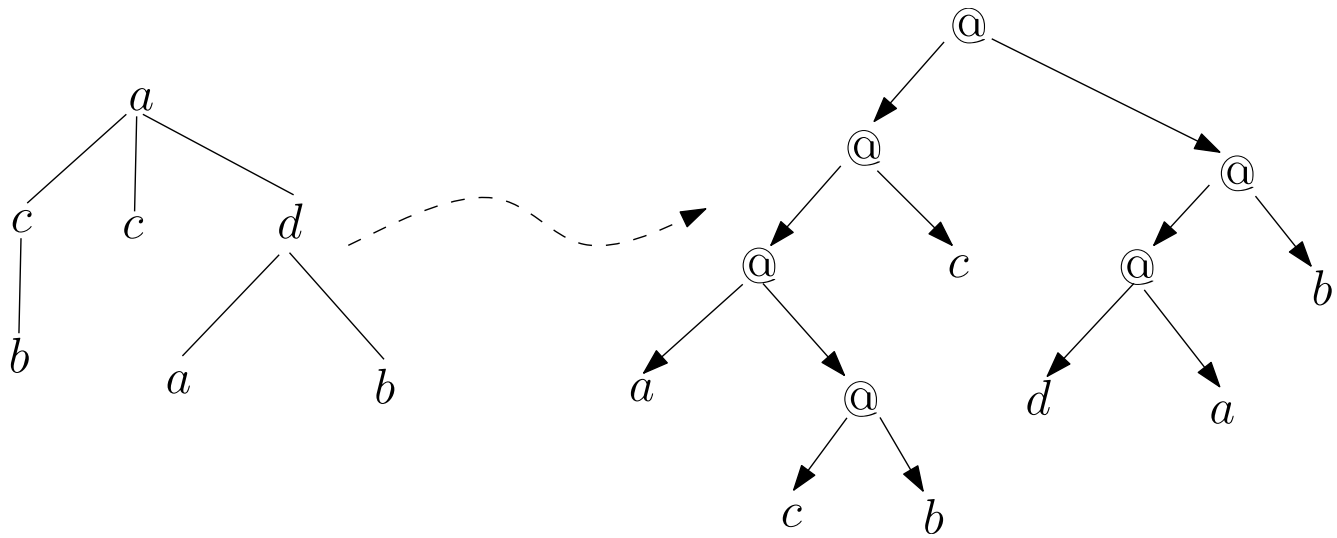
$$L'(x) = \delta_a^*(L'(x0) \ldots L'(xk)).$$

Extension operator

Extension operator @:

Given two trees $t, t' \in U_\Sigma$ s.t. $t = a(t_1, \ldots, t_n)$, then $t@t' = a(t_1, \ldots, t_n, t')$.

Encoding of unranked trees by extension operator:



The *extension encoding* $ext : U_\Sigma \to T_{\{@\}\cup\Sigma}$,

- $ext(a) = a$,
- $ext(a(t_1, \ldots, t_n)) = @(ext(a(t_1, \ldots, t_{n-1})), ext(t_n))$.

Proposition. $ext : U_\Sigma \to T_{\{@\}\cup\Sigma}$ is a bijection.

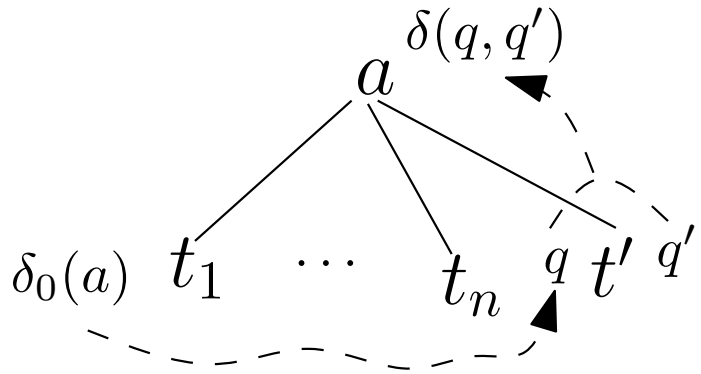
For $L \subseteq U_\Sigma$, define $ext(L) = \{ext(t) \mid t \in L\}$.

DSHA as DBUTA over $T_{\{@\}\cup\Sigma}$

The intuition:

ranked	stepwise
(a, q)	$\delta_0(a) = q$
$(q_1, q_2, @, q)$	$\delta(q_1, q_2) = q$

Lemma. Let $\mathcal{A} = (Q, \Sigma, \delta_0, F, \delta)$ be a DSHA and $t, t' \in U_\Sigma$ s.t. $r_{\mathcal{A},t}(\varepsilon) = q$ and $r_{\mathcal{A},t'}(\varepsilon) = q'$, then $r_{\mathcal{A},t@t'}(\varepsilon) = \delta(q, q')$.



DSHA as DBUTA over $T_{\{@\}\cup\Sigma}$

The intuition:

ranked	stepwise
(a, q)	$\delta_0(a) = q$
$(q_1, q_2, @, q)$	$\delta(q_1, q_2) = q$

Lemma. Let $\mathcal{A} = (Q, \Sigma, \delta_0, F, \delta)$ be a DSHA and $t, t' \in U_\Sigma$ s.t. $r_{\mathcal{A},t}(\varepsilon) = q$ and $r_{\mathcal{A},t'}(\varepsilon) = q'$, then $r_{\mathcal{A},t@t'}(\varepsilon) = \delta(q, q')$.

For a DSHA $\mathcal{A} = (Q, \Sigma, \delta_0, F, \delta)$, define $ext(\mathcal{A}) = (Q, \{@\} \cup \Sigma, \delta', F)$, where

- for every $a \in \Sigma$, $(a, q) \in \delta'$ iff $\delta_0(a) = q$,
- $(q_1, q_2, @, q) \in \delta'$ iff $\delta(q_1, q_2) = q$.

Proposition. For every DSHA $\mathcal{A}$, $L(ext(\mathcal{A})) = ext(L(\mathcal{A}))$.

Minimization of DSHA

Theorem. For each regular language $L \subseteq U_\Sigma$, there is a unique minimum DSHA recognizing L.

Proof.

Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are two minimum DSHAs for L.
Then $ext(\mathcal{A}_1)$ and $ext(\mathcal{A}_2)$ are two minimum DBTUAs for $ext(L)$.
Therefore, $ext(\mathcal{A}_1) \cong ext(\mathcal{A}_2)$, so $\mathcal{A}_1 \cong \mathcal{A}_2$. □

Minimization of DSHA

Theorem. For each regular language $L \subseteq U_\Sigma$, there is a unique minimum DSHA recognizing L.

Define a congruence $\equiv_L$ (wrt. @) over U_Σ as follows:

$$t_1 \equiv_L t_2 \textit{ iff } ext(t_1) \equiv_{ext(L)} ext(t_2).$$

Theorem. Let $L \subseteq U_\Sigma$. Then L is regular iff $\equiv_L$ is of finite index.

Proof.

Suppose $\equiv_L$ is of finite index. Then L is recognized by the DSHA $(Q, \Sigma, \delta_0, F, \delta)$, where

- $Q = \{[t] \mid t \in U_\Sigma\}$, $F = \{[t] \mid t \in L\}$,
- $\delta_0(a) = [a]$, $\delta([t_1], [t_2]) = [t_1@t_2]$.

Suppose L is regular. Then $ext(L)$ is also regular.
So $\equiv_{ext(L)}$, and thus $\equiv_L$, is of finite index. □

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Membership

Theorem. The membership problem for NFHA(NFA) is in PTIME.

Proof.

Let $\mathcal{A} = (Q, \Sigma, \delta, F)$ be a NFHA(NFA) and t be a tree.
The following problem can be solved in PTIME.

> *Given a NFA* $\mathcal{B} = (Q', Q, \delta', q_0, F')$, *a word* $S_1 \dots S_n \in (2^Q)^*$,
> *decide whether* $\exists q_1 \dots q_n$ *s.t.* $\forall i.\ q_i \in S_i$, *and* $q_1 \dots q_n$ *is accepted by* $\mathcal{B}$.

1. Compute the set of reachable states
   - $P_0 = \{q_0\}$,
   - for $i > 0$, $P_i = \{q' \mid \exists p' \in P_{i-1}, q \in S_i, (p', q, q') \in \delta'\}$.
2. Check whether $P_n \cap F' \neq \varnothing$.

Then the set of reachable states of $\mathcal{A}$ after reading t in bottom-up can be computed in PTIME. □

Emptiness

Theorem. The emptiness problem for NFHA(NFA) is in PTIME.

Proof.

Let $\mathcal{A} = (Q, \Sigma, \delta, F)$ be a normalised NFHA(NFA).

A state q is reachable if $\exists$ a run of $\mathcal{A}$ over a tree t, say r, s.t. $r(\varepsilon) = q$.

Compute the set of reachable states $R_{\mathcal{A}}$ as follows, until $R_{\mathcal{A},i} = R_{\mathcal{A},i-1}$.

- $R_{\mathcal{A},0} = \{q \mid \exists a, R.\ a(R) \to q \in \delta, \varepsilon \in R\}$,
- $R_{\mathcal{A},i} = R_{\mathcal{A},i-1} \cup \{q \mid \exists a, R.\ a(R) \to q, R \cap (R_{\mathcal{A},i-1})^* \neq \varnothing\}$.

Claim. $L(\mathcal{A})$ is nonempty iff $R_{\mathcal{A}} \cap F \neq \varnothing$. □

Inclusion

Theorem. The inclusion problem of NFHA(NFA) is EXPTIME-complete.

Proof.

Upper bound:
Let $\mathcal{A}, \mathcal{A}'$ be two NFHA(NFA).

1. Determinize $\mathcal{A}'$: In EXPTIME, obtaining a complete DFHA(DFA) $\mathcal{A}''$,
2. Complement $\mathcal{A}''$ to get $\mathcal{A}'''$: Just complementing the set of accepting states,
3. Decide whether $L(\mathcal{A}) \cap L(\mathcal{A}''')$ is empty: In polynomial time over the size of $\mathcal{A}$ and $\mathcal{A}'''$.

Lower bound:
Reduction from APSPACE TMs to the universality of NFHA(NFA).
Similar to the lower bound for the universality of NBTUA over ranked trees. □

Determinism

Theorem. Checking whether a NFHA(NFA) is deterministic is in PTIME.

Proof.

Let $\mathcal{A} = (Q, \Sigma, \delta, F)$ be a normalised NFHA(NFA) s.t. every $R_{a,q}$ is given by a NFA $\mathcal{B}_{a,q}$.
The following computation is in PTIME:

$$\forall a(R_{a,q_1}) \to q_1, a(R_{a,q_2}) \to q_2 \in \delta \ s.t. \ q_1 \neq q_2, \ test \ R_{a,q_1} \cap R_{a,q_2} = \varnothing.$$

Completeness

A *complete* NFHA is a NFHA $\mathcal{A} = (Q, \Sigma, \delta, F)$ s.t.

$$\forall t,\ \exists \textit{ at least one run of } \mathcal{A} \textit{ over } t.$$

Theorem. Completeness of NFHA(NFA) (resp. DFHA(DFA)) is PSPACE-c.

Proof.

PSPACE-hardness: Reduction from intersection of DFAs.
Suppose $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are n DFAs s.t. $\mathcal{A}_i = (Q_i, \Sigma, \delta_i, q_{0,i}, F_i)$.
Construct a DFHA(DFA) $\mathcal{B} = (Q', \Sigma', \delta', F')$, where

- $Q' = \{q_1, q_r\} \cup \{q_a \mid a \in \Sigma\}$, $\Sigma' = \Sigma \cup \{r\}$, $F' = \{q_r\}$,
- δ' is defined as follows.
  Let $R_i = prj(L(\overline{\mathcal{A}_i}))$ (where $prj(a) = q_a$), $R' = Q'^*(\{q_r, q_1\})Q'^*$.
  Then $\delta' = \{a(R_i) \to q_a, a(R') \to q_1, r(R_i) \to q_r, r(R') \to q_1\}$.

Claim.
$\mathcal{B}$ is complete iff $\bigcup_i R_i = (\{q_a \mid a \in \Sigma\})^*$ iff $\bigcup_i L(\overline{\mathcal{A}_i}) = \Sigma^*$ iff $\bigcap_i L(\mathcal{A}_i) = \varnothing$.

Completeness

A *complete* NFHA is a NFHA $\mathcal{A} = (Q, \Sigma, \delta, F)$ s.t.

$\forall t$, $\exists$ *at least one run of* $\mathcal{A}$ *over* t.

Theorem. Completeness of NFHA(NFA) (resp. DFHA(DFA)) is PSPACE-c.

Proof.

In PSPACE:

Let $\mathcal{A} = (Q, \Sigma, \delta, F)$ be a normalised NFHA(NFA)

s.t. the languages $R_{a,q}$ *are defined by NFAs* $\mathcal{B}_{a,q}$.

W.l.o.g assume that all the states of $\mathcal{A}$ are reachable.

A state q *is reachable if* $\exists$ *a run of* $\mathcal{A}$ *over a tree* t, *say* r, *s.t.* $r(\varepsilon) = q$.

Claim. $\mathcal{A}$ is complete iff $\forall a \in \Sigma. \bigcup_{q \in Q} R_{a,q} = Q^*$.

From the NFAs $\mathcal{B}_{a,q}$,

- an NFA $\mathcal{C}_a$ can be constructed in PTIME to recognize $\bigcup_{q \in Q} R_{a,q}$,
- the universality of $\mathcal{C}_a$ can be checked in PSPACE.

□

(Possibly) Open questions

DFHA(DFA):

The complexity of inclusion problem

Next lecture

Applications to model checking