On the Expressive Power of QLTL*

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Abstract. LTL cannot express the whole class of ω-regular languages and several extensions have been proposed. Among them, Quantified propositional Linear Temporal Logic ($QLTL$), proposed by Sistla, extends LTL by quantifications over the atomic propositions. The expressive power of LTL and its fragments have been made relatively clear by numerous researchers. However, there are few results on the expressive power of $QLTL$ and its fragments (besides those of LTL). In this paper we get some initial results on the expressive power of $QLTL$. First, we show that both $Q(U)$ (the fragment of $QLTL$ in which "Until" is the only temporal operator used, without restriction on the use of quantifiers) and $Q(F)$ (similar to $Q(U)$, with temporal operator "Until" replaced by "Future") can express the whole class of ω-regular languages. Then we compare the expressive power of various fragments of $QLTL$ in detail and get a panorama of the expressive power of fragments of $QLTL$. Finally, we consider the quantifier hierarchy of $Q(U)$ and $Q(F)$, and show that one alternation of existential and universal quantifiers is necessary and sufficient to express the whole class of ω-regular languages.

1 Introduction

Linear Temporal Logic (LTL) was first defined by the philosopher A. Prior in 1957 [9] as a tool to reason about the temporal information. Later, in 1977, A. Pnueli introduced LTL into computer science to reason about the behaviors of reactive systems [8]. Since then, it has become one of the most popular temporal logics used in the specification and verification of reactive systems.

Expressive power is one of the main concerns of temporal logics. Perhaps because of their popularity, the expressive power of LTL and its fragments have been made relatively clear by numerous researchers. A well-known result is that an ω-regular language is LTL-definable iff it is first order definable iff it is ω-star free iff its syntactic monoid is aperiodic [5, 4, 14, 15, 7]. Since the class of ω-star-free languages is a strict subclass of the class of ω-regular languages,

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some natural temporal properties such as the property that the proposition p holds at all even positions cannot be expressed in LTL [18]. Consequently several extensions of LTL have been proposed to define the whole class of ω-regular languages. Among them we mention Extended Temporal Logic (ETL) [19], linear μ-calculus (νTL)[17] and Quantified propositional Linear Temporal Logic ($QLTL$, also known as $QPTL$) [11].

$QLTL$ extends LTL by quantifications over atomic propositions. While the expressive power of LTL and its fragments have been made relatively clear, there are few results on the expressive power of $QLTL$ and its fragments (besides those of LTL). A well-known result is that ω-regular languages can be expressed by X, F operators and existential quantifiers in $QLTL$ [2, 12], which, nevertheless, is almost all we know about the expressive power of $QLTL$ and its fragments besides those of LTL. We do not even know whether several natural fragments of $QLTL$, e.g. $Q(U)$ (the fragment of $QLTL$ in which "Until" is the only temporal operator used, without restriction on the use of quantifiers) and $Q(F)$ (similar to $Q(U)$, with temporal operator "Until" replaced by "Future"), are expressively equivalent to $QLTL$ or not. Consequently we believe that the expressive power of $QLTL$ could be made clearer, which is the main theme of this paper.

In this paper, we first give a positive answer to the question whether $Q(U)$ and $Q(F)$ can define the whole class of ω-regular languages. Then we compare the expressive power of various fragments of $QLTL$ in detail and get a panorama of the expressive power of fragments of $QLTL$. In particular, we show that the expressive power of $EQ(F)$(the fragments of $QLTL$ containing formulas of the form $\exists q_1 ... \exists q_k \psi$, where ψ is the LTL formula in which "Future" is the only temporal operator used) is strictly weaker than that of LTL; and the expressive power of $EQ(U)$ (the fragments of $QLTL$ containing formulas of the form $\exists q_1 ... \exists q_k \psi$, where ψ is the LTL formula in which "Until" is the only temporal operator used) is incompatible with that of LTL. Finally, we consider the quantifier hierarchy of $Q(U)$ and $Q(F)$, and show that one alternation of existential and universal quantifiers is necessary and sufficient to express the whole class of ω-regular languages.

Compared to ETL and νTL, $QLTL$ is more natural and easier to use for those people already familiar with LTL. As it was pointed out in [6, 3], $QLTL$ has important applications in the verification of complex systems because quantifications have the ability to reason about refinement relations between programs.

However, the complexity of $QLTL$ is very high: $QLTL$ is not elementarily decidable [12]. So from a practical point of view, it seems that it is unnecessary to bother to clarify the expressive power of $QLTL$. Our main motivation of the exploration of the expressive power of $QLTL$ is from a theoretical point of view, that is, the analogy between $QLTL$ and $S1S$ [16], monadic second order logic over words.

The formulas of $S1S$ are constructed from atomic propositions $x = y$, $x < y$ and $P_\sigma(x)$ (P_σ is the unary relation symbol for each letter σ in the alphabet of words) by boolean combinations, first and second order quantifications. $S1S$ defines exactly the class of ω-regular languages. $QLTL$ can be seen as a variant of $S1S$ because the quantifications over atomic propositions in $QLTL$ are essentially second order quantifications over positions of the ω-words.

In $S1S$, second order quantifications are so powerful that the first order vocabulary can be suppressed into the single successor relation ("$S(x,y)$") since the linear order relation ("$<$") can be defined by the successor relation with the help of second order quantifications:

$$x < y \equiv \neg(x = y) \wedge \forall X((X(x) \wedge \forall z \forall z'(X(z) \wedge S(z,z') \to X(z'))) \to X(y)).$$

Then, analogously we may think that in $QLTL$ the LTL part (the first order part) can also be suppressed to the temporal operator X ("Next"), the counterpart of successor relation $S(x,y)$. However, because in $S1S$ the positions of words can be referred to directly by first order variables while in $QLTL$ they cannot, it turns out that in $QLTL$ the LTL part cannot be suppressed into the single temporal operator X (As a matter of fact, the fragment of $QLTL$ with only X operators used has the same expressive power as the fragment of LTL with only X operator used). However, we still want to know to what extent the LTL part of $QLTL$ can be suppressed. So we consider $Q(U)$ and $Q(F)$, the fragment of $QLTL$ with only U and F operator used respectively, to see whether they can still express the whole class of ω-regular languages. When we find out that they can do so, we then want to know whether they can also do so when only the existential quantifiers are available. The answer is negative, and naturally, we then consider the quantifier hierarchy of $Q(U)$ and $Q(F)$ to see how many alternations of existential and universal quantifiers are necessary and sufficient to express the whole class of ω-regular languages.

The rest of the paper is organized as follows: in Section 2, we give some notation and definitions; then in Section 3, we recall some relevant results on the expressive power of $QLTL$ and its fragments; in Section 4, we establish the main results of this paper; finally in Section 5, we give some conclusions.

2 Notation and definitions

2.1 Syntax of QLTL

Let $\mathcal{P}$ denote the set of propositional variables $\{p_1, p_2, ...\}$. Formulas of $QLTL$ are defined by the following rules:

$$\varphi := q(q \in \mathcal{P}) \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid X\varphi_1 \mid \varphi_1 U \varphi_2 \mid \exists q \varphi_1 (q \in \mathcal{P})$$

Let φ be a $QLTL$ formula, the subformulas of φ is denoted by $Sub(\varphi)$, and the closure of φ, denoted by $Cl(\varphi)$, is $Sub(\varphi) \cup \{\neg\psi | \psi \in Sub(\varphi)\}$.

Let φ be a $QLTL$ formula. The free-variables-set and bound-variables-set of φ, denoted by $Free(\varphi)$ and $Bound(\varphi)$ respectively, are defined similar to that of first order logic.

The set of variables occurring in a formula φ, denoted by $Var(\varphi)$, is $Free(\varphi) \cup Bound(\varphi)$.

In the remaining part of this paper, we assume that all $QLTL$ formulas φ are well-named: i.e., for all φ, $Free(\varphi) \cap Bound(\varphi) = \emptyset$, and for any $q \in Bound(\varphi)$, there is a unique quantified formula $\exists q \psi$ in $Cl(\varphi)$.

We define several abbreviations of $QLTL$ formulas as follows: $true = q \vee \neg q (q \in \mathcal{P})$, $false = \neg true$, $\varphi_1 \wedge \varphi_2 = \neg(\neg\varphi_1 \vee \neg\varphi_2)$, $\varphi_1 \rightarrow \varphi_2 = \neg\varphi_1 \vee \varphi_2$ $F\varphi_1 = trueU\varphi_1$, $G\varphi_1 = \neg F \neg \varphi_1$, $\forall q \varphi_1 = \neg\left(\exists q(\neg\varphi_1)\right)$.

Moreover, we introduce the following abbreviations. Let AP be a given nonempty finite subset of $\mathcal{P}$. Then, for $a \in 2^{AP}$,

$$\mathcal{B}(a)^{AP} = \left(\bigwedge_{p \in a} p\right) \wedge \left(\bigwedge_{p \in AP \backslash a} \neg p\right);$$

and for $A \subseteq 2^{AP}$,

$$\mathcal{B}(A)^{AP} = \bigvee_{a \in A} \mathcal{B}(a)^{AP}.$$

2.2 Semantics of QLTL

$QLTL$ formulas are interpreted as follows. Let $u \in \left(2^{\mathcal{P}}\right)^{\omega}$. Denote the suffix of u starting from the i-th position (the first position is 0) as u^i and the letter in the i-th position of u as u_i.

- $u \models q$ if $q \in u_0$.
- $u \models \varphi_1 \vee \varphi_2$ if $u \models \varphi_1$ or $u \models \varphi_2$.
- $u \models \neg\varphi_1$ if $u \not\models \varphi_1$.
- $u \models X\varphi_1$ if $u^1 \models \varphi_1$.
- $u \models \varphi_1 U \varphi_2$ if there is $i \geq 0$ such that $u^i \models \varphi_2$ and for all $0 \leq j < i$, $u^j \models \varphi_1$.
- $u \models \exists q \varphi_1$ if there is some $v \in \left(2^{\mathcal{P}}\right)^{\omega}$ such that v differs from u only in the assignments of q (namely for all $i \geq 0$ and for all $q' \in \mathcal{P}\backslash\{q\}$, $q' \in v_i$ iff $q' \in u_i$) and $v \models \varphi_1$.

Let $AP \subseteq AP' \subseteq \mathcal{P}$. If $a \in 2^{AP}$, $a' \in 2^{AP'}$, and $a' \cap AP = a$, then we say that the restriction of a' to AP is a, denoted by $a'|_{AP} = a$. If $A \subseteq 2^{AP}$, $A' \subseteq 2^{AP'}$, and $A = \{a'|_{AP} \big| \; a' \in A'\}$, then we say that the restriction of A' to AP is A, denoted by $A'|_{AP} = A$. If $u \in \left(2^{AP}\right)^{\omega}$, $u' \in \left(2^{AP'}\right)^{\omega}$ and for all $i \geq 0$, $u'_i|_{AP} = u_i$, then we say that the restriction of u' to AP is u, denoted by $u'|_{AP} = u$. Let $L \subseteq \left(2^{AP}\right)^{\omega}$ and $L' \subseteq \left(2^{AP'}\right)^{\omega}$, we say that the restriction of L' to AP is L, denoted by $L'|_{AP} = L$, if $L = \left\{u \in \left(2^{AP}\right)^{\omega} | \exists u' \in L', u'|_{AP} = u\right\}$.

Proposition 1. *Let AP be a nonempty finite subset of $\mathcal{P}$ and φ be a $QLTL$ formula such that $Free(\varphi) \subseteq AP$. Then, for any $u, v \in \left(2^{\mathcal{P}}\right)^{\omega}$ with $u|_{AP} = v|_{AP}$, we have that $u \models \varphi$ iff $v \models \varphi$.*

Let φ_1, φ_2 be two $QLTL$ formulas. φ_1 and φ_2 are said to be equivalent, denoted by $\varphi_1 \equiv \varphi_2$, if for all $u \in \left(2^{\mathcal{P}}\right)^{\omega}$, $u \models \varphi_1$ iff $u \models \varphi_2$.

Proposition 2. *Let AP be a nonempty finite subset of $\mathcal{P}$, φ_1 and φ_2 be two formulas such that $Free(\varphi_1), Free(\varphi_2) \subseteq AP$. Then $\varphi_1 \equiv \varphi_2$ iff (for all $u \in \left(2^{AP}\right)^\omega$, $u \models \varphi_1$ iff $u \models \varphi_2$).*

For a $QLTL$ formula, the bound variables are usually seen as auxiliary variables. Consequently if AP is the set of propositional variables that we are concerned about, and if we want to use $QLTL$ formula φ to define a language of $\left(2^{AP}\right)^\omega$, naturally we may require that $Free(\varphi) \subseteq AP$ and $Bound(\varphi) \cap AP = \emptyset$. So we introduce the following definition.

Definition 1 (Compatibility of AP and φ). *Let AP be a given nonempty finite subset of $\mathcal{P}$ and φ be a formula of $QLTL$. AP and φ are said to be compatible if $Free(\varphi) \subseteq AP$ and $Bound(\varphi) \cap AP = \emptyset$.*

Let AP be a nonempty finite subset of $\mathcal{P}$ and φ be a formula such that AP and φ are compatible. The language of $\left(2^{AP}\right)^\omega$ defined by φ, denoted by $\mathcal{L}(\varphi)^{AP}$, is $\left\{u \in \left(2^{AP}\right)^\omega | u \models \varphi\right\}$.

Proposition 3. *Let AP be a nonempty finite subset of $\mathcal{P}$ and $\varphi = \exists q_1 ... \exists q_k \psi$ be a formula such that AP and φ are compatible. Let $AP' = AP \cup \{q_1, ..., q_k\}$, then AP' and ψ are compatible and $\mathcal{L}(\varphi)^{AP} = \mathcal{L}(\psi)^{AP'}|_{AP}$.*

2.3 Fragments of QLTL and expressive power of logics

Let $O_1, O_2, ... \in \{X, F, G, U\}$. We use $L(O_1, O_2, ...)$ to denote the fragment of $QLTL$ containing temporal operators $\{O_1, O_2, ...\}$ but containing no quantifiers, and use $Q(O_1, O_2, ...)$ to denote the fragment of $QLTL$ containing both temporal operators $\{O_1, O_2, ...\}$ and quantifiers. Moreover we denote the fragment of $QLTL$ containing exactly formulas of the form $\exists q_1 ... \exists q_k \psi$ (or $\forall q_1 ... \forall q_k \psi$), where $\psi \in L(O_1, O_2, ...)$, as $EQ(O_1, O_2, ...)$ (or $AQ(O_1, O_2, ...)$).

For instance, LTL is $L(X, U)$ and $QLTL$ is $Q(X, U)$.

Let φ be a formula in $QLTL$ and $\mathcal{SL}$ be one fragment of $QLTL$. We say that φ is expressible in $\mathcal{SL}$ iff there is a formula ψ in $\mathcal{SL}$ such that $\varphi \equiv \psi$.

Let AP be a nonempty finite subset of $\mathcal{P}$, $L \subseteq \left(2^{AP}\right)^\omega$, and $\mathcal{SL}$ be one fragment of $QLTL$ (e.g., $Q(F)$). We say that L is expressible in $\mathcal{SL}$ if there is a formula φ in $\mathcal{SL}$ such that AP and φ are compatible and $\mathcal{L}(\varphi)^{AP} = L$.

Let $\mathcal{SL}_1$ and $\mathcal{SL}_2$ be two fragments of $QLTL$. We say that $\mathcal{SL}_1$ is less expressive than $\mathcal{SL}_2$, denoted by $\mathcal{SL}_1 \leq \mathcal{SL}_2$, if for any formula $\varphi_1 \in \mathcal{SL}_1$, there exists a formula $\varphi_2 \in \mathcal{SL}_2$ such that $\varphi_1 \equiv \varphi_2$, and we say that $\mathcal{SL}_1$ and $\mathcal{SL}_2$ are expressively equivalent, denoted by $\mathcal{SL}_1 \equiv \mathcal{SL}_2$, if $\mathcal{SL}_1 \leq \mathcal{SL}_2$ and $\mathcal{SL}_2 \leq \mathcal{SL}_1$. Moreover we say that $\mathcal{SL}_1$ is strictly less expressive than $\mathcal{SL}_2$, denoted by $\mathcal{SL}_1 < \mathcal{SL}_2$, if $\mathcal{SL}_1 \leq \mathcal{SL}_2$ but not $\mathcal{SL}_2 \leq \mathcal{SL}_1$. Finally we say that the expressive power of $\mathcal{SL}_1$ and $\mathcal{SL}_2$ are incompatible, denoted by $\mathcal{SL}_1 \perp \mathcal{SL}_2$, if neither $\mathcal{SL}_1 \leq \mathcal{SL}_2$ nor $\mathcal{SL}_2 \leq \mathcal{SL}_1$, namely there are two formulas $\varphi_1 \in \mathcal{SL}_1$ and $\varphi_2 \in \mathcal{SL}_2$ such that there exists no formula in $\mathcal{SL}_2$ equivalent to φ_1 and there exists no formula in $\mathcal{SL}_1$ equivalent to φ_2.

2.4 Büchi automaton and ω-languages

A Büchi automaton $\mathcal{B}$ is a quintuple $(Q, \Sigma, \delta, q_0, T)$, where Q is the finite state set, Σ is the finite set of letters, $\delta \subseteq Q \times \Sigma \times Q$ is the transition relation, $q_0 \in Q$ is the initial state, and $T \subseteq Q$ is the accepting state set. Let $u \in \Sigma^\omega$, a run of $\mathcal{B}$ on u is an infinite state sequence $s_0 s_1 ... \in Q^\omega$ such that $s_0 = q_0$ and $(s_i, u_i, s_{i+1}) \in \delta$ for all $i \geq 0$. A run of $\mathcal{B}$ on u is accepting if some accepting state occurs in it infinitely often. u is accepted by $\mathcal{B}$ if $\mathcal{B}$ has an accepting run on u. The language defined by $\mathcal{B}$, denoted by $\mathcal{L}(\mathcal{B})$, is the set of ω-words accepted by $\mathcal{B}$.

An ω-language is said to be ω-regular if it can be defined by some Büchi automaton.

An ω-language $L \subseteq \Sigma^\omega$ is said to be stutter invariant if for all $u \in \Sigma^\omega$ and function $f : \mathbf{N} \to \mathbf{N} \backslash \{0\}$ ($\mathbf{N}$ is the set of natural numbers), we have that $u \in L$ iff $u^{f(0)} u^{f(1)} ... \in L$.

Let $L \subseteq \Sigma^\omega$ be ω-regular. The syntactic congruence of L, denoted by $\approx_L$, is a congruence on Σ^* defined as follows: let $u, v \in \Sigma^*$, then, $u \approx_L v$ if for all $x, y, z \in \Sigma^*$, ($xuyz^\omega \in L$ iff $xvyz^\omega \in L$) and ($x(yuz)^\omega \in L$ iff $x(yvz)^\omega \in L$). The syntactic monoid of L, denoted by $M(L)$, is the division monoid $\Sigma^* / \approx_L$.

An ω-language $L \subseteq \Sigma^\omega$ is said to be non-counting if there is $n \geq 0$ such that for all $x, y, z, u \in \Sigma^*$, ($xu^n yz^\omega \in L$ iff $xu^{n+1} yz^\omega \in L$) and ($x(yu^n z)^\omega \in L$ iff $x(yu^{n+1} z)^\omega \in L$).

A monoid M is said to be aperiodic if there is $k \geq 0$ such that for all $m \in M$, $m^k = m^{k+1}$.

Let $L \subseteq \Sigma^\omega$. It is not hard to show that $M(L)$ is aperiodic iff L is non-counting.

3 Known results on the expressive power of $QLTL$ and LTL

In the remaining part of this paper, we always assume that AP is a nonempty finite subset of $\mathcal{P}$.

Proposition 4 ([2, 12]). *An ω-language is ω-regular iff it is expressible in $QLTL$.*

Corollary 1. $Q(X, U) \equiv EQ(X, F)$.

Proposition 5 ([1]).

(i) Xp_1 *is not expressible in* $L(U)$*;*
(ii) Fp_1 *is not expressible in* $L(X)$*;*
(iii) $p_1 U p_2$ *is not expressible in* $L(X, F)$*.*

In the following we recall three propositions characterizing the expressive power of LTL(namely $L(X, U)$), $L(U)$ and $L(F)$ respectively.

In the remaining part of this subsection, we assume that $L \subseteq (2^{AP})^\omega$.

Proposition 6 **(Characterization of LTL, [5, 4, 14, 15, 7]).** *Suppose that L is ω-regular, then the following two conditions are equivalent:*

- *L is expressible in LTL;*
- *The syntactic monoid of L, $M(L)$, is aperiodic.*

Proposition 7 **(Characterization of $L(U)$, [10]).** *Let φ be a formula in $L(X,U)$ and $Free(\varphi) \subseteq AP$. Then φ is expressible in $L(U)$ iff $\mathcal{L}(\varphi)^{AP}$ is stutter invariant.*

Definition 2 **(Restricted ω-regular set).** *L is said to be a restricted ω-regular set if it is of the form*

$$S_1^* s_1 S_2^* s_2 ... S_{m-1}^* s_{m-1} S_m^\omega, \tag{1}$$

where $S_i \subseteq 2^{AP}$ $(1 \leq i \leq m)$, and $s_i \in S_i \backslash S_{i+1}$ $(1 \leq i < m)$.

For instance, let $AP = \{p_1\}$, then, $\left(2^{AP}\right)^\omega$ and $\left(2^{AP}\right)^* \{p_1\}\emptyset^\omega$ are both restricted ω-regular sets.

Definition 3. *Let $s_0 \in 2^{AP}$ and $S' \subseteq 2^{AP}$. We define $L_{inf(S')}^{init(s_0)}$ as follows:*

$$L_{inf(S')}^{init(s_0)} = \{u \in L | u_0 = s_0, \textit{ each element of } S' \textit{ occurs infinitely often in } u\}$$

Proposition 8 **(Characterization of $L(F)$, [13]).** *Let L be nonempty. Then, L is expressible in $L(F)$ iff L is a finite union of nonempty languages of the form $M_{inf(S')}^{init(s_0)}$, where $M \subseteq \left(2^{AP}\right)^\omega$ is a restricted ω-regular set, $s_0 \in 2^{AP}$ and $S' \subseteq 2^{AP}$.*

For instance, let $AP = \{p_1\}$, then, $\mathcal{L}(Fp_1)^{AP} \subseteq \left(2^{AP}\right)^\omega$ is exactly the union of languages $(L_1)_{inf(\emptyset)}^{init(\{p_1\})}$, $(L_1)_{inf(\{\{p_1\}\})}^{init(\emptyset)}$, and $(L_2)_{inf(\emptyset)}^{init(\emptyset)}$, where $L_1 = \left(2^{AP}\right)^\omega$ and $L_2 = \left(2^{AP}\right)^* \{p_1\}\emptyset^\omega$.

4 Our results on the expressive power of $QLTL$ and its fragments

According to Proposition 4, $Q(X,U)$, $Q(X,F)$, $EQ(X,U)$ and $EQ(X,F)$ are all expressively equivalent, which, nevertheless, is almost all we know about the expressive power of $QLTL$ besides those of LTL. For instance, we do not know whether several natural fragments of $QLTL$, e.g., $Q(U)$ and $Q(F)$, can define the whole class of ω-regular languages or not.

In this section, we first give a positive answer to the above question, namely, we show that $Q(U)$ and $Q(F)$ can define the whole class of ω-regular languages. Then, since $EQ(X,U)$ and $EQ(X,F)$ can also do so, analogously, we want to know whether $EQ(U)$ and $EQ(F)$ can do so or not. However, the answer is negative. As a matter of fact, we show that $EQ(F) < LTL$ and $EQ(U) \perp$

LTL. Furthermore, we compare the expressive power of $EQ(U)$ and $EQ(F)$ with that of other fragments of $QLTL$ and get a panorama of the expressive power of various fragments of $QLTL$ (Fig. 1). Since neither $EQ(U)$ nor $EQ(F)$ can express the whole class of ω-regular languages, we want to know how many alternations of existential and universal quantifiers are necessary and sufficient to do that. The answer is one, which will be shown in the end of this section.

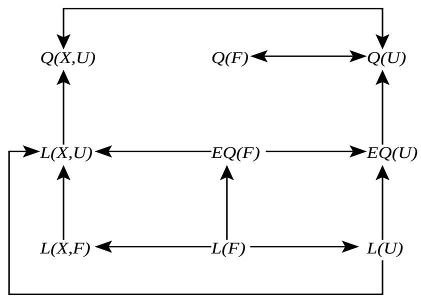

Fig. 1. Expressive power of $QLTL$ and its fragments

Remark 1 (Notation in Fig. 1). Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two nodes in Fig. 1. If $\mathcal{L}_2$ is reachable from $\mathcal{L}_1$ but not vice versa, then $\mathcal{L}_1 < \mathcal{L}_2$, e.g. $EQ(F) < EQ(U)$. If neither $\mathcal{L}_2$ is reachable from $\mathcal{L}_1$ nor $\mathcal{L}_1$ is reachable from $\mathcal{L}_2$, then $\mathcal{L}_1 \perp \mathcal{L}_2$, e.g. $EQ(F) \perp L(U)$. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are reachable from each other (namely, in the same Strongly Connected Component), then $\mathcal{L}_1 \equiv \mathcal{L}_2$, e.g. $Q(U) \equiv Q(F)$. □

4.1 Expressive power of $Q(U)$ and $Q(F)$

In the following we will show that, with the help of quantifiers, the operator X can be expressed by the operator U and the operator U can be expressed by the operator F.

Lemma 1. *Let $\varphi \in QLTL$, $q_1, q_2 \in \mathcal{P} \backslash Var(\varphi)$ and $q_1 \neq q_2$. Then*

$$X\varphi \equiv \big(\varphi \wedge \exists q_1 \left(\neg q_1 \wedge (\varphi \wedge \neg q_1)\ U\ (\varphi \wedge q_1)\right)\big) \vee \\ \big(\neg\varphi \wedge \neg\exists q_2 \left(\neg q_2 \wedge (\neg\varphi \wedge \neg q_2)\ U\ (\neg\varphi \wedge q_2)\right)\big).$$

Lemma 2. *Let φ_1 and φ_2 be two formulas of $QLTL$ and $q \in \mathcal{P} \backslash (Var(\varphi_1) \cup Var(\varphi_2))$. Then*

$$\varphi_1 U \varphi_2 \equiv \exists q \left(F(\varphi_2 \wedge q) \wedge G(\neg q \rightarrow G\neg q) \wedge G(\varphi_1 \vee \varphi_2 \vee \neg q)\right).$$

From Lemma 1 and Lemma 2, we have the following theorem.

Theorem 1. $Q(X, U) \equiv Q(U) \equiv Q(F)$.

4.2 Expressive power of $EQ(F)$ and $EQ(U)$

Both $EQ(X,U)$ and $EQ(X,F)$ can define the whole class of ω-regular languages (Corollary 1). Then a natural question to ask is whether this is true for $EQ(U)$ and $EQ(F)$ as well. We will give a negative answer to this question in this subsection. Moreover, in this subsection, we will compare the expressive power of $EQ(F)$ and $EQ(U)$ with that of other fragments of $QLTL$.

We first show that $EQ(F)$ cannot define the whole class of ω-regular languages. In fact we show that $EQ(F)$ is strictly less expressive than LTL.

Lemma 3. *Let $AP \subseteq AP' \subseteq \mathcal{P}$ and $L \subseteq \left(2^{AP'}\right)^\omega$. If L is a restricted ω-regular set, $s_0 \in 2^{AP'}$, $S' \subseteq 2^{AP'}$ and $L^{init(s_0)}_{inf(S')} \neq \emptyset$, then, $\left(L^{init(s_0)}_{inf(S')}\right)\Big|_{AP} = (L|_{AP})^{init(s_0|_{AP})}_{inf(S'|_{AP})}$.*

Lemma 4. *For any formula $\varphi = \exists q_1 ... \exists q_k \psi \in EQ(F)$, there exists some formula $\theta \in L(X,U)$ such that $\varphi \equiv \theta$.*

Proof of Lemma 4.

Suppose that $\varphi = \exists q_1 ... \exists q_k \psi \in EQ(F)$, where $\psi \in L(F)$.

Suppose that φ and AP are compatible and $AP' = AP \cup \{q_1, ..., q_k\}$.

Then, according to Proposition 3, we have that ψ and AP' are compatible, and $\mathcal{L}(\varphi)^{AP} = \mathcal{L}(\psi)^{AP'}|_{AP}$.

If $\mathcal{L}(\psi)^{AP'} = \emptyset$, then $\varphi \equiv false$. So we assume that $\mathcal{L}(\psi)^{AP'} \neq \emptyset$.

According to Proposition 8, $\mathcal{L}(\psi)^{AP'}$ is a finite union of nonempty languages of the form $L^{init(s_0)}_{inf(S')}$, where $L \subseteq \left(2^{AP'}\right)^\omega$ is a restricted ω-regular set, $s_0 \in 2^{AP'}$ and $S' \subseteq 2^{AP'}$.

In the remaining part of the proof of this lemma, we always suppose that L is a restricted ω-regular set, specifically, $S_1^* s_1 S_2^* s_2 ... S_{m-1}^* s_{m-1} S_m^\omega$, where $S_i \subseteq 2^{AP}$ $(1 \leq i \leq m)$, and $s_i \in S_i \backslash S_{i+1}$ $(1 \leq i < m)$.

From Lemma 3, we know that $\mathcal{L}(\varphi)^{AP} = \mathcal{L}(\psi)^{AP'}|_{AP}$ is a finite union of nonempty languages of the form $(L|_{AP})^{init(s_0|_{AP})}_{inf(S'|_{AP})}$.

In the following we will show that there is a formula ξ in $L(X,U)$ such that $Var(\xi) = Free(\xi) \subseteq AP$ and $\mathcal{L}(\xi)^{AP} = (L|_{AP})^{init(s_0|_{AP})}_{inf(S'|_{AP})}$. Let θ be the disjunction of all these ξ's. Then $\mathcal{L}(\varphi)^{AP} = \mathcal{L}(\theta)^{AP}$. Because $Free(\varphi) \subseteq AP$ and $Free(\theta) \subseteq AP$, according to Proposition 2, we conclude that φ and θ are equivalent.

In order to define ξ, we define a sequence of formulas η_i $(1 \leq i \leq m)$ as follows:

$$\eta_i = \begin{cases} G\left(\mathcal{B}\left(S_m|_{AP}\right)^{AP}\right) & \text{if} \quad i = m \\ \mathcal{B}\left(S_i|_{AP}\right)^{AP} \;\; U \;\; \left(\mathcal{B}(s_i|_{AP})^{AP} \wedge X\eta_{i+1}\right) & \text{if} \quad 1 \leq i < m \end{cases}$$

It is not hard to show that for all $1 \leq i \leq m$,

$$\mathcal{L}(\eta_i)^{AP} = \left(S_i|_{AP}\right)^* \left(s_i|_{AP}\right) ... \left(S_m|_{AP}\right)^\omega .$$

Thus, $L|_{AP} = \mathcal{L}(\eta_1)^{AP}$.

We can define ξ by the formula

$$\mathcal{B}\left(s_0|_{AP}\right)^{AP} \wedge \eta_1 \wedge \bigwedge_{a\in(S'|_{AP})} GF\left(\mathcal{B}(a)^{AP}\right).$$

□

Lemma 5. *Let φ be a formula in $EQ(U)$ and AP be compatible with φ. Then for any $u \in (2^{AP})^{\omega}$, any function $f : \boldsymbol{N} \to \boldsymbol{N}\backslash\{0\}$, if $u \models \varphi$, then, $u_0^{f(0)}...u_i^{f(i)}... \models \varphi$.*

Lemma 6. *Let $AP = \{p_1\}$. Then Xp_1 is not expressible in $EQ(U)$.*

Proof of Lemma 6.

To the contrary, suppose that Xp_1 is expressible in $EQ(U)$.

We know that $\emptyset\{p_1\}^{\omega} \models Xp_1$, then according to Lemma 5, we have that $\emptyset^2\{p_1\}^{\omega} \models Xp_1$, a contradiction. □

Theorem 2. *$EQ(F) < LTL$.*

Proof.

It follows directly from Lemma 4 and Lemma 6. □

Theorem 3. *$EQ(F) \perp L(X,F)$.*

Proof.

From Lemma 2, we know that p_1Up_2 is expressible in $EQ(F)$. While it is not expressible in $L(X,F)$ according to Proposition 5.

Xp_1 is not expressible in $EQ(F)$ according to Lemma 6.

So, $EQ(F) \perp L(X,F)$. □

From Lemma 6, we already know that $EQ(U)$ cannot define the whole class of ω-regular languages. In the following, we will show that the expressive power of $EQ(U)$ and LTL are incompatible.

Lemma 7. *Let $AP = \{p_1\}$ and*

$$L = \{u \in \left(2^{AP}\right)^{\omega} | (\emptyset\{p_1\}) \text{ occurs an odd number of times in } u\}.$$

L is expressible in $EQ(U)$, while it is not expressible in LTL.

Remark 2. A language similar to L in Lemma 7 is used in Proposition 2 of [2].

□

Theorem 4. *$EQ(U) \perp LTL$.*

Proof.

It follows from Lemma 6 and Lemma 7. □

Now we compare the expressive power of $EQ(F)$ and $EQ(U)$ with that of $L(F)$ and $L(U)$.

Lemma 8. *Let $AP = \{p_1\}$. Then*

$$L = \{\emptyset, \{p_1\}\}^* \{p_1\}\{p_1\} \{\emptyset, \{p_1\}\}^* \emptyset^\omega \subseteq \left(2^{AP}\right)^\omega$$

is expressible in $EQ(F)$, while it is not expressible in $L(U)$.

The following theorem can be derived from Lemma 8 easily.

Theorem 5. *$L(F) < EQ(F)$ and $L(U) < EQ(U)$.*

But how about the expressive power of $EQ(F)$ and $L(U)$? In Lemma 8, we have shown that there is a language expressible in $EQ(F)$, but not expressible in $L(U)$. In the following we will show that there is a language expressible in $L(U)$, but not expressible in $EQ(F)$.

Lemma 9. *Let $AP = \{p_1, p_2, p_3\}$ and*

$$L = (\{p_1\}\{p_1\}^*\{p_2\}\{p_2\}^*\{p_3\}\{p_3\}^*)^\omega .$$

Then L is expressible in $L(U)$, while it is not expressible in $EQ(F)$.

Proof of Lemma 9.

We first define the formula φ in $L(U)$ such that AP and φ are compatible and $\mathcal{L}(\varphi)^{AP} = L$:

$$\begin{aligned}\varphi \equiv\ & \mathcal{B}(\{p_1\})^{AP} \wedge G\left(\mathcal{B}(\{p_1\})^{AP} \to \mathcal{B}(\{p_1\})^{AP}\ U\ \mathcal{B}(\{p_2\})^{AP}\right) \wedge \\ & G\left(\mathcal{B}(\{p_2\})^{AP} \to \mathcal{B}(\{p_2\})^{AP}\ U\ \mathcal{B}(\{p_3\})^{AP}\right) \wedge \\ & G\left(\mathcal{B}(\{p_3\})^{AP} \to \mathcal{B}(\{p_3\})^{AP}\ U\ \mathcal{B}(\{p_1\})^{AP}\right) .\end{aligned}$$

Now we show that L is not expressible in $EQ(F)$.

To the contrary, suppose that there is an $EQ(F)$ formula $\psi = \exists q_1 ... \exists q_k \xi$ such that ψ and AP are compatible and $L = \mathcal{L}(\psi)^{AP}$.

Let $AP' = AP \cup \{q_1, ..., q_k\}$. Then, according to Proposition 3, we have that ξ and AP' are compatible, $\mathcal{L}(\psi)^{AP} = \mathcal{L}(\xi)^{AP'}|_{AP}$.

According to Proposition 8, $\mathcal{L}(\xi)^{AP'}$ is a finite union of nonempty languages of the form $M^{init(s_0)}_{inf(S')}$, where M is a restricted ω-regular set, $s_0 \in 2^{AP'}$, $S' \subseteq 2^{AP'}$.

From Lemma 3, we know that $L = \mathcal{L}(\xi)^{AP'}|_{AP}$ is a finite union of nonempty languages of the form $(M|_{AP})^{init(s_0|_{AP})}_{inf(S'|_{AP})}$.

Let $u = (\{p_1\}\{p_2\}\{p_3\})^\omega \in L$. Then, $u \in (M|_{AP})^{init(s_0|_{AP})}_{inf(S'|_{AP})}$ for some restricted ω-regular set M, $s_0 \in \left(2^{AP'}\right)^\omega$ and $S' \subseteq \left(2^{AP'}\right)^\omega$.

Suppose that $M = S_1^* s_1 ... S_{m-1}^* s_{m-1} S_m^\omega$, where $S_i \subseteq 2^{AP}$ $(1 \leq i \leq m)$, and $s_i \in S_i \backslash S_{i+1}$ $(1 \leq i < m)$. Then,

$$M|_{AP} = (S_1|_{AP})^* (s_1|_{AP}) ... (S_{m-1}|_{AP})^* (s_{m-1}|_{AP}) (S_m|_{AP})^\omega .$$

Since $\{p_1\}$, $\{p_2\}$ and $\{p_3\}$ occur infinitely often in $u \in M|_{AP}$, we have that $\{\{p_1\}, \{p_2\}, \{p_3\}\} \subseteq S_m|_{AP}$.

If $m = 1$, then $M|_{AP} = (S_m|_{AP})^\omega$. In this case, let

$$u' = \{p_1\}\{p_2\}\{p_3\}(\{p_2\}\{p_1\}\{p_3\})^\omega.$$

Evidently $u' \in M|_{AP}$. Moreover, $u_0 = u'_0$, and the elements of 2^{AP} occurring infinitely often in u and u' are the same. So, $u' \in (M|_{AP})^{init(s_0)}_{init(S')} \subseteq L$, a contradiction.

Now we assume that $m > 1$.

Since $u \in M_{AP}$, we have that $u = x(s_{m-1}|_{AP})y\left(\{p_1\}\{p_2\}\{p_3\}\right)^\omega$, where

$x \in (S_1|_{AP})^* \, (s_1|_{AP}) \ldots (S_{m-1}|_{AP})^*$ and $y\left(\{p_1\}\{p_2\}\{p_3\}\right)^\omega \in (S_m|_{AP})^\omega$.

Let $u' = x(s_{m-1}|_{AP})y\left(\{p_2\}\{p_1\}\{p_3\}\right)^\omega$.

Then, $u' \in (S_1|_{AP})^* \, (s_1|_{AP}) \ldots (s_{m-1}|_{AP}) \, (S_m|_{AP})^\omega$. Moreover, $u'_0 = u_0$ and the elements of 2^{AP} occurring infinitely often in u and u' are the same. So, $u' \in (M|_{AP})^{init(s_0)}_{inf(S')} \subseteq L$, a contradiction as well.

So, we conclude that L is not expressible in $EQ(F)$. □

Theorem 6. *$L(U) \perp EQ(F)$.*

Proof.
It follows from Lemma 8 and Lemma 9. □

Also we have the following theorem according to Lemma 9.

Theorem 7. *$EQ(F) < EQ(U)$.*

The expressive power of $QLTL$ and its fragments are summarized into Fig. 1.

4.3 Quantifier hierarchy of $Q(U)$ and $Q(F)$

In Subsection 4.2, we have known that $EQ(F)$ and $EQ(U)$ can not define the whole class of ω-regular languages. It follows easily that $AQ(F)$ and $AQ(U)$ can not define the whole class of ω-regular languages as well. Moreover since $\neg X p_1 \equiv X(\neg p_1)$ is not expressible in $EQ(U)$ (similar to the proof of Lemma 6), Xp_1 is not expressible in $AQ(U)$ or in $AQ(F)$. Consequently Xp_1 is expressible in neither $EQ(U) \cup AQ(U)$ nor in $EQ(F) \cup AQ(F)$. Thus we conclude that alternations of existential and universal quantifiers are necessary to define the whole class of ω-regular languages in $Q(U)$ and $Q(F)$. A natural question then occurs: how many alternations of existential and universal quantifiers are sufficient to define the whole class of ω-regular languages? The answer is one.

Now we define the quantifier hierarchy in $Q(U)$ and $Q(F)$.

The definitions of hierarchy of Σ_k, Π_k and $\triangle_k$ in $Q(U)$ and $Q(F)$ are similar to the quantifier hierarchy of first order logic. Σ_k (Π_k resp.) contains the formulas of the prenex normal form such that there are k-blocks of quantifiers and the quantifiers in each block are of the same type (all existential or all universal); the consecutive blocks are of different types; the first block is existential (universal resp.). $\triangle_k = \Sigma_k \cap \Pi_k$, namely $\triangle_k$ contains those formulas both equivalent to some Σ_k formula and to some Π_k formula. In addition, we define $\bigtriangledown_k = \Sigma_k \cup \Pi_k$.

Lemma 10. *Σ_2^U and Σ_2^F define the whole class of ω-regular languages.*

Proof of Lemma 10.

Let $\mathcal{B} = (Q, 2^{AP}, \delta, q_0, T)$ be a Büchi automaton. Suppose that $Q = \{q_0, ..., q_n\}$, $\mathcal{L}(\mathcal{B})$ can be defined by the following formula φ.

$$\varphi := \exists q_0 ... \exists q_n \left(q_0 \wedge G\left(\bigwedge_{i \neq j} \neg(q_i \wedge q_j) \right) \wedge \right.$$
$$\left. G\left(\bigvee_{(q_i, a, q_j) \in \delta} \left(q_i \wedge \mathcal{B}(a)^{AP} \wedge X q_j\right) \right) \wedge \left(\bigvee_{q_i \in T} GF q_i \right) \right)$$

Let $AP' = AP \cup Q$. If we can find a formula ψ in Π_1^U (Π_1^F, resp.) such that ψ and AP' are compatible and

$$\psi \equiv G\left(\bigvee_{(q_i, a, q_j) \in \delta} (q_i \wedge \mathcal{B}(a)^{AP} \wedge X q_j) \right),$$

then, we are done.

We first show that such a ψ in Π_1^U exists.

We observe that $\bigvee_{(q_i, a, q_j) \in \delta} (q_i \wedge \mathcal{B}(a)^{AP} \wedge X q_j)$ can be rewritten into its conjunctive normal form and the conjunctions can be moved to the outside of "G":

$$G\left(\bigvee_{(q_i, a, q_j) \in \delta} (q_i \wedge \mathcal{B}(a)^{AP} \wedge X q_j) \right)$$

$$\equiv \bigwedge_{\substack{i_1, ..., i_k \\ a_1, ..., a_l \\ j_1, ..., j_m}} G\left(q_{i_1} \vee ... \vee q_{i_k} \vee \mathcal{B}(a_1)^{AP} \vee ... \vee \mathcal{B}(a_l)^{AP} \vee X q_{j_1} \vee ... \vee X q_{j_m}\right)$$

It is sufficient to show that there is a Π_1^U formula such that the formula and AP' are compatible and the formula is equivalent to

$$G\left(q_{i_1} \vee ... \vee q_{i_k} \vee \mathcal{B}(a_1)^{AP} \vee ... \vee \mathcal{B}(a_l)^{AP} \vee X q_{j_1} \vee ... \vee X q_{j_m}\right). \quad (2)$$

The negation of the formula (2) is of the form $F(\varphi_1 \wedge X\varphi_2)$, where φ_1, φ_2 are boolean combinations of propositional variables in AP'. If we can prove that for any formula of the form $F(\varphi_1 \wedge X\varphi_2)$, there is a formula ξ in Σ_1^U such that ξ and AP' are compatible, and $\xi \equiv F(\varphi_1 \wedge X\varphi_2)$, then, we are done.

Let

$$S_i = \left\{ a \in 2^{AP'} \middle| a \text{ satisfies the boolean formula } \varphi_i \right\}, \text{ where } i = 1, 2.$$

Then, for any $u \in \left(2^{AP'}\right)^\omega$,

$$u \models F(\varphi_1 \wedge X\varphi_2) \text{ iff } u \models F\left(\mathcal{B}(S_1)^{AP'} \wedge X\mathcal{B}(S_2)^{AP'}\right).$$

From Proposition 2, we know that

$$F(\varphi_1 \wedge X\varphi_2) \equiv F\left(\mathcal{B}(S_1)^{AP'} \wedge X\mathcal{B}(S_2)^{AP'}\right).$$

Let $q' \in \mathcal{P}\backslash AP'$, and $AP'' = AP' \cup \{q'\}$, $S_1' = S_1$, and $S_2' = \{a \cup \{q'\} \big| a \in S_2\}$. We have that $S_i'|_{AP'} = S_i$ $(i = 1, 2)$ and $S_1' \cap S_2' = \emptyset$.

Then, Σ_1^U formula

$$\chi := \exists q' F\left(\mathcal{B}(S_1')^{AP''} \wedge \mathcal{B}(S_1')^{AP''}\ U\ \mathcal{B}(S_2')^{AP''}\right)$$

satisfies that χ and AP' are compatible, and

$$\chi \equiv F\left(\mathcal{B}(S_1)^{AP'} \wedge X\mathcal{B}(S_2)^{AP'}\right) \equiv F\left(\varphi_1 \wedge X\varphi_2\right).$$

Now we show that there is also a formula $\chi' \in \Sigma_1^F$ equivalent to $F(\varphi_1 \wedge X\varphi_2)$.

According to Lemma 2, there are $q'' \in \mathcal{P}\backslash AP''$ and $\xi \in L(F)$ such that $\exists q''\xi \equiv \mathcal{B}(S_1')^{AP''}\ U\ \mathcal{B}(S_2')^{AP''}$.

Let

$$\chi' := \exists q' \exists q'' F\left(\mathcal{B}(S_1')^{AP''} \wedge \xi\right).$$

Then $\chi' \in \Sigma_1^F$, χ' and AP' are compatible and

$$\chi' \equiv \chi \equiv F\left(\varphi_1 \wedge X\varphi_2\right).$$

□

The following theorem is a direct consequence of Lemma 10.

Theorem 8. *$Q(U) \equiv \Sigma_2^U \equiv \Pi_2^U \equiv \triangle_2^U \equiv \triangledown_2^U$ and $Q(F) \equiv \Sigma_2^F \equiv \Pi_2^F \equiv \triangle_2^F \equiv \triangledown_2^F$.*

5 Conclusions

In this paper, we first showed that $Q(U)$ and $Q(F)$ can define the whole class of ω-regular languages. Then we compared the expressive power of $EQ(F)$, $EQ(U)$ and other fragments of $QLTL$ in detail and got a panorama of the expressive power of fragments of $QLTL$. In particular, we showed that $EQ(F)$ is strictly less expressive than LTL and that the expressive power of $EQ(U)$ and LTL are incompatible. Furthermore, we showed that one alternation of existential and universal quantifiers is necessary and sufficient to express the whole class of ω-regular languages.

The results established in this paper can be easily adapted to the regular languages on finite words.

There are several open problems. For instance, since we discovered that neither $EQ(U)$ nor $EQ(F)$ can define the whole class of ω-regular languages, a natural problem is to find (effective) characterizations for those languages expressible in $EQ(U)$ and $EQ(F)$ respectively.

We can also consider similar problems for the other temporal operators, such as the strict "Until" and "Future" operators.

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A Proof of Lemma 1

Denote the right part formula of the equation in the lemma as ψ.

Let $AP = Free(X\varphi) = Free(\varphi)$. It is sufficient to prove that for all $u \in \left(2^{AP}\right)^{\omega}$, $u \models X\varphi$ iff $u \models \psi$ according to Proposition 2.

"$\Rightarrow$": Let $u \in \left(2^{AP}\right)^{\omega}$ and $u \models X\varphi$.

Then $u^1 \models \varphi$.There are the following two cases:

Case I: $u \models \varphi$. Define v as follows:

$$v_i = \begin{cases} u_i \cup \{q_1\} & \text{if} \quad i = 1 \\ u_i & \text{if} \quad \text{otherwise} \end{cases}$$

It is obvious that $v \models \varphi \wedge \neg q_1$, $v^1 \models \varphi \wedge q_1$. Consequently $v \models (\varphi \wedge \neg q_1)\ U\ (\varphi \wedge q_1)$. Then $u \models \exists q_1 \left(\neg q_1 \wedge (\varphi \wedge \neg q_1)\ U\ (\varphi \wedge q_1)\right)$.

As a result we conclude that $u \models \varphi \wedge \exists q_1 \left(\neg q_1 \wedge (\varphi \wedge \neg q_1)\ U\ (\varphi \wedge q_1)\right)$. Then $u \models \psi$.

Case II: $u \models \neg\varphi$.

Now we show that $u \models \neg\exists q_2 \left(\neg q_2 \wedge (\neg\varphi \wedge \neg q_2) U (\neg\varphi \wedge q_2)\right)$.

To the contrary, suppose that $u \models \exists q_2 \left(\neg q_2 \wedge (\neg\varphi \wedge \neg q_2) U (\neg\varphi \wedge q_2)\right)$.

Then there is a $v \in \left(2^{AP\cup\{q_2\}}\right)^{\omega}$ such that $v|_{AP} = u$ and $v \models \neg q_2 \wedge (\neg\varphi \wedge \neg q_2) U (\neg\varphi \wedge q_2)$.

There is $i \geq 0$ such that $v^i \models \neg\varphi \wedge q_2$ and for all $0 \leq j < i$, $v^j \models \neg\varphi \wedge \neg q_2$. Then $v^i \models q_2$ and for all $0 \leq j < i$, $v^j \models \neg q_2$, thus $i \geq 1$ since $v \models \neg q_2$. But then $v^j \models \neg\varphi$ for all $0 \leq j \leq i$, and consequently $v^1 \models \neg\varphi$, as a result $u^1 \models \neg\varphi$, a contradiction.

Finally we conclude that $u \models \neg\varphi \wedge \neg\exists q_2 \left(\neg q_2 \wedge (\neg\varphi \wedge \neg q_2) U (\neg\varphi \wedge q_2)\right)$, $u \models \psi$.

"$\Leftarrow$": Suppose that $u \in \left(2^{AP}\right)^{\omega}$ and $u \models \psi$.

There are two cases:

Case I: $u \models \varphi \wedge \exists q_1 \left(\neg q_1 \wedge (\varphi \wedge \neg q_1) U (\varphi \wedge q_1)\right)$.

Then there is a $v \in \left(2^{AP\cup\{q_1\}}\right)^{\omega}$ such that $v|_{AP} = u$ and $v \models \neg q_1 \wedge (\varphi \wedge \neg q_1) U (\varphi \wedge q_1)$.

There is $i \geq 0$, $v^i \models \varphi \wedge q_1$ and for all $0 \leq j < i$, $v^j \models \varphi \wedge \neg q_1$. It is evident that $i > 0$, thus $v^1 \models \varphi$, $v \models X\varphi$. Consequently we conclude that $u \models X\varphi$.

Case II: $u \models \neg\varphi \wedge \neg\exists q_2 \left(\neg q_2 \wedge (\neg\varphi \wedge \neg q_2) U (\neg\varphi \wedge q_2)\right)$.

Now we show that $u \models X\varphi$, namely $u^1 \models \varphi$.

To the contrary suppose that $u^1 \models \neg\varphi$. Define v as follows:

$$v_i = \begin{cases} u_i \cup \{q_2\} & \text{if} \quad i = 1 \\ u_i & \text{if} \quad otherwise \end{cases}$$

Then $v \models \neg\varphi \wedge \neg q_2$ and $v^1 \models \neg\varphi \wedge q_2$. Consequently $v \models \neg q_2 \wedge (\neg\varphi \wedge \neg q_2)\ U\ (\neg\varphi \wedge q_2)$, $u \models \exists q_2 \left(\neg q_2 \wedge (\neg\varphi \wedge \neg q_2)\ U\ (\neg\varphi \wedge q_2)\right)$, a contradiction.

B Proof of Lemma 2

Let ψ denote $F(\varphi_2 \wedge q) \wedge G(\varphi_1 \vee \varphi_2 \vee \neg q) \wedge G(\neg q \rightarrow G\neg q)$.

Let $AP = Free(\varphi_1) \cup Free(\varphi_2)$. It is sufficient to prove that for all $u \in \left(2^{AP}\right)^\omega$, $u \models \varphi_1 U \varphi_2$ iff $u \models \exists q \psi$ according to Proposition 2.

"$\Rightarrow$": Suppose that $u \in \left(2^{AP}\right)^\omega$ and $u \models \varphi_1 U \varphi_2$.

There is $i \geq 0$ such that $u^i \models \varphi_2$ and for all $0 \leq j < i$, $u^j \models \varphi_1$.

Define v as follows:

$$v_j = \begin{cases} u_j \cup \{q\} & \text{if} \quad 0 \leq j \leq i \\ u_j & \text{if} \quad j > i \end{cases}$$

Then for all $j \leq i$, $q \in v_j$, and for all $j > i$, $q \notin v_j$ since $q \notin AP$ and $u \in \left(2^{AP}\right)^\omega$.

Thus $v \models G(\neg q \rightarrow G\neg q)$, and $v_i \models \varphi_2 \wedge q$, as a consequence $v \models F(\varphi_2 \wedge q)$.

For all $0 \leq j \leq i$, $v^j \models \varphi_1 \vee \varphi_2$ since $u|_{Free(\varphi_1)} = v|_{Free(\varphi_1)}$ and $u|_{Free(\varphi_2)} = v|_{Free(\varphi_2)}$. And for all $j > i$, $v^j \models \neg q$. Consequently $v \models G(\varphi_1 \vee \varphi_2 \vee \neg q)$.

As a result we conclude that $v \models \psi$. Then $u \models \exists q \psi$ according to the definition of semantics of "$\exists$" quantifier.

"$\Leftarrow$": Suppose that $u \in \left(2^{AP}\right)^\omega$ and $u \models \exists q \psi$.

There is some $v \in \left(2^{\mathcal{P}}\right)^\omega$ such that v differs from u only in the assignments of q, and $v \models \psi$.

Then there is $i \geq 0$ such that $v^i \models \varphi_2 \wedge q$.

Because $v^i \models q$ and $v \models G(\neg q \rightarrow G\neg q)$, then for all $0 \leq j < i$, $v^j \models q$ as well.

It is also true that $v \models G(\varphi_1 \vee \varphi_2 \vee \neg p)$, then for all $0 \leq j < i$, $v^j \models \varphi_1 \vee \varphi_2$.

Consequently we have $v^i \models \varphi_2$, and for all $0 \leq j < i$, $v^j \models \varphi_1 \vee \varphi_2$. Then we conclude that $v \models \varphi_1 U \varphi_2$.

$u \models \varphi_1 U \varphi_2$ follows from the fact that $u|_{2^{Free(\varphi_1 U \varphi_2)}} = v|_{2^{Free(\varphi_1 U \varphi_2)}}$ because u and v only differs from each other in the assignments of q and $q \notin Free(\varphi_1 U \varphi_2)$.

C Proof of Lemma 3

Suppose that L is $S_1^* s_1 S_2^* s_2 ... S_{m-1}^* s_{m-1} S_m^\omega$, where $S_i \subseteq 2^{AP}$ $(1 \leq i \leq m)$, and $s_i \in S_i \backslash S_{i+1}$ $(1 \leq i < m)$.

Since $L_{inf(S')}^{init(s_0)} \neq \emptyset$, we have that $s_0 \in S_1$ and $S' \subseteq S_m$.

$\left(L_{inf(S')}^{init(s_0)}\right)\Big|_{AP} \subseteq (L|_{AP})_{inf(S'|_{AP})}^{init(s_0|_{AP})}$:

Suppose that $u \in \left(L_{inf(S')}^{init(s_0)}\right)\Big|_{AP}$. Then, there is $v \in L_{inf(S')}^{init(s_0)}$ such that $v|_{AP} = u$. So, $v \in L$, $v_0 = s_0$ and each element of S' occurs infinitely often in v.

Then, we know that $u \in L|_{AP}$, $u_0 = s_0|_{AP}$, and each element of $S'|_{AP}$ occurs infinitely often in u. Consequently, $u \in (L|_{AP})_{inf(S'|_{AP})}^{init(s_0|_{AP})}$.

$(L|_{AP})_{inf(S'|_{AP})}^{init(s_0|_{AP})} \subseteq \left(L_{inf(S')}^{init(s_0)}\right)\Big|_{AP}$:

Suppose that $u \in (L|_{AP})^{init(s_0|_{AP})}_{inf(S'|_{AP})}$. Then, $u \in L|_{AP}$, $u_0 = s_0|_{AP}$ and each element in $S'|_{AP}$ occurs infinitely often in u.

Since

$$L|_{AP} = (S_1|_{AP})^* (s_1|_{AP}) ... (S_{m-1}|_{AP})^* (s_{m-1}|_{AP}) (S_m|_{AP})^\omega ,$$

we have that $u = x_1(s_1|_{AP})...x_{m-1}(s_{m-1}|_{AP})x_m$, where $x_i \in (S_i|_{AP})^*$ $(1 \leq i < m)$ and $x_m \in (S_m|_{AP})^\omega$.

Because $s_0 \in S_1$ and $S' \subseteq S_m$, we can change the assignment of $q_1, ..., q_k$ on u to get a $v = x'_1 s_1 ... x'_{m-1} s_{m-1} x'_m$ such that $v|_{AP} = u$, $x'_i \in S_i^*$ $(1 \leq i < m)$, $x'_m \in S_m^\omega$, $v_0 = s_0$ and each element of S' occurs infinitely often on v. So, $v \in L^{init(s_0)}_{inf(S')}$ and $u \in \left(L^{init(s_0)}_{inf(S')}\right)\Big|_{AP}$.

D Proof of Lemma 5

Suppose that $\varphi = \exists q_1 ... \exists q_k(\psi)$, where ψ is a formula in $L(U)$.

Let $AP' = AP \cup \{q_1, ..., q_k\}$. Then, $\mathcal{L}(\varphi)^{AP} = \mathcal{L}(\psi)^{AP'}|_{AP}$ according to Proposition 3.

Let $u \in (2^{AP})^\omega$, $f : \mathbf{N} \to \mathbf{N} \backslash \{0\}$, and $u \models \varphi$. Then, there is $v \in \left(2^{AP'}\right)^\omega$ such that $v \models \psi$ and $v|_{AP} = u$. The languages defined by $L(U)$ formulas are stutter invariant according to Proposition 5. So, $v_0^{f(0)} ... v_i^{f(i)} ... \models \psi$. Evidently $u_0^{f(0)} ... u_i^{f(i)} ... = \left(v_0^{f(0)} ... v_i^{f(i)} ...\right)\Big|_{AP}$. We conclude that $u_0^{f(0)} ... u_i^{f(i)} ... \models \varphi$.

E Proof of Lemma 7

We first show that L is not non-counting. Since L is non-counting iff its syntactic monoid is aperiodic, according to Proposition 6, we know that L is not expressible in LTL.

To the contrary, suppose that L is non-counting. Then there is $n \geq 0$ such that for all $x, y, z, u \in \left(2^{AP}\right)^*$, ($xu^n yz^\omega \in L$ iff $xu^{n+1}yz^\omega \in L$) and ($x\left(yu^n z\right)^\omega \in L$ iff $x\left(yu^{n+1}z\right)^\omega \in L$). Let $x = y = z = \emptyset$ and $u = \emptyset\{p_1\}$. Then,

$$xu^n yz^\omega = \emptyset\left(\emptyset\{p_1\}\right)^n \emptyset\emptyset^\omega \in L \text{ iff } xu^{n+1}yz^\omega = \emptyset\left(\emptyset\{p_1\}\right)^{n+1} \emptyset\emptyset^\omega \in L,$$

contradicting to definition of L.

Now we show that L is expressible in $EQ(U)$.

Let $u \in L$. Then, $(\emptyset\{p_1\})$ occurs an odd number of times in u. There are three cases.

Case 1: $\emptyset$ occurs in the first position of u and $(\emptyset\{p_1\})$ occurs at least three times in u,
Case 2: $\{p_1\}$ occurs in the first position of u,
Case 3: $\emptyset$ occurs in the first position of u and $(\emptyset\{p_1\})$ occurs ony once in u.

Let $L_i = \{u \in L | u$ satisfies the condition of the Case i above$\}$, where $i = 1, 2, 3$.

Let

$$x = \emptyset\emptyset^*\{p_1\}\{p_1\}^*\emptyset\emptyset^*\{p_1\}\{p_1\}^*,$$
$$y = \emptyset\emptyset^*\{p_1\}\{p_1\}^*\left(\{p_1\}^\omega \cup \emptyset^\omega\right).$$

Then $L_1 = xx^*y$, $L_2 = \{p_1\}\{p_1\}^*x^*y$ and $L_3 = y$.

We introduce new variables p_2, p_3, p_4 and let $AP' = AP \cup \{p_2, p_3, p_4\}$.

Define $L'_i \subseteq \left(2^{AP'}\right)^\omega$ such that $L'_i|_{AP} = L_i$ $(i = 1, 2, 3)$ as follows.

$L'_1 = x'(x')^*y'$, $L'_2 = \{p_1, p_2\}\{p_1, p_2\}^*(x')^*y'$ and $L'_3 = y'$, where

$$x' = \emptyset\emptyset^*\{p_1\}\{p_1\}^*\{p_2\}\{p_2\}^*\{p_1, p_2\}\{p_1, p_2\}^*,$$

$$y' = \{p_3\}\{p_3\}^*\{p_1, p_3\}\{p_1, p_3\}^*\left(\{p_1, p_3\}^\omega \cup \{p_4\}^\omega\right).$$

It is easy to verify that $L'_i|_{AP} = L_i$ $(i = 1, 2, 3)$ since $x'|_{AP} = x$ and $y'|_{AP} = y$.

Let $L' = L'_1 \cup L'_2 \cup L'_3$. Then L' is accepted by the Büchi automaton $\mathcal{B}$ illustrated in Fig. 2.

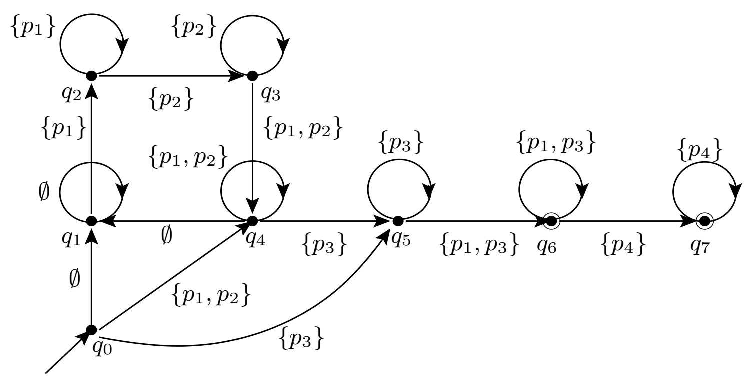

Fig. 2. Büchi automaton $\mathcal{B}$ for L'

There are eight states in $\mathcal{B}$, q_0 is the initial state, q_6, q_7 are the accepting states.

q_0 has three out-edges labeled by $\emptyset$, $\{p_1, p_2\}$ and $\{p_3\}$ respectively, corresponding to the three distinct letters occurring in the first position of ω-words in L'_1, L'_2, L'_3 respectively.

When the run of $\mathcal{B}$ on an ω-word $u \in \left(2^{AP'}\right)^\omega$ reaches q_4 or q_5, $(\emptyset\{p_1\})$ must have occurred even number of times in $u|_{AP}$.

When a run of $\mathcal{B}$ reaches q_4, it has two choices: one is to stay in the square cycle (containing states q_1, q_2, q_3, q_4), the other is to leave the square cycle and visit q_5. If we want a run to be accepting, then eventually we must visit q_5

since the accepting states are q_6, q_7. So, along an ω-word accepted by $\mathcal{B}$, p_3 will eventually become true.

When a run of $\mathcal{B}$ reaches q_6, we have two choices to make the run accepting: either stay in q_6 forever or eventually visit q_7 and stay in q_7 forever.

Now we define formula ψ in $L(U)$ such that $\mathcal{L}(\psi)^{AP'} = L'$. Then, $L = L'|_{AP} = \mathcal{L}(\psi)^{AP'}|_{AP} = \mathcal{L}(\exists p_2 \exists p_3 \exists p_4 \psi)^{AP}$ according to Proposition 3. Consequently, we conclude that L is expressible in $EQ(U)$.

We define the formula ψ in $L(U)$ as follows:

$$\begin{aligned}
\psi := & \Big(\mathcal{B}(\emptyset)^{AP'} \vee \mathcal{B}(\{p_1, p_2\})^{AP'} \vee \mathcal{B}(\{p_3\})^{AP'}\Big) \wedge Fp_3 \wedge \\
& G\Big(\mathcal{B}(\emptyset)^{AP'} \rightarrow \mathcal{B}(\emptyset)^{AP'}\ U\ \mathcal{B}(\{p_1\})^{AP'}\Big) \wedge \\
& G\Big(\mathcal{B}(\{p_1\})^{AP'} \rightarrow \mathcal{B}(\{p_1\})^{AP'}\ U\ \mathcal{B}(\{p_2\})^{AP'}\Big) \wedge \\
& G\Big(\mathcal{B}(\{p_2\})^{AP'} \rightarrow \mathcal{B}(\{p_2\})^{AP'}\ U\ \mathcal{B}(\{p_1, p_2\})^{AP'}\Big) \wedge \\
& G\left(\mathcal{B}(\{p_1, p_2\})^{AP'} \rightarrow \mathcal{B}(\{p_1, p_2\})^{AP'}\ U\ \left(\mathcal{B}(\emptyset)^{AP'} \vee \mathcal{B}(\{p_3\})^{AP'}\right)\right) \wedge \\
& G\left(\mathcal{B}(\{p_3\})^{AP'} \rightarrow \mathcal{B}(\{p_3\})^{AP'}\ U\ \left(G\left(\mathcal{B}(\{p_1, p_3\})^{AP'}\right) \vee \right.\right. \\
& \left.\left.\left(\mathcal{B}(\{p_1, p_3\})^{AP'} \wedge \mathcal{B}(\{p_1, p_3\})^{AP'}\ U\ G\left(\mathcal{B}(\{p_4\})^{AP'}\right)\right)\right)\right).
\end{aligned}$$

F Proof of Lemma 8

Since L is not stutter invariant, it follows that L is not expressible in $L(U)$ according to Proposition 5.

Let $AP' = AP \cup \{p_2\}$,

$$L' = (\{\{p_2\}, \{p_1, p_2\}\})^* \{p_1, p_2\} (\{p_1\})^* \{p_1\} (\{\{p_2\}, \{p_1, p_2\}\})^* \{p_2\} \emptyset^\omega.$$

It is easy to see that L' is a restricted ω-regular set and $L'|_{AP} = L$.

Let $S' = \{\emptyset\}$. Then, $L' = (L')^{init(\{p_2\})}_{inf(S')} \bigcup (L')^{init(\{p_1, p_2\})}_{inf(S')}$. So, L' is expressible in $L(F)$ according to Proposition 5. We conclude that L is expressible in $EQ(F)$ according to Proposition 3.