Safe Probabilistic Invariance Verification for Stochastic Discrete-time Dynamical Systems

Yiqing Yu, Taoran Wu, Bican Xia, Ji Wang, and Bai Xue

***Abstract*— Ensuring safety through set invariance has proven a useful method in a variety of applications in robotics and control. In this paper, we focus on the safe probabilistic invariance verification problem for discrete-time dynamical systems subject to stochastic disturbances over the infinite time horizon. Our goal is to compute the lower and upper bounds of the liveness probability for a given safe set and set of initial states. This probability represents the likelihood that the system will remain within the safe set for all time. To address this problem, we draw inspiration from stochastic barrier certificates for safety verification and build upon the findings in [21], where an equation was presented for exact probability analysis. We present two sets of optimizations and demonstrate their effectiveness through two examples, using semi-definite programming tools.**

***Index Terms*— Stochastic Discrete-time Systems; Safe Probabilistic Invariance Verification.**

I. Introduction

The rapid development of modern technology has led to an increase in intelligent autonomous systems. Ensuring the safe operation of these systems is essential, but external disturbances can cause uncertainties, making it necessary to consider their impact on system safety. Safe robust invariance is commonly used to formalize the impact of unknown perturbations and guarantee that a system will remain inside a specific safe set for all time, regardless of bounded external disturbances. Many studies have been published on certifying safe robust invariance over the past few decades [16], [22].

While bounding disturbances is useful for perturbation analysis, many systems have more information available, such as a probability distribution [7]. In such cases, safe probabilistic invariance complements safe robust invariance by ensuring that a system will remain inside a specified safe set with a certain probability [12]. This approach reduces inherent conservatism by allowing probabilistic violations and has gained increasing attention [13]. Probabilistic invariance can be evaluated by examining its dual, probabilistic reachability. [2], [1] investigated the finite-time probabilistic invariance problem for discrete-time stochastic (hybrid) systems via reachability analysis. Meanwhile, [17], [18] studied the infinite-time horizon probabilistic invariance by defining it as a finite-time reach-avoid property in combination with infinite-time invariance around absorbing sets over the state space and provided a lower bound for infinite-time probabilistic invariance. However, no known general automatic procedure enables computing absorbing sets exactly.

This paper focuses on the safe probabilistic invariance verification of stochastic discrete-time systems. The objective is to establish lower and upper bounds on the liveness probability of the system remaining inside a bounded safe set for all time, given a specific initial set. While lower bounds have been commonly studied in existing literature, this paper also introduces an approach to compute upper bounds. The utilization of upper bounds can enhance our comprehension of the invariance properties of the system and provide a more precise estimation of the exact liveness probability. Additionally, it can also help us deal with scenarios in which the safe set remains safe but becomes uncomfortable. However, it is important to note that this aspect is not the main focus of our work, and therefore we will refrain from providing an in-depth discussion of it in this context. We propose optimizations for solving the safe probabilistic invariance verification problem using an auxiliary switched system that freezes outside the safe set. It is shown that verifying safe set invariance for the original system is equivalent to verifying it for the switched system. Firstly, a set of optimizations is proposed to address the safe probabilistic invariance verification problem using classical barrier certificates and the switched system. While the barrier certificate based optimizations require the existence of non-negative supermartingales, which can be challenging in practice, new optimizations that are subject to weaker constraints are further proposed. These are inspired by the equation presented in [21], [20], which characterizes the exact probability of safely reaching a target set with bounded solutions. To demonstrate the theoretical developments, two numerical examples are solved using the semi-definite programming tool.

The contributions of this work are summarized below.

1) The safe probabilistic invariance verification in stochastic discrete-time systems is investigated. Given a specific initial set, lower and upper bounds on the liveness probability of the system remaining inside a bounded safe set for all time are studied.
2) Based on an auxiliary switched system, two sets of optimizations are proposed for addressing the safe set invariance verification problem. One is adapted from

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the classical barrier certificates for safety verification. The other is inspired by the equation in [21], which is shown to be weaker than the previous one.

II. Related Work

While there is a significant body of research on the verification of stochastic (hybrid) systems, this section focuses specifically on works closely related to the topic at hand, with the exception of the works mentioned earlier. For readers interested in a broader survey of the field, we recommend consulting [13].

The problem studied in this work is closely related to the computation of probabilistic invariant sets, which define a set of states for which a system starting from any point in that set must remain inside a specified region of interest with a certain probability. By computing probabilistic invariant sets, we can demonstrate that certain properties hold for an initial set if it is a subset of the computed probabilistic invariant sets. Previous works, such as [10], [11], and [9], have approximated polyhedral probabilistic invariant sets using Chebyshev's inequality for linear systems with Gaussian noise. However, these methods are limited to computing lower bounds of the liveness probability and cannot be applied to compute upper bounds. Furthermore, previous works have primarily focused on linear systems with Gaussian disturbances, whereas this paper considers nonlinear systems. Recently, [8] proposed an algorithm to compute infinite-horizon probabilistic controlled invariant sets based on the dynamic program in [2]. These probabilistic invariant sets are computed using the stochastic backward reachable set from a robust invariant set.

Another two closely related works to the present one are [3], which studies the temporal verification based on stochastic barrier certificates, and [21], which derives a set of equations being able to characterize the exact probability of reaching a specified target set while avoiding unsafe states. With an assumption that the evolution space $\mathcal{X}$ is a robust invariant (i.e., $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{d}):\mathcal{X}\times\mathcal{D}\rightarrow\mathcal{X}$), barrier certificates inspired by the ones in [14] and [15] were formulated for temporal verification of stochastic discrete-time systems in [3]. The first set of optimizations in this paper was adapted from them. The assumption that the evolution space $\mathcal{X}$ (i.e., the safe set in this paper) is a robust invariant is abandoned in our method. Instead, an auxiliary switched system with a robust invariant set is borrowed to construct our constraints for addressing the safe set invariance verification problem. Recently, [6], [19] extended the control barrier certificates from deterministic setting to the synthesis of controllers for enforcing invariance of a safe set with at least a certain probability. However, continuous-time systems modelled by stochastic differential equations were considered in [6], [19]. The second set of optimizations proposed in this paper, which are subject to weaker constraints than the first ones, is inspired by the results in [21]. A new equation which is able to characterize the exact probability of leaving the safe set $\mathcal{X}$ is formulated and constraints for addressing the safe set invariance problem are constructed via relaxing this equation.

This paper is structured as follows. In Section III, we formalize the stochastic system and associated safe probabilistic invariance verification of interest. In Section IV we present two sets of optimizations for addressing the safe probabilistic invariance verification, and demonstrate them on two examples in Section V. Finally, this paper is concluded in Section VI.

III. Preliminaries

We begin by introducing the concept of discrete-time systems that are subject to stochastic disturbances, as well as the problem of verifying safe probabilistic invariance. Throughout this paper, we will refer to several basic notions. For example, $\mathbb{N}$ is the set of nonnegative integers, while $\mathbb{N}_{\leq k}$ is the set of nonnegative integers that are less than or equal to k. Additionally, we use the notation Δ^c and $\partial\Delta$ to represent the complement and boundary of a set Δ, respectively; $\sum[\boldsymbol{x}]$ denotes the set of sum-of-squares polynomials over variables $\boldsymbol{x}$, i.e., $\sum[\boldsymbol{x}]=\{p\in\mathbb{R}[\boldsymbol{x}]\mid p=\sum_{i=1}^{k}q_i^2(\boldsymbol{x}),q_i(\boldsymbol{x})\in\mathbb{R}[\boldsymbol{x}],i=1,\dots,k\}$. Furthermore, $\mathbb{R}_{\geq 0}$ is the set of nonnegative real numbers. Finally, we use the indicator function $1_A(\boldsymbol{x})$ to denote whether or not $\boldsymbol{x}$ is an element of a set A. Specifically, if $\boldsymbol{x}\in A$, then $1_A(\boldsymbol{x})=1$, and if $\boldsymbol{x}\notin A$, then $1_A(\boldsymbol{x})=0$.

A. Problem Statement

In this paper we are examining stochastic discrete-time systems that are described by stochastic difference equations of the form:

$$\begin{aligned}&\boldsymbol{x}(l+1)=\boldsymbol{f}(\boldsymbol{x}(l),\boldsymbol{d}(l)),\quad\forall l\in\mathbb{N},\\&\boldsymbol{x}(0)=\boldsymbol{x}_0\in\mathbb{R}^n.\end{aligned}\tag{1}$$

Here, $\boldsymbol{x}(\cdot):\mathbb{N}\rightarrow\mathbb{R}^n$ represents the states, and $\boldsymbol{d}(\cdot):\mathbb{N}\rightarrow\mathcal{D}$ with $\mathcal{D}\subseteq\mathbb{R}^m$ represents the stochastic disturbances. The random vectors $\boldsymbol{d}(0),\boldsymbol{d}(1),\dots$ are independent and identically distributed (i.i.d), and take values in $\mathcal{D}$ with the probability distribution:

$$\mathrm{Prob}(\boldsymbol{d}(l)\in B)=\mathbb{P}(B),\quad\forall l\in\mathbb{N},\quad\forall B\subseteq\mathcal{D}.$$

In addition, $\mathbb{E}[\cdot]$ is the expectation induced by $\mathbb{P}$.

To prepare for defining the trajectory of system (1), we first need to define a disturbance signal.

Definition 1: A disturbance signal π is an ordered sequence $\{\boldsymbol{d}(i),i\in\mathbb{N}\}$, where $\boldsymbol{d}(\cdot):\mathbb{N}\rightarrow\mathcal{D}$.

The disturbance signal π is a stochastic process defined on the canonical sample space $\Omega=\mathcal{D}^{\infty}$ with the probability measure $\mathbb{P}^{\infty}$, and is denoted by $\{\boldsymbol{d}(i),i\in\mathbb{N}\}$. We use $\mathbb{E}^{\infty}[\cdot]$ to represent an expectation with respect to the probability measure $\mathbb{P}^{\infty}$.

Given a disturbance signal π and an initial state $\boldsymbol{x}_0\in\mathbb{R}^n$, a unique trajectory $\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(\cdot):\mathbb{N}\rightarrow\mathbb{R}^n$ is induced with $\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(0)=\boldsymbol{x}_0$. Specifically, we have

$$\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(l+1)=\boldsymbol{f}(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(l),\boldsymbol{d}(l))$$

for all $l\in\mathbb{N}$.

Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0\subseteq\mathcal{X}$, the safe probabilistic invariance verification problem is to determine lower and upper bounds of the liveness probability of remaining within the safe set $\mathcal{X}$ for system (1), starting from $\mathcal{X}_0$.

Definition 2: The safe set invariance verification is to compute lower and upper bounds, denoted by $\epsilon_1 \in [0,1]$ and $\epsilon_2 \in [0,1]$ respectively, for the liveness probability that the system, starting from any state in $\mathcal{X}_0$, will remain inside the safe set $\mathcal{X}$ for all time, i.e., to compute ϵ_1 and ϵ_2 such that

$$\epsilon_1 \leq \mathbb{P}^{\infty}(\forall k \in \mathbb{N}.\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(k) \in \mathcal{X} \mid \boldsymbol{x}_0 \in \mathcal{X}_0) \leq \epsilon_2. \tag{2}$$

By computing these bounds, we can determine the degree of safety that the system provides and ensure that it meets the desired specifications.

Remark 1: If the set $\mathcal{X}^c$ represents the desired (or, more comfortable) states, then ϵ_2 can be used as an upper bound for the probability of the system (1) getting trapped within $\mathcal{X}$. This means that if we can ensure that the liveness probability is below ϵ_2, we can be confident that the system will eventually reach the desired states with high probability.

This paper focuses on addressing the safe probabilistic invariance verification problem as defined in Definition 2.

B. Reachability Probability Characterization in [21]

In this subsection, we will recall an equation that was derived for probabilistic reach-avoid analysis. The equation's bounded solution is equivalent to the precise probability of the system entering a specified target set within a finite time while remaining inside a given safe set before the first target is reached. We will adopt this equation to address the safe invariance verification problem outlined in Definition 2.

Theorem 1 (Theorem 1, [21]): Given a bounded safe set $\mathcal{X}$, a target set $\mathcal{X}_r$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0, \mathcal{X}_r \subseteq \mathcal{X}$, if there exist bounded functions $v(\boldsymbol{x}) : \widehat{\mathcal{X}} \rightarrow \mathbb{R}$ and $w(\boldsymbol{x}) : \widehat{\mathcal{X}} \rightarrow \mathbb{R}$ such that for $\boldsymbol{x} \in \widehat{\mathcal{X}}$,

$$\begin{cases} v(\boldsymbol{x}) = \mathbb{E}^{\infty}[v(\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}}(1))], \\ v(\boldsymbol{x}) = 1_{\mathcal{X}_r}(\boldsymbol{x}) + \mathbb{E}^{\infty}[w(\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}}(1))] - w(\boldsymbol{x}), \end{cases} \tag{3}$$

then

$$\begin{aligned} &\mathbb{P}^{\infty}\big(\exists k \in \mathbb{N}.\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(k) \in \mathcal{X}_r \bigwedge \forall l \in \mathbb{N}_k.\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(l) \in \mathcal{X} \mid \boldsymbol{x}_0 \in \mathcal{X}\big) \\ &= \mathbb{P}^{\infty}\big(\exists k \in \mathbb{N}.\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}_0}(k) \in \mathcal{X}_r \mid \boldsymbol{x}_0 \in \mathcal{X}\big) \\ &= \lim_{i \rightarrow \infty} \frac{\mathbb{E}^{\infty}[\sum_{j=0}^{i-1} 1_{\widehat{\mathcal{X}} \setminus \mathcal{X}}(\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}_0}(j))]}{i} = v(\boldsymbol{x}), \end{aligned}$$

where $\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}_0}(\cdot) : \mathbb{N} \rightarrow \mathbb{R}^n$ is the trajectory to the system

$$\begin{cases} \boldsymbol{x}(j+1) = 1_{\mathcal{X} \setminus \mathcal{X}_r}(\boldsymbol{x}(j)) \cdot \boldsymbol{f}(\boldsymbol{x}(j), \boldsymbol{d}(j)) \\ +1_{\mathcal{X}_r}(\boldsymbol{x}(j)) \cdot \boldsymbol{x}(j) + 1_{\widehat{\mathcal{X}} \setminus \mathcal{X}}(\boldsymbol{x}(j)) \cdot \boldsymbol{x}(j), \forall j \in \mathbb{N}, \\ \boldsymbol{x}(0) = \boldsymbol{x}_0, \end{cases}$$

and $\widehat{\mathcal{X}}$ is a set satisfying $\widehat{\mathcal{X}} \supset \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{d}), \boldsymbol{x}_0 \in \mathcal{X}, \boldsymbol{d} \in \mathcal{D}\} \cup \mathcal{X}$.

A sufficient condition for certifying lower bounds of the probability $\mathbb{P}^{\infty}\big(\exists k \in \mathbb{N}.\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(k) \in \mathcal{X}_r \bigwedge \forall l \in \mathbb{N}_k.\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(l) \in \mathcal{X} \mid \boldsymbol{x}_0 \in \mathcal{X}_0\big)$ can be derived via relaxing (3). It is obtained by adding a constraint $v(\boldsymbol{x}) \geq \epsilon_1, \forall \boldsymbol{x} \in \mathcal{X}_0$ into the ones in Corollary 1 in [21].

Proposition 1: Given a safe set $\mathcal{X}$, a target set $\mathcal{X}_r$ and an initial set $\mathcal{X}_0$, where $\mathcal{X}_0, \mathcal{X}_r \subseteq \mathcal{X}$, if there exist bounded functions $v(\boldsymbol{x}) : \widehat{\mathcal{X}} \rightarrow \mathbb{R}$ and $w(\boldsymbol{x}) : \widehat{\mathcal{X}} \rightarrow \mathbb{R}$ such that

$$\begin{cases} v(\boldsymbol{x}) \geq \epsilon_1, \forall \boldsymbol{x} \in \mathcal{X}_0, \\ v(\boldsymbol{x}) \leq \mathbb{E}^{\infty}[v(\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}}(1))], \forall \boldsymbol{x} \in \widehat{\mathcal{X}}, \\ v(\boldsymbol{x}) \leq 1_{\mathcal{X}_r}(\boldsymbol{x}) + \mathbb{E}^{\infty}[w(\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}}(1))] - w(\boldsymbol{x}), \forall \boldsymbol{x} \in \widehat{\mathcal{X}} \end{cases}$$

which is equivalent to

$$\begin{cases} v(\boldsymbol{x}) \geq \epsilon_1, \forall \boldsymbol{x} \in \mathcal{X}_0, \\ v(\boldsymbol{x}) \leq \mathbb{E}^{\infty}[v(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}}(1))], \forall \boldsymbol{x} \in \mathcal{X} \setminus \mathcal{X}_r, \\ v(\boldsymbol{x}) \leq \mathbb{E}^{\infty}[w(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}}(1))] - w(\boldsymbol{x}), \forall \boldsymbol{x} \in \mathcal{X} \setminus \mathcal{X}_r, \\ v(\boldsymbol{x}) \leq 1, \forall \boldsymbol{x} \in \mathcal{X}_r, \\ v(\boldsymbol{x}) \leq 0, \forall \boldsymbol{x} \in \widehat{\mathcal{X}} \setminus \mathcal{X}, \end{cases} \tag{4}$$

then $\mathbb{P}^{\infty}\big(\exists k \in \mathbb{N}.\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(k) \in \mathcal{X}_r \bigwedge \forall l \in \mathbb{N}_k.\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(l) \in \mathcal{X} \mid \boldsymbol{x}_0 \in \mathcal{X}_0\big) = \mathbb{P}^{\infty}\big(\exists k \in \mathbb{N}.\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}_0}(k) \in \mathcal{X}_r \mid \boldsymbol{x}_0 \in \mathcal{X}_0\big) \geq \epsilon_1$.

IV. Safe Probabilistic Invariance Verification

In this section we present two sets of optimizations for addressing the safe probabilistic invariance verification problem in Definition 2. The first set of optimizations is adapted from the classical stochastic barrier certificates for safety and reachability verification. The second set of optimizations, which are subject to weaker constraints than the first one, is inspired by Theorem 1 and Proposition 1.

Similar to [21], in constructing our optimizations we need an auxiliary system as follows:

$$\begin{cases} \boldsymbol{x}(j+1) = \widehat{\boldsymbol{f}}(\boldsymbol{x}(j), \boldsymbol{d}(j)), \forall j \in \mathbb{N}, \\ \boldsymbol{x}(0) = \boldsymbol{x}_0, \end{cases} \tag{5}$$

where $\widehat{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{d}) = 1_{\mathcal{X}}(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{d}) + 1_{\widehat{\mathcal{X}} \setminus \mathcal{X}}(\boldsymbol{x}) \cdot \boldsymbol{x}$ and $\widehat{\mathcal{X}}$ is a set containing the union of the set $\mathcal{X}$ and all reachable states starting from $\mathcal{X}$ within one step, i.e.,

$$\widehat{\mathcal{X}} \supset \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{d}), \boldsymbol{x}_0 \in \mathcal{X}, \boldsymbol{d} \in \mathcal{D}\} \cup \mathcal{X}. \tag{6}$$

Given a disturbance signal π, we define the trajectory to system (5) as $\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}_0}(\cdot) : \mathbb{N} \rightarrow \mathbb{R}^n$, where $\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}_0}(0) = \boldsymbol{x}_0$. It is easy to observe that $\widehat{\mathcal{X}}$ is a robust invariant of system (5) according to $\widehat{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{d}) \in \widehat{\mathcal{X}}, \forall (\boldsymbol{x}, \boldsymbol{d}) \in \widehat{\mathcal{X}} \times \mathcal{D}$.

Also, since $\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}_0}(1) = \boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(1)$ for $\boldsymbol{x} \in \mathcal{X}$, we have that

$$\begin{aligned} &\mathbb{P}^{\infty}(\exists k \in \mathbb{N}.\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(k) \in \widehat{\mathcal{X}} \setminus \mathcal{X} \wedge \forall i \in \mathbb{N}_{k-1}.\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(i) \in \mathcal{X}) \\ &= \mathbb{P}^{\infty}(\exists k \in \mathbb{N}.\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}_0}(k) \in \widehat{\mathcal{X}} \setminus \mathcal{X} \wedge \forall i \in \mathbb{N}_{k-1}.\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}_0}(i) \in \mathcal{X}) \end{aligned}$$

and

$$\mathbb{P}^{\infty}(\forall k \in \mathbb{N}.\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(k) \in \mathcal{X}) = \mathbb{P}^{\infty}(\forall k \in \mathbb{N}.\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}_0}(k) \in \mathcal{X}).$$

Given a disturbance signal π and an initial state $\boldsymbol{x}_0 \in \mathcal{X}$, the resulting trajectory $\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}_0}(\cdot) : \mathbb{N} \rightarrow \mathbb{R}^n$ either enters the unsafe set $\widehat{\mathcal{X}} \setminus \mathcal{X}$ in finite time (i.e., $\exists k \in \mathbb{N}.\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}_0}(k) \in \widehat{\mathcal{X}} \setminus \mathcal{X} \wedge \forall i \in \mathbb{N}_{\leq k-1}.\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}_0}(i) \in \mathcal{X}$) or stays inside the safe set $\mathcal{X}$ always (i.e., $\forall k \in \mathbb{N}.\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}_0}(k) \in \mathcal{X}$). Thus,

$$\begin{aligned} &\mathbb{P}^{\infty}(\exists k \in \mathbb{N}.\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}_0}(k) \in \widehat{\mathcal{X}} \setminus \mathcal{X} \wedge \forall i \in \mathbb{N}_{\leq k-1}.\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}_0}(i) \in \mathcal{X}) \\ &+ \mathbb{P}^{\infty}(\forall k \in \mathbb{N}.\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}_0}(k) \in \mathcal{X}) = 1. \end{aligned}$$

Therefore, if the upper and lower bounds of the probability $\mathbb{P}^{\infty}(\exists k \in \mathbb{N}.\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}_0}(k) \in \widehat{\mathcal{X}} \setminus \mathcal{X} \wedge \forall i \in \mathbb{N}_{\leq k-1}.\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}_0}(i) \in \mathcal{X})$ are gained, one can obtain the lower and upper bounds of the liveness probability of staying inside the safe set $\mathcal{X}$.

A. Barrier Certificates Based Invariance Verification

In this subsection we propose optimizations for certifying lower and upper bounds of the liveness probability by adapting the classical barrier certificate for safety verification.

Proposition 2 provides a straightforward sufficient condition for lower bounds on liveness probability, as demonstrated in Theorem 5 of [3].

Proposition 2: Under the assumption that $\Omega \subseteq \mathbb{R}^n$ is a robust invariant set for system (1), i.e., $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{d}) : \Omega \times \mathcal{D} \rightarrow \Omega$, and $\mathcal{X} \subseteq \Omega$, if there exists $v(\boldsymbol{x}) : \Omega \rightarrow \mathbb{R}_{\geq 0}$ such that

$$\begin{cases} v(\boldsymbol{x}) \leq 1 - \epsilon_1, \forall \boldsymbol{x} \in \mathcal{X}_0, \\ v(\boldsymbol{x}) \geq 1, \forall \boldsymbol{x} \in \mathcal{X}_{unsafe} (= \Omega \setminus \mathcal{X}), \\ \mathbb{E}^{\infty}[v(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}})] - v(\boldsymbol{x}) \leq 0, \forall \boldsymbol{x} \in \Omega, \end{cases} \tag{7}$$

then $\mathbb{P}^{\infty}\big(\exists k \in \mathbb{N}.\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(k) \in \mathcal{X}_{unsafe} \mid \boldsymbol{x}_0 \in \mathcal{X}_0\big) \leq 1 - \epsilon_1$. Thus, $\mathbb{P}^{\infty}\big(\forall k \in \mathbb{N}.\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(k) \in \mathcal{X} \mid \boldsymbol{x}_0 \in \mathcal{X}_0\big) \geq \epsilon_1$.

Similarly, a sufficient condition for determining upper bounds of the liveness probability can be obtained straightforwardly from Theorem 16 in [3]. However, finding a robust invariant set Ω for many systems, except for the trivial case of $\Omega = \mathbb{R}^n$, can be challenging and computationally intensive, if it even exists. Moreover, when $\Omega = \mathbb{R}^n$ in Proposition 2, the resulting constraint (7) may be too strong, leading to an overly conservative upper bound. We use an example to illustrate this below.

Example 1: In this example we consider a computer-based model, which is modified from the reversed-time Van der Pol oscillator based on Euler's method with the time step 0.01:

$$\begin{cases} x(l+1) = x(l) - 0.02y(l), \\ y(l+1) = y(l) + 0.01\Big((0.8 + d(l))x(l) \\ \qquad\qquad + 10(x^2(l) - 0.21)y(l)\Big), \end{cases} \tag{8}$$

where $d(\cdot) : \mathbb{N} \rightarrow \mathcal{D} = [-0.1, 0.1]$, $\mathcal{X} = \{(x, y)^{\top} \mid h(\boldsymbol{x}) \leq 0\}$ with $h(\boldsymbol{x}) = x^2 + y^2 - 1$, and $\mathcal{X}_0 = \{(x, y)^{\top} \mid g(\boldsymbol{x}) < 0\}$ with $g(\boldsymbol{x}) = x^2 + y^2 - 0.01$.

Via solving **Op0**, which is encoded into a semi-definite program **SDP0** (shown in Appendix) via the sum of squares decomposition for multivariate polynomials:

$$\textbf{Op0} \quad \max_{v(\boldsymbol{x}), \epsilon_1} \epsilon_1$$

$$\text{s.t.} \begin{cases} v(\boldsymbol{x}) \leq 1 - \epsilon_1, \forall \boldsymbol{x} \in \mathcal{X}_0, \\ v(\boldsymbol{x}) \geq \mathbb{E}^{\infty}[v(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}}(1))], \forall \boldsymbol{x} \in \mathbb{R}^n, \\ v(\boldsymbol{x}) \geq 1, \forall \boldsymbol{x} \in \mathbb{R}^n \setminus \mathcal{X}, \\ v(\boldsymbol{x}) \geq 0, \forall \boldsymbol{x} \in \mathbb{R}^n, \\ \epsilon_1 \geq 0. \end{cases}$$

we obtain a lower bound of the liveness probability, which is 2.1368e-07. This is too conservative to be useful in practice. The resulting semi-definite program is addressed when unknown polynomials of degree 8 are used. ■

In the following we will present weaker sufficient conditions for certifying lower and upper bounds using the switched system (5). They are respectively formulated in Proposition 3 and 4.

Proposition 3: Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$ with $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist a barrier certificate $v(\boldsymbol{x}) : \widehat{\mathcal{X}} \rightarrow \mathbb{R}$ satisfying

$$\begin{cases} v(\boldsymbol{x}) \leq 1 - \epsilon_1, \forall \boldsymbol{x} \in \mathcal{X}_0, \\ v(\boldsymbol{x}) \geq \mathbb{E}^{\infty}[v(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}}(1))], \forall \boldsymbol{x} \in \mathcal{X}, \\ v(\boldsymbol{x}) \geq 1, \forall \boldsymbol{x} \in \widehat{\mathcal{X}} \setminus \mathcal{X}, \\ v(\boldsymbol{x}) \geq 0, \forall \boldsymbol{x} \in \widehat{\mathcal{X}}, \end{cases} \tag{9}$$

then $\mathbb{P}^{\infty}\big(\exists k \in \mathbb{N}.\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(k) \in \widehat{\mathcal{X}} \setminus \mathcal{X} \mid \boldsymbol{x}_0 \in \mathcal{X}_0\big) \leq 1 - \epsilon_1$. Consequently, $\mathbb{P}^{\infty}\big(\forall k \in \mathbb{N}.\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(k) \in \mathcal{X} \mid \boldsymbol{x}_0 \in \mathcal{X}_0\big) \geq \epsilon_1$.

Proof: Constraint (9) is equivalent to the following constraint

$$\begin{cases} v(\boldsymbol{x}) \leq 1 - \epsilon_1, \forall \boldsymbol{x} \in \mathcal{X}_0, \\ v(\boldsymbol{x}) \geq \mathbb{E}^{\infty}[v(\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}}(1))], \forall \boldsymbol{x} \in \widehat{\mathcal{X}}, \\ v(\boldsymbol{x}) \geq 1, \forall \boldsymbol{x} \in \mathcal{X}_{unsafe} (= \widehat{\mathcal{X}} \setminus \mathcal{X}), \\ v(\boldsymbol{x}) \geq 0, \forall \boldsymbol{x} \in \widehat{\mathcal{X}}. \end{cases}$$

Therefore, $v(\boldsymbol{x})$ is a classical stochastic barrier certificate for system (5) with the invariant set $\widehat{\mathcal{X}}$ and the unsafe set $\widehat{\mathcal{X}} \setminus \mathcal{X}$. Therefore, according to Theorem 5 in [3], we have the conclusion that the probability of reaching the unsafe set $\widehat{\mathcal{X}} \setminus \mathcal{X}$ for system (5) starting from $\mathcal{X}_0$ is less than or equal to $1 - \epsilon_1$, i.e., $\mathbb{P}^{\infty}\big(\exists k \in \mathbb{N}.\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}_0}(k) \in \widehat{\mathcal{X}} \setminus \mathcal{X} \mid \boldsymbol{x}_0 \in \mathcal{X}_0\big) \leq 1 - \epsilon_1$. Therefore,

$$\mathbb{P}^{\infty}\big(\forall k \in \mathbb{N}.\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}_0}(k) \in \mathcal{X} \mid \boldsymbol{x}_0 \in \mathcal{X}_0\big) \geq \epsilon_1.$$

Since if $\boldsymbol{x}_0 \in \mathcal{X}$, $\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}_0}(1) = \boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(1)$ holds. Consequently, $\mathbb{P}^{\infty}\big(\forall k \in \mathbb{N}.\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(k) \in \mathcal{X} \mid \boldsymbol{x}_0 \in \mathcal{X}_0\big) \geq \epsilon_1$. ■

According to Proposition 3, a lower bound of the liveness probability can be computed via solving the following optimization:

$$\textbf{Op1} \quad \max_{v(\boldsymbol{x}), \epsilon_1} \epsilon_1$$

$$\text{s.t.} \begin{cases} v(\boldsymbol{x}) \leq 1 - \epsilon_1, \forall \boldsymbol{x} \in \mathcal{X}_0, \\ v(\boldsymbol{x}) \geq \mathbb{E}^{\infty}[v(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}}(1))], \forall \boldsymbol{x} \in \mathcal{X}, \\ v(\boldsymbol{x}) \geq 1, \forall \boldsymbol{x} \in \widehat{\mathcal{X}} \setminus \mathcal{X}, \\ v(\boldsymbol{x}) \geq 0, \forall \boldsymbol{x} \in \widehat{\mathcal{X}}, \\ \epsilon_1 \geq 0. \end{cases}$$

Example 2: Consider Example 1 once again. By solving **Op1** using $\widehat{\mathcal{X}} = \{(x, y)^{\top} \mid \widehat{h}(\boldsymbol{x}) \leq 0\}$ with $\widehat{h}(\boldsymbol{x}) = x^2 + y^2 - 2$, we encode it into a semi-definite program **SDP1**(shown in Appendix) via the sum of squares decomposition for multivariate polynomials. The solution yields a lower bound for the liveness probability, which is 0.9465. **SDP1** is addressed when unknown polynomials of degree 8 are used. ■

Proposition 4: Assume that $\mathcal{X}_{unsafe} = \widehat{\mathcal{X}} \setminus \mathcal{X}$ and $\mathcal{X}$ is a closed set[1]. If there exists a function $v(\boldsymbol{x}) : \widehat{\mathcal{X}} \to \mathbb{R}$ satisfying

$$\begin{cases} v(\boldsymbol{x}) \leq \epsilon_2, \forall \boldsymbol{x} \in \mathcal{X}_0, \\ v(\boldsymbol{x}) \geq 1, \forall \boldsymbol{x} \in \partial\widehat{\mathcal{X}} \setminus \partial\mathcal{X}_{unsafe}, \\ \mathbb{E}^{\infty}[v(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}}(1))] - v(\boldsymbol{x}) \leq -\delta, \forall \boldsymbol{x} \in \mathcal{X}, \\ v(\boldsymbol{x}) \geq 0, \forall \boldsymbol{x} \in \widehat{\mathcal{X}}, \end{cases} \tag{10}$$

where $\delta > 0$ is a user-defined value, then $\mathbb{P}^{\infty}\big(\forall k \in \mathbb{N}.\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(k) \in \mathcal{X} \mid \boldsymbol{x}_0 \in \mathcal{X}_0\big) \leq \epsilon_2$.

Proof: Constraint (10) is equivalent to the following constraint

$$\begin{cases} v(\boldsymbol{x}) \leq \epsilon_2, \forall \boldsymbol{x} \in \mathcal{X}_0, \\ v(\boldsymbol{x}) \geq 1, \forall \boldsymbol{x} \in \partial\widehat{\mathcal{X}} \setminus \partial\mathcal{X}_{unsafe}, \\ \mathbb{E}^{\infty}[v(\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}}(1))] - v(\boldsymbol{x}) \leq -\delta, \forall \boldsymbol{x} \in \overline{\widehat{\mathcal{X}} \setminus \mathcal{X}_{unsafe}}, \\ v(\boldsymbol{x}) \geq 0, \forall \boldsymbol{x} \in \widehat{\mathcal{X}}, \end{cases} \tag{11}$$

According to Theorem 16 in [3] and following the proof of Proposition 3, we have the conclusion $\mathbb{P}^{\infty}\big(\forall k \in \mathbb{N}.\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(k) \in \mathcal{X} \mid \boldsymbol{x}_0 \in \mathcal{X}_0\big) \leq \epsilon_2$ will be obtained. ■

Remark 2: If $v(\boldsymbol{x})$ is bounded over $\widehat{\mathcal{X}}$ in (10), constraint (10) provides strong guarantees of leaving the safe set $\mathcal{X}$ almost surely, i.e., $\mathbb{P}^{\infty}\big(\forall k \in \mathbb{N}.\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(k) \in \mathcal{X} \mid \boldsymbol{x}_0 \in \mathcal{X}_0\big) = 0$. This conclusion is justified as follows.

From $\mathbb{E}^{\infty}[v(\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}}(1))] - v(\boldsymbol{x}) \leq -\delta, \forall \boldsymbol{x} \in \overline{\widehat{\mathcal{X}} \setminus \mathcal{X}_{unsafe}}$, where $\mathcal{X}_{unsafe} = \widehat{\mathcal{X}} \setminus \mathcal{X}$, we have

$$\mathbb{E}^{\infty}[v(\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}}(1))] - v(\boldsymbol{x}) - \delta 1_{\mathcal{X}_{unsafe}}(\boldsymbol{x}) \leq -\delta, \forall \boldsymbol{x} \in \widehat{\mathcal{X}}.$$

Thus, for $\boldsymbol{x} \in \widehat{\mathcal{X}}$, we have that

$$\mathbb{E}^{\infty}[v(\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}}(k))] - v(\boldsymbol{x}) - \delta \sum_{i=0}^{k-1} \mathbb{E}^{\infty}[1_{\mathcal{X}_{unsafe}}(\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}}(i))] \leq -k\delta,$$

which implies $\mathbb{P}^{\infty}\big(\exists k \in \mathbb{N}.\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}}(k) \in \widehat{\mathcal{X}} \setminus \mathcal{X} \mid \boldsymbol{x} \in \mathcal{X}\big) = \lim_{k \to \infty} \frac{\sum_{i=0}^{k-1} \mathbb{E}^{\infty}[1_{\mathcal{X}_{unsafe}}(\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}}(i))]}{k} \geq 1$ (according to Lemma 3 in [21]) Thus, we have the conclusion.

It is worth noting that if $\partial\widehat{\mathcal{X}} \cap \partial\mathcal{X} = \emptyset$, the set $\partial\widehat{\mathcal{X}} \setminus \partial\mathcal{X}_{unsafe}$ in (10) is empty. As a result, the constraint $v(\boldsymbol{x}) \geq 1, \forall \boldsymbol{x} \in \partial\widehat{\mathcal{X}} \setminus \partial\mathcal{X}_{unsafe}$ becomes redundant and can be removed. Throughout this paper, unless explicitly stated otherwise, we assume that $\partial\widehat{\mathcal{X}} \cap \partial\mathcal{X} = \emptyset$. It is worth mentioning that this assumption is not overly strict and can be easily satisfied by enlarging the set that satisfies (6). The primary role of this assumption is to facilitate solving constraint (10). Accordingly, based on Proposition 4, we can calculate an upper bound for the liveness probability by solving the following optimization problem:

$$\textbf{Op2} \qquad \min_{v(\boldsymbol{x}), \epsilon_2} \ \epsilon_2$$

$$\text{s.t.} \begin{cases} v(\boldsymbol{x}) \leq \epsilon_2, \forall \boldsymbol{x} \in \mathcal{X}_0, \\ \mathbb{E}^{\infty}[v(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}}(1))] - v(\boldsymbol{x}) \leq -\delta, \forall \boldsymbol{x} \in \mathcal{X}, \\ v(\boldsymbol{x}) \geq 0, \forall \boldsymbol{x} \in \widehat{\mathcal{X}}, \\ \epsilon_2 \geq 0. \end{cases}$$

Remark 3: Another condition, which is analogous to the one in Proposition 4, was proposed in [5].

Proposition 5: If there exist a function $v(\boldsymbol{x}) : \mathcal{X} \to \mathbb{R}_{\geq 0}$ and constant $c > 0$ such that

$$\begin{cases} v(\boldsymbol{x}) \geq c, \forall \boldsymbol{x} \in \mathcal{X}, \\ v(\boldsymbol{x}) < c, \forall \boldsymbol{x} \in \mathcal{X}_{unsafe}(= \widehat{\mathcal{X}} \setminus \mathcal{X}), \\ \mathbb{E}^{\infty}[v(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}}(1))] - v(\boldsymbol{x}) \leq -1, \forall \boldsymbol{x} \in \mathcal{X}, \end{cases} \tag{12}$$

then $\mathbb{P}^{\infty}\big(\forall k \in \mathbb{N}.\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}}(k) \in \mathcal{X} \mid \boldsymbol{x} \in \mathcal{X}_0\big) = 0$.

Proof: The proof is similar to the one of Proposition 3, and the conclusion is justified from Theorem 19 in [3]. ■

Remark 4: It is worth remarking here that we do not extend the k-Inductive Barrier certificates proposed in [3] for addressing the problem in this paper. A set of sufficient conditions, which is similar to the one in Proposition 2 can be easily obtained based on k-Inductive barrier certificates. However, the gain of sufficient conditions being analogous to the ones in Proposition 3 and 4 should be carefully treated and will be considered in the future work.

B. Equations Relaxation Based Invariance Verification

In this subsection, we present the second set of optimizations for addressing the set invariance verification problem in Definition 2. We begin by introducing an equation that characterizes the exit probability of leaving the safe set $\mathcal{X}$. This equation is adapted from (3), where the unsafe set $\widehat{\mathcal{X}} \setminus \mathcal{X}$ is regarded as the target set $\mathcal{X}_r$ in (3). We then propose two sufficient conditions for certifying lower and upper bounds of the liveness probability by relaxing the derived equation.

Theorem 2: Given a safe set $\mathcal{X}$, if there exist a function $v(\boldsymbol{x}) : \widehat{\mathcal{X}} \to \mathbb{R}$ [2] and a bounded function $w(\boldsymbol{x}) : \widehat{\mathcal{X}} \to \mathbb{R}$ such that for $\boldsymbol{x} \in \widehat{\mathcal{X}}$,

$$\begin{cases} v(\boldsymbol{x}) = \mathbb{E}^{\infty}[v(\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}}(1))], \\ v(\boldsymbol{x}) = 1_{\widehat{\mathcal{X}} \setminus \mathcal{X}}(\boldsymbol{x}) + \mathbb{E}^{\infty}[w(\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}}(1))] - w(\boldsymbol{x}), \end{cases} \tag{13}$$

then

$$\begin{aligned} v(\boldsymbol{x}_0) &= \mathbb{P}^{\infty}\big(\exists k \in \mathbb{N}.\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(k) \in \widehat{\mathcal{X}} \setminus \mathcal{X} \mid \boldsymbol{x}_0 \in \mathcal{X}\big) \\ &= \mathbb{P}^{\infty}\big(\exists k \in \mathbb{N}.\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}_0}(k) \in \widehat{\mathcal{X}} \setminus \mathcal{X} \mid \boldsymbol{x}_0 \in \mathcal{X}\big) \\ &= \lim_{i \to \infty} \frac{\mathbb{E}^{\infty}[\sum_{j=0}^{i-1} 1_{\widehat{\mathcal{X}} \setminus \mathcal{X}}(\widehat{\boldsymbol{\phi}}_{\pi}^{\boldsymbol{x}_0}(j))]}{i}. \end{aligned}$$

Thereby, $\mathbb{P}^{\infty}\big(\forall k \in \mathbb{N}.\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(k) \in \mathcal{X} \mid \boldsymbol{x}_0 \in \mathcal{X}\big) = 1 - v(\boldsymbol{x}_0)$.

Proof: The conclusion can be assured by following the proof of Theorem 1 in [21]. ■

Like Proposition 1, two sufficient conditions can be obtained for certifying lower and upper bounds of the liveness probability via directly relaxing equation (13), respectively.

Proposition 6: Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$ with $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist a function $v(\boldsymbol{x}) : \widehat{\mathcal{X}} \to \mathbb{R}$ and a

[1] The requirement that $\mathcal{X}$ is closed is reflected in (11) in the proof.

[2] Comparing with Theorem 1, the explicit requirement that $v(\boldsymbol{x})$ is bounded is abandoned here. If following the proof of Theorem 1 in [21], one can find that this requirement is not necessary.

bounded function $w(\boldsymbol{x}): \widehat{\mathcal{X}} \rightarrow \mathbb{R}$ satisfying

$$\begin{cases} v(\boldsymbol{x}) \leq 1-\epsilon_1, \forall \boldsymbol{x} \in \mathcal{X}_0, \\ v(\boldsymbol{x}) \geq \mathbb{E}^{\infty}[v(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}}(1))], \forall \boldsymbol{x} \in \mathcal{X}, \\ v(\boldsymbol{x}) \geq \mathbb{E}^{\infty}[w(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}}(1))] - w(\boldsymbol{x}), \forall \boldsymbol{x} \in \mathcal{X}, \\ v(\boldsymbol{x}) \geq 1, \forall \boldsymbol{x} \in \widehat{\mathcal{X}} \setminus \mathcal{X}, \end{cases} \tag{14}$$

then

$$\mathbb{P}^{\infty}\left(\exists k \in \mathbb{N}. \boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(k) \in \widehat{\mathcal{X}} \setminus \mathcal{X} \mid \boldsymbol{x}_0 \in \mathcal{X}\right) \leq v(\boldsymbol{x}_0) \leq 1-\epsilon_1.$$

Consequently, $\mathbb{P}^{\infty}\left(\forall k \in \mathbb{N}. \boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(k) \in \mathcal{X} \mid \boldsymbol{x}_0 \in \mathcal{X}_0\right) \geq \epsilon_1$.

Proof: The conclusion can be assured by following the proof of Corollary 1 in [21], with the inequality signs reversed. ■

According to Proposition 6, a lower bound of the liveness probability can be computed via solving the following optimization:

$$\textbf{Op3} \quad \max_{v(\boldsymbol{x}), w(\boldsymbol{x}), \epsilon_1} \epsilon_1$$
$$\text{s.t.} \begin{cases} v(\boldsymbol{x}) \leq 1-\epsilon_1, \forall \boldsymbol{x} \in \mathcal{X}_0, \\ v(\boldsymbol{x}) \geq \mathbb{E}^{\infty}[v(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}}(1))], \forall \boldsymbol{x} \in \mathcal{X}, \\ v(\boldsymbol{x}) \geq \mathbb{E}^{\infty}[w(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}}(1))] - w(\boldsymbol{x}), \forall \boldsymbol{x} \in \mathcal{X}, \\ v(\boldsymbol{x}) \geq 1, \forall \boldsymbol{x} \in \widehat{\mathcal{X}} \setminus \mathcal{X}, \\ \epsilon_1 \geq 0, \\ w(\boldsymbol{x}) \text{ is bounded over } \widehat{\mathcal{X}}. \end{cases}$$

The comparison between (9) and (14) implies that (14) is weaker than (9), since if $v(\boldsymbol{x})$ satisfying (9) also satisfies (14) with $w(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \widehat{\mathcal{X}}$.

Proposition 7: Given a safe set $\mathcal{X}$ and an initial set $\mathcal{X}_0$ with $\mathcal{X}_0 \subseteq \mathcal{X}$, if there exist a function $v(\boldsymbol{x}): \widehat{\mathcal{X}} \rightarrow \mathbb{R}$ and a bounded function $w(\boldsymbol{x}): \widehat{\mathcal{X}} \rightarrow \mathbb{R}$ satisfying

$$\begin{cases} v(\boldsymbol{x}) \geq 1-\epsilon_2, \forall \boldsymbol{x} \in \mathcal{X}_0, \\ v(\boldsymbol{x}) \leq \mathbb{E}^{\infty}[v(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}}(1))], \forall \boldsymbol{x} \in \mathcal{X}, \\ v(\boldsymbol{x}) \leq \mathbb{E}^{\infty}[w(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}}(1))] - w(\boldsymbol{x}), \forall \boldsymbol{x} \in \mathcal{X}, \\ v(\boldsymbol{x}) \leq 1, \forall \boldsymbol{x} \in \widehat{\mathcal{X}} \setminus \mathcal{X}, \end{cases} \tag{15}$$

then

$$\mathbb{P}^{\infty}\left(\exists k \in \mathbb{N}. \boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(k) \in \widehat{\mathcal{X}} \setminus \mathcal{X} \mid \boldsymbol{x}_0 \in \mathcal{X}_0\right) \geq v(\boldsymbol{x}_0) \geq 1-\epsilon_2.$$

Consequently, $\mathbb{P}^{\infty}\left(\forall k \in \mathbb{N}. \boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(k) \in \mathcal{X} \mid \boldsymbol{x}_0 \in \mathcal{X}_0\right) \leq \epsilon_2$.

Proof: The conclusion can be assured by following the proof of Corollary 1 in [21]. ■

By comparing constraints (10) and (15), we can conclude when $\mathcal{X}$ is closed and $v(\boldsymbol{x})$ is bounded over $\widehat{\mathcal{X}}$ that (15) is weaker than (10). This is because if $v(\boldsymbol{x})$ satisfies (10), $1-v(\boldsymbol{x})$ satisfies (15) with $w(\boldsymbol{x}) = M(1-v(\boldsymbol{x}))$ for $\boldsymbol{x} \in \widehat{\mathcal{X}}$, where $M\delta \geq \sup_{\boldsymbol{x} \in \widehat{\mathcal{X}}}(1-v(x))$.

According to Proposition 7, an upper bound of the liveness probability can be computed via solving the following optimization:

$$\textbf{Op4} \quad \min_{v(\boldsymbol{x}), w(\boldsymbol{x}), \epsilon_2} \epsilon_2$$
$$\text{s.t.} \begin{cases} v(\boldsymbol{x}) \geq 1-\epsilon_2, \forall \boldsymbol{x} \in \mathcal{X}_0, \\ v(\boldsymbol{x}) \leq \mathbb{E}^{\infty}[v(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}}(1))], \forall \boldsymbol{x} \in \mathcal{X}, \\ v(\boldsymbol{x}) \leq \mathbb{E}^{\infty}[w(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}}(1))] - w(\boldsymbol{x}), \forall \boldsymbol{x} \in \mathcal{X}, \\ v(\boldsymbol{x}) \leq 1, \forall \boldsymbol{x} \in \widehat{\mathcal{X}} \setminus \mathcal{X}, \\ \epsilon_2 \geq 0, \\ w(\boldsymbol{x}) \text{ is bounded over } \widehat{\mathcal{X}}. \end{cases}$$

V. Examples

In this section we demonstrate our theoretical developments on two examples. Specifically, we encode the problems **Op0-Op4** as semi-definite programs using the sum of squares decomposition for multivariate polynomials, and solve the resulting semi-definite programs (**SDP0-SDP4**, see Appendix) using the Mosek 10.0 tool [4]. To ensure numerical stability during the solution of the semi-definite programs, we impose a constraint on the coefficients of the unknown polynomials $(v(\boldsymbol{x}), w(\boldsymbol{x}), s_i(\boldsymbol{x}), i = 0, \dots, 4)$. Specifically, we restrict these coefficients to the interval $[-100, 100]$.

Example 3: In this example we consider the one-dimensional discrete-time system:

$$x(l+1) = (-0.5 + d(l))x(l),$$

where $d(\cdot): \mathbb{N} \rightarrow \mathcal{D} = [-1, 1]$, $\mathcal{X} = \{x \mid h(x) \leq 0\}$ with $h(x) = x^2 - 1$, and $\mathcal{X}_0 = \{x \mid (x+0.8)^2 = 0\}$. Besides, we assume that the probability distribution on $\mathcal{D}$ is the uniform distribution.

The set $\widehat{\mathcal{X}} = \{x \mid \widehat{h}(x) \leq 0\}$ with $\widehat{h}(x) = x^2 - 2$ is used in solving **SDP1-SDP4**. The computed lower and upper bounds are summarized in Table I and Fig. 1. It is concluded that tighter lower and upper bounds of the liveness probability can be obtained when polynomials of higher degree are used for performing computations. According to Remark 2, **SDP2** cannot be used to compute upper bounds of the liveness probability, which is consistent with our experimental results. However, we can obtain upper bounds of the liveness probability via solving **SDP4**. Meanwhile, it is also observed that the lower bounds computed from **SDP1** and **SDP3** are identical, and the ones generated by solving **SDP0** are the most conservative.

Example 4: In this example we consider the discrete-time Lotka-Volterra model:

$$\begin{cases} x(l+1) = rx(l) - ay(l)x(l), \\ y(l+1) = sy(l) + acy(l)x(l), \end{cases} \tag{16}$$

where $r = 0.5$, $a = 1$, $s = -0.5 + d(l)$ with $d(\cdot): \mathbb{N} \rightarrow \mathcal{D} = [-1, 1]$ and $c = 1$, $\mathcal{X} = \{(x, y)^{\top} \mid h(\boldsymbol{x}) \leq 0\}$ $h(\boldsymbol{x}) = x^2 + y^2 - 4$, and $\mathcal{X}_0 = \{(x, y)^{\top} \mid g(\boldsymbol{x}) \leq 0\}$ with $g(\boldsymbol{x}) = (x+0.6)^2 + (y+0.5)^2 - 0.1$. Besides, we assume that the probability distribution imposed on $\mathcal{D}$ is the uniform distribution. Three trajectories starting from $(-0.6, -0.5)^{\top}$ are visualized in Fig. 2.

Op3 and Op4													
d	2	4	6	8	10	12	14	16	18	20	22	24	26
ϵ_1	0.3600	0.5922	0.6704	0.6914	0.6937	0.7289	0.7372	0.7552	0.7578	0.7600	0.7625	0.7633	0.7651
ϵ_2	1.0000	0.9843	0.9504	0.9489	0.9487	0.9473	0.9240	0.9141	0.8988	0.8988	0.8924	0.8801	0.8763
Op1													
ϵ_1	0.3600	0.5922	0.6704	0.6914	0.6937	0.7289	0.7372	0.7552	0.7578	0.7600	0.7625	0.7633	0.7651
Op0													
ϵ_1	0.3600	0.5922	0.5922	0.5927	0.5942	0.6379	0.6777	0.6883	0.6885	0.6907	0.7090	0.7258	0.7283

TABLE I
COMPUTED LOWER AND UPPER BOUNDS OF THE LIVENESS PROBABILITY
(d DENOTES THE DEGREE OF UNKNOWN POLYNOMIALS INVOLVED IN THE RESULTING SDPS)

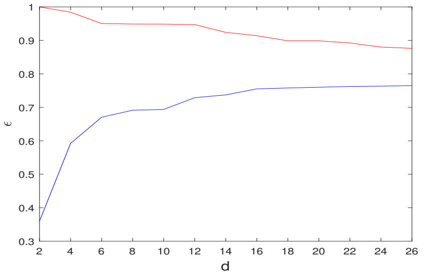

Fig. 1. A visualized illustration of computed lower and upper bounds (shown in Table I) of the liveness probability via solving **SDP3** and **SDP4**

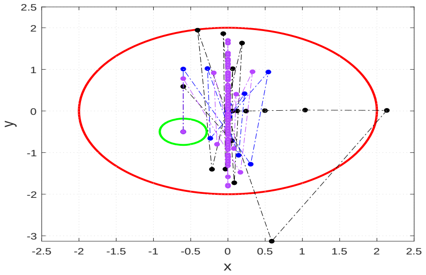

Fig. 2. An illustration of four trajectories for system (16) starting from $(-0.6, -0.5)^\top$ (red curve - $\partial\mathcal{X}$; green curve - $\partial\mathcal{X}_0$).

The set $\widehat{\mathcal{X}} = \{(x,y)^\top \mid \widehat{h}(\boldsymbol{x}) \leq 0\}$ with $\widehat{h}(\boldsymbol{x}) = x^2 + y^2 - 30$ is used in solving **SDP1-SDP4**. The computed lower and upper bounds are summarized in Table II. **SDP3** is able to compute tighter lower bounds than **SDP1** and **SDP0**, consistent with our theoretical conclusion that constraint (14) is more expressive than constraints (9) and (7). Note that **SDP2** cannot be used to compute upper bounds of the liveness probability, as mentioned in Remark 2. Instead, one can obtain upper bounds by solving **SDP4**.

Op3 and Op4		
d	8	10
ϵ_1	0.2772	0.4852
ϵ_2	1	0.9999
Op1		
ϵ_1	0.1668	0.4613
Op0		
ϵ_1	9.5417e-05	0.2603

TABLE II
COMPUTED LOWER AND UPPER BOUNDS OF THE LIVENESS PROBABILITY (d DENOTE THE DEGREE OF UNKNOWN POLYNOMIALS INVOLVED IN THE RESULTING SDPS)

Op3		
d	8	10
ϵ_1	0.1085	0.5669
Op1		
ϵ_1	0.1431	0.4837

TABLE III
COMPUTED LOWER AND UPPER BOUNDS OF THE LIVENESS PROBABILITY WHEN THE SET $\widehat{\mathcal{X}} = \{(x,y)^\top \mid x^2 + y^2 \leq 36\}$ IS USED (d DENOTE THE DEGREE OF UNKNOWN POLYNOMIALS INVOLVED IN THE RESULTING SDPS)

Using higher-degree polynomials (i.e., $v(\boldsymbol{x}), w(\boldsymbol{x})$) or a larger set $\widehat{\mathcal{X}}$ can lead to tighter bounds on the liveness probability when solving problems **Op0**, **Op1**, **Op3**, and **Op4**. However, directly solving these problems is challenging, if not impossible. Therefore, we relax them into semi-definite programs (**SDP0**, **SDP1**, **SDP3**, and **SDP4**) using the sum of squares decomposition for multivariate polynomials, which allows for efficient solving. Nonetheless, this relaxation may yield unexpected outcomes. For instance, when we use a larger set $\widehat{\mathcal{X}} = \{(x,y)^\top \mid x^2 + y^2 \leq 36\}$ to solve **SDP1** and **SDP3** with unknown polynomials of degree 10 in Example 4, we obtain larger lower bounds; when unknown polynomials of degree 8 are taken for solving **SDP1** and **SDP3**, the lower bound computed from **SDP1** is larger than the one from **SDP3**. They are summarized in Table III. These results may be affected by floating point errors. Therefore, besides enhancing the accuracy of SDP solving algorithms, it is imperative to develop advanced algorithms that can solve **Op0-Op4** in a nonlinear form that goes beyond polynomial. Moreover, while our theoretical analysis demonstrates that the constraint in **Op3** is weaker than that in **Op1**, we cannot guarantee that **SDP3** is weaker than **SDP1**. Hence, it is uncertain whether the lower bound obtained by solving **SDP3** is tighter than that obtained by solving **SDP1**.

VI. CONCLUSION

In this paper, we propose two sets of optimizations for computing lower and upper bounds of the liveness probability. These optimizations ensure that the system remains inside a specified safe set for all time, starting from each state in a specified initial set. The first optimization is based on classical barrier certificates used for safety and reachability verification. The second uses an equation that characterizes the exact reachability probability. Our theoretical analysis shows that the second optimization is more expressive and can provide tighter lower and upper bounds. To demonstrate the proposed optimizations, we apply semi-definite programming to two examples.

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Appendix

$$\textbf{SDP0} \quad \max_{\epsilon_1, v(\boldsymbol{x}), s_i(\boldsymbol{x}), i=0,\dots,1} -\epsilon_1$$

$$\text{s.t.} \begin{cases} 1-\epsilon_1 - v(\boldsymbol{x}) + s_0(\boldsymbol{x}) g(\boldsymbol{x}) \in \sum[\boldsymbol{x}], \\ v(\boldsymbol{x}) - \mathbb{E}^{\infty}[v(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}}(1))] \in \sum[\boldsymbol{x}], \\ v(\boldsymbol{x}) - 1 - s_1(\boldsymbol{x}) h(\boldsymbol{x}) \in \sum[\boldsymbol{x}], \\ v(\boldsymbol{x}) \in \sum[\boldsymbol{x}], \\ \epsilon_1 \geq 0, \\ s_0(\boldsymbol{x}) \in \sum[\boldsymbol{x}], s_1(\boldsymbol{x}) \in \sum[\boldsymbol{x}]. \end{cases}$$

$$\textbf{SDP1} \quad \max_{\epsilon_1, v(\boldsymbol{x}), s_i(\boldsymbol{x}), i=0,\dots,4} -\epsilon_1$$

$$\text{s.t.} \begin{cases} 1-\epsilon_1 - v(\boldsymbol{x}) + s_0(\boldsymbol{x}) g(\boldsymbol{x}) \in \sum[\boldsymbol{x}], \\ v(\boldsymbol{x}) - \mathbb{E}^{\infty}[v(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}}(1))] + s_1(\boldsymbol{x}) h(\boldsymbol{x}) \in \sum[\boldsymbol{x}], \\ v(\boldsymbol{x}) - 1 + s_2(\boldsymbol{x}) \widehat{h}(\boldsymbol{x}) - s_3(\boldsymbol{x}) h(\boldsymbol{x}) \in \sum[\boldsymbol{x}], \\ v(\boldsymbol{x}) + s_4(\boldsymbol{x}) \widehat{h}(\boldsymbol{x}) \in \sum[\boldsymbol{x}], \\ \epsilon_1 \geq 0, \\ s_i(\boldsymbol{x}) \in \sum[\boldsymbol{x}], i = 0, \dots, 4. \end{cases}$$

$$\textbf{SDP2} \quad \min_{v(\boldsymbol{x}), \epsilon_2, s_i(\boldsymbol{x}), i=0,\dots,2} \epsilon_2$$

$$\text{s.t.} \begin{cases} \epsilon_2 - v(\boldsymbol{x}) + s_0(\boldsymbol{x}) g(\boldsymbol{x}) \in \sum[\boldsymbol{x}], \\ -\delta - \mathbb{E}^{\infty}[v(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}}(1))] + v(\boldsymbol{x}) + s_1(\boldsymbol{x}) h(\boldsymbol{x}) \in \sum[\boldsymbol{x}], \\ v(\boldsymbol{x}) + s_2(\boldsymbol{x}) \widehat{h}(\boldsymbol{x}) \in \sum[\boldsymbol{x}], \\ \epsilon_2 \geq 0, \\ s_i(\boldsymbol{x}) \in \sum[\boldsymbol{x}], i = 0, \dots, 2, \end{cases}$$

where $\delta = 10^{-6}$.

$$\textbf{SDP3} \quad \max_{\epsilon_1, v(\boldsymbol{x}), w(\boldsymbol{x}), s_i(\boldsymbol{x}), i=0,\dots,4} -\epsilon_1$$

$$\begin{cases} 1-\epsilon_1 - v(\boldsymbol{x}) + s_0(\boldsymbol{x}) g(\boldsymbol{x}) \in \sum[\boldsymbol{x}], \\ v(\boldsymbol{x}) - \mathbb{E}^{\infty}[v(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}}(1))] + s_1(\boldsymbol{x}) h(\boldsymbol{x}) \in \sum[\boldsymbol{x}], \\ v(\boldsymbol{x}) - \mathbb{E}^{\infty}[w(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}}(1))] + w(\boldsymbol{x}) + s_2(\boldsymbol{x}) h(\boldsymbol{x}) \in \sum[\boldsymbol{x}], \\ v(\boldsymbol{x}) - 1 + s_3(\boldsymbol{x}) \widehat{h}(\boldsymbol{x}) - s_4(\boldsymbol{x}) h(\boldsymbol{x}) \in \sum[\boldsymbol{x}], \\ \epsilon_1 \geq 0, \\ s_i(\boldsymbol{x}) \in \sum[\boldsymbol{x}], i = 0, \dots, 4. \end{cases}$$

$$\textbf{SDP4} \quad \min_{\epsilon_2, v(\boldsymbol{x}), w(\boldsymbol{x}), s_i(\boldsymbol{x}), i=0,\dots,4} \epsilon_2$$

$$\begin{cases} v(\boldsymbol{x}) - 1 + \epsilon_2 + s_0(\boldsymbol{x}) g(\boldsymbol{x}) \in \sum[\boldsymbol{x}], \\ \mathbb{E}^{\infty}[v(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}}(1))] - v(\boldsymbol{x}) + s_1(\boldsymbol{x}) h(\boldsymbol{x}) \in \sum[\boldsymbol{x}], \\ -v(\boldsymbol{x}) + \mathbb{E}^{\infty}[w(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}}(1))] - w(\boldsymbol{x}) + s_2(\boldsymbol{x}) h(\boldsymbol{x}) \in \sum[\boldsymbol{x}], \\ 1 - v(\boldsymbol{x}) + s_3(\boldsymbol{x}) \widehat{h}(\boldsymbol{x}) - s_4(\boldsymbol{x}) h(\boldsymbol{x}) \in \sum[\boldsymbol{x}], \\ \epsilon_2 \geq 0, \\ s_i(\boldsymbol{x}) \in \sum[\boldsymbol{x}], i = 0, \dots, 4. \end{cases}$$