Sufficient and Necessary Barrier-like Conditions for Safety and Reach-avoid Verification of Stochastic Discrete-time Systems

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***Abstract*—In this paper, we examine sufficient and necessary barrier-like conditions for the safety verification and reach-avoid verification of stochastic discrete-time systems. Safety verification aims to certify the satisfaction of the safety property, which stipulates that the probability of the system, starting from a specified initial state, remaining within a safe set is greater than or equal to a specified lower bound. A sufficient and necessary barrier-like condition is formulated for safety verification. In contrast, reach-avoid verification extends beyond safety to include reachability, seeking to certify the satisfaction of the reach-avoid property. It requires that the probability of the system, starting from a specified initial state, reaching a target set eventually while remaining within a safe set until the first hit of the target, is greater than or equal to a specified lower bound. Two sufficient and necessary barrier-like conditions are formulated under certain assumptions. These conditions are derived via relaxing Bellman equations.**

I. Introduction

Temporal verification is crucial in modern systems analysis, particularly in complex systems where temporal behavior is of paramount importance [14]. It involves rigorously examining a system's adherence to temporal properties, including safety and reach-avoid guarantees, to ensure desired outcomes and avoid undesirable events. Formal methods like model checking [5] and theorem proving [10] are indispensable tools in this process, allowing for precise and comprehensive analysis of temporal specifications.

Over the past two decades, barrier certificates have emerged as a powerful tool for safety and reach-avoid verification of dynamical systems. They provide a Lyapunov-like assurance regarding system behavior, with the mere existence of a barrier certificate being sufficient to establish the satisfiability of safety and reach-avoid specifications, as demonstrated in [14]. This approach simplifies the verification process and offers a formal mathematical framework for ensuring safety and correctness of a system without the need to explicitly evolve it over time. Especially, with the advancement in polynomial optimization, particularly sum-of-squares polynomial optimization, the problem of finding barrier certificates can be addressed through convex optimization, especially when the system of interest is polynomial, further motivating the development of barrier certificate-based methods. Barrier certificates were initially proposed for deterministic systems as a popular formal approach to safety verification in [11]. Subsequent efforts have focused on adapting and enhancing barrier functions, as well as broadening their applications [2], [3], [6], [7]. However, many real-world applications are susceptible to stochastic disturbances and are thus modeled as stochastic systems. In the continuous-time stochastic setting, safety verification over the infinite time horizon via barrier certificates was introduced alongside its deterministic counterpart in [12]. Based on Ville's Inequality [19] and a stopped process, [12] developed a non-negative barrier function and established a sufficient condition for safety verification over the infinite time horizon, i.e., to certify upper bounds of probabilities of eventually entering an unsafe region from specific initial states, while ensuring that the system remains within the interior of a state-constrained set until the first encounter with the unsafe set. Subsequently, drawing inspiration from [8], [16] formulated a sufficient barrier-like condition for upper-bounding the probability of entering an unsafe region from certain initial states within finite time frames, while maintaining the system's presence within the interior of a state-constrained set until the initial contact with the unsafe set. The systems in [16] involve both continuous-time and discrete-time ones. Especially, when the state-constrained set is a robust invariant set, sufficient barrier-like conditions for safety verification of stochastic discrete-time systems were studied in [4], [27]. Another commonly studied safety property is related to set invariance. It is to justify lower bounds of liveness probabilities over either the infinite time horizon (i.e., the system stays within a specified safe set for all time) or finite time horizons (i.e., the system stays within a given safe set during a specified finite time period) [1]. In other words, it is justifying upper bounds of exit probabilities that the system exits a specified safe set either eventually or within a bounded time horizon. Correspondingly, sufficient barrier-like conditions were formulated in [18], [25] for such safety verification. It is emphasized that the present work focuses on verifying lower bounds of liveness probabilities over the infinite time horizon, although the proposed method can also be applied to the safety verification scenario over the infinite time horizon in [12]. Afterwards, control barrier functions for synthesizing controllers to guarantee safety were explored in [17], [20]. As to the reach-avoid verification, more recently, a new sufficient barrier-like condition was proposed in [23] for reach-avoid analysis of stochastic discrete-time dynamical systems over the infinite time horizon and later, extended to stochastic continuous-time dynamical systems [24]. This condition was constructed by relaxing a set of equations whose solution characterizes the exact reach-avoid probability of eventually entering a desired target set from an initial state while adhering to safety constraints. Another barrier-like function, termed reach-avoid supermartingales, was proposed to guarantee reach-avoid specifications as well as facilitate controllers synthesis for stochastic discrete-time systems in [28] under the assumption that the system is evolving within a robust invariant set. These barrier-like conditions aim to lower bound reach-avoid probabilities in [23], [24], [28]. On the other hand, converse theorems for barrier certificates, which focus on the existence of barrier certificates, have significantly contributed to elucidating how safety and reach-avoid criteria can be represented by barrier certificates. Consequently, these concepts have garnered growing interest since the inception of barrier certificates, and have been investigated in [9], [13]–[15], [21]. Nonetheless, there exists a paucity of research exploring the existence of barrier certificates for stochastic dynamical systems. This work aims to address this void.

In this paper, we explore the development of sufficient and necessary barrier-like conditions for safety and reach-avoid verification of stochastic discrete-time systems over the infinite time horizon. The safety verification process involves assessing whether the liveness probability that a system, starting from an initial state, will stay within a safe set for all time is greater than or equal to a specified threshold. By relaxing a Bellman equation, one of whose solutions characterizes the exact liveness probability, we construct a sufficient and necessary barrier-like condition for safety verification. On the other hand, the reach-avoid verification concerns verifying whether the reach-avoid probability that the system, starting from an initial state, will enter a target set eventually while avoiding unsafe sets before hitting the target, is greater than or equal to a specified threshold. We consider two cases for the reach-avoid verification. In the first case, we assume that the system will either enter the target set or leave the safe set in finite time almost surely. Under this context, by relaxing a Bellman equation, which possesses a unique bounded solution that characterizes the exact reach-avoid probability, we construct a sufficient and necessary barrier-like condition for the reach-avoid verification. In the second case, we assume that the specified threshold is strictly smaller than the exact reach-avoid probability. Under this context, by relaxing a Bellman equation featuring a unique bounded solution that provides a lower bound of the exact reach-avoid probability, we construct a sufficient and necessary barrier-like condition for the reach-avoid verification.

This paper is structured as follows: In Section II, we formulate the stochastic discrete-time systems of interest and the safety and reach-avoid verification problems. Section III presents sufficient and necessary barrier-like conditions for safety verification and their derivation. Afterward, Section IV presents sufficient and necessary barrier-like conditions for reach-avoid verification and their derivation. Finally, we conclude this paper in Section V.

II. Preliminaries

We start the exposition by a formal introduction of stochastic discrete-time systems and safety/reach-avoid verification problems of interest. Before posing the problem studied, let me introduce some basic notions used throughout this paper: $\mathbb{R}$ denotes the set of real values; $\mathbb{N}$ denotes the set of nonnegative integers; $\mathbb{N}_{\leq k}$ is the set of non-negative integers being less than or equal to k; $\mathbb{N}_{\geq k}$ is the set of non-negative integers being larger than or equal to k; for sets Δ_1 and Δ_2, $\Delta_1 \setminus \Delta_2$ denotes the difference of sets Δ_1 and Δ_2, which is the set of all elements in Δ_1 that are not in Δ_2; $1_A(\boldsymbol{x})$ denotes the indicator function in the set A, where, if $\boldsymbol{x} \in A$, then $1_A(\boldsymbol{x}) = 1$ and if $\boldsymbol{x} \notin A$, $1_A(\boldsymbol{x}) = 0$.

A. Problem Statement

This paper considers stochastic discrete-time systems that are modeled by stochastic difference equations of the following form:

$$\boldsymbol{x}(l+1) = \boldsymbol{f}(\boldsymbol{x}(l), \boldsymbol{\theta}(l)), \forall l \in \mathbb{N}, \tag{1}$$

where $\boldsymbol{x}(l) \in \mathbb{R}^n$ is the state at time l and $\boldsymbol{\theta}(l) \in \Theta$ with $\Theta \subseteq \mathbb{R}^m$ is the stochastic disturbance at time l. In addition, let $\boldsymbol{\theta}(0), \boldsymbol{\theta}(1), \dots$ be i.i.d. (independent and identically distributed) random variables on a probability space $(\Theta, \mathcal{F}, \mathbb{P}_{\boldsymbol{\theta}})$, and take values in Θ with the following probability distribution: for any measurable set $B \subseteq \Theta$,

$$\text{Prob}(\boldsymbol{\theta}(l) \in B) = \mathbb{P}_{\boldsymbol{\theta}}(B), \quad \forall l \in \mathbb{N}.$$

The corresponding expectation is denoted as $\mathbb{E}_{\boldsymbol{\theta}}[\cdot]$.

Before defining the trajectory of system (1), we define a disturbance signal.

Definition 1: A disturbance signal π is a sample path of the stochastic process $\{\boldsymbol{\theta}(i) : \Theta \to \Theta, i \in \mathbb{N}\}$, which is defined on the canonical sample space Θ^{∞}, endowed with its product topology $\mathcal{B}(\Theta^{\infty})$, with the probability measure $\mathbb{P}_{\pi} := \mathbb{P}_{\boldsymbol{\theta}}^{\infty}$. The expectation associated with the probability measure $\mathbb{P}_{\pi}$ is denoted by $\mathbb{E}_{\pi}[\cdot]$.

A disturbance signal π together with an initial state $\boldsymbol{x}_0 \in \mathbb{R}^n$ induces a unique discrete-time trajectory as follows.

Definition 2: Given a disturbance signal π and an initial state $\boldsymbol{x}_0 \in \mathbb{R}^n$, a trajectory of system (1) is denoted as $\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(\cdot) \colon \mathbb{N} \to \mathbb{R}^n$ with $\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(0) = \boldsymbol{x}_0$, i.e.,

$$\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(l+1) = \boldsymbol{f}(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(l), \boldsymbol{\theta}(l)), \forall l \in \mathbb{N}.$$

The safety and reach-avoid verification for the system governed by (1) over the infinite time horizon are defined below.

Definition 3 (Safety Verification): Given a safe set $\mathcal{X} \subseteq \mathbb{R}^n$, an initial state $\boldsymbol{x}_0 \in \mathcal{X} \setminus \mathcal{X}_r$, a target set $\mathcal{X}_r \subseteq \mathcal{X}$, and a lower bound $\epsilon_1 \in [0,1]$, the safety verification aims to certify that the liveness probability $\mathbb{P}_{\pi}(S_{\boldsymbol{x}_0})$, which denotes the probability that the system (1), starting from the initial state $\boldsymbol{x}_0$, will stay within the safe set $\mathcal{X}$ for all time, is greater than or equal to ϵ_1, i.e.,

$$\mathbb{P}_{\pi}(S_{\boldsymbol{x}_0}) \geq \epsilon_1.$$

where $S_{\boldsymbol{x}_0} = \{\pi \mid \forall i \in \mathbb{N}. \boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(i) \in \mathcal{X}\}$.

Definition 4 (Reach-avoid Verification): Given a safe set $\mathcal{X} \subseteq \mathbb{R}^n$, an initial state $\boldsymbol{x}_0 \in \mathcal{X} \setminus \mathcal{X}_r$, a target set $\mathcal{X}_r \subseteq \mathcal{X}$, and a lower bound $\epsilon_2 \in [0,1]$, the reach-avoid verification aims to certify that the reach-avoid probability $\mathbb{P}_{\pi}(RA_{\boldsymbol{x}_0})$, which denotes the probability that system (1), starting from the initial state $\boldsymbol{x}_0$, will reach the target set $\mathcal{X}_r$ eventually while staying within the safe set $\mathcal{X}$, is greater than or equal to ϵ_2, i.e.,

$$\mathbb{P}_{\pi}(RA_{\boldsymbol{x}_0}) \geq \epsilon_2.$$

where $RA_{\boldsymbol{x}_0} = \{\pi \mid \exists k \in \mathbb{N}. \boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(k) \in \mathcal{X}_r \wedge \forall i \in \mathbb{N}_{\leq k}. \boldsymbol{\phi}_{\pi}^{\boldsymbol{x}_0}(i) \in \mathcal{X}\}$.

In the sequel, we will formulate sufficient and necessary barrier-like conditions for certifying $\epsilon_1 \leq \mathbb{P}_{\pi}(S_{\boldsymbol{x}_0})$. It is worth noting here that the proposed method can also be used to construct sufficient and necessary conditions for the safety verification scenario in [12], which involves certifying upper bounds of the probability that the system eventually enters unsafe sets from an initial state while adhering to state-constrained sets. Please refer to Remark 2 in Subsection IV-A. In contrast, under certain assumptions, we will formulate sufficient and necessary barrier-like conditions for certifying $\mathbb{P}_{\pi}(RA_{\boldsymbol{x}_0}) \geq \epsilon_2$.

III. Safety Verification

This section will introduce sufficient and necessary barrier-like conditions for certifying lower bounds in safety verification and will detail their construction process. The construction involves constructing and relaxing a Bellman equation, one of whose solutions characterizes the exact liveness probability $\mathbb{P}_{\pi}(S_{\boldsymbol{x}})$ for $\boldsymbol{x} \in \mathbb{R}^n$. The Bellman equation is derived from a value function.

Let's start with the value function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$, which is able to characterize the exact liveness probability $\mathbb{P}_{\pi}(S_{\boldsymbol{x}})$ for $\boldsymbol{x} \in \mathbb{R}^n$,

$$V(\boldsymbol{x}) := \mathbb{E}_{\pi}\big[g(\boldsymbol{x})\big] \tag{2}$$

where

$$g(\boldsymbol{x}) = 1_{\mathbb{R}^n \setminus \mathcal{X}}(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}}(0)) + \sum_{i \in \mathbb{N}_{\geq 1}} \prod_{j=0}^{i-1} 1_{\mathcal{X}}(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}}(j)) 1_{\mathbb{R}^n \setminus \mathcal{X}}(\boldsymbol{\phi}_{\pi}^{\boldsymbol{x}}(i)).$$

Lemma 1: The value function $V(\boldsymbol{x})$ in (2) is equal to one minus the liveness probability $\mathbb{P}(S_{\boldsymbol{x}})$, i.e.,

$$V(\boldsymbol{x}) = 1 - \mathbb{P}_{\pi}(S_{\boldsymbol{x}})$$

for $\boldsymbol{x} \in \mathbb{R}^n$.

Proof: Clearly, $\mathbb{E}_\pi[1_{\mathbb{R}^n\setminus\mathcal{X}}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0))]=1_{\mathbb{R}^n\setminus\mathcal{X}}(\boldsymbol{x})$. In addition, since

$$\begin{aligned}&\mathbb{E}_\pi[\prod_{j=0}^{i-1}1_{\mathcal{X}}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(j))1_{\mathbb{R}^n\setminus\mathcal{X}}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(i))]\\&=\mathbb{P}_\pi(\wedge_{j=1}^{i-1}[\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(j)\in\mathcal{X}]\wedge[\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(i)\in\mathbb{R}^n\setminus\mathcal{X}])\end{aligned}$$

is the probability that the system (1) starting from $\boldsymbol{x}$ will exit the safe set $\mathcal{X}$ at time $t=i$ while stay within $\mathcal{X}$ before i, where $i\in\mathbb{N}_{\geq 1}$, we have $\mathbb{P}_\pi(S_{\boldsymbol{x}})=1-1_{\mathbb{R}^n\setminus\mathcal{X}}(\boldsymbol{x})-\sum_{i\in\mathbb{N}_{\geq 1}}\mathbb{P}_\pi(\wedge_{j=1}^{i-1}[\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(j)\in\mathcal{X}]\wedge[\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(i)\in\mathbb{R}^n\setminus\mathcal{X}])=1-V(\boldsymbol{x})$. ■

According to Lemma 1, $V(\boldsymbol{x})$ falls within [0,1] for $\boldsymbol{x}\in\mathbb{R}^n$ and thus it is bounded over $\mathbb{R}^n$. We next will show that the value function (2) can be reduced to a bounded solution to a Bellman equation (or, dynamic programming equation) via the dynamic programming principle. A value function characterizes the exact liveness probability over finite time horizons and its related dynamic programming equations can be found in [1].

Proposition 1: The value function $V(\cdot):\mathbb{R}^n\to\mathbb{R}$ in (2) satisfies the following Bellman equation

$$V(\boldsymbol{x})=1_{\mathbb{R}^n\setminus\mathcal{X}}(\boldsymbol{x})+1_{\mathcal{X}}(\boldsymbol{x})\mathbb{E}_{\boldsymbol{\theta}}[V(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}))],\forall\boldsymbol{x}\in\mathbb{R}^n. \tag{3}$$

Proof: Since $g(\boldsymbol{x})=1_{\mathbb{R}^n\setminus\mathcal{X}}(\boldsymbol{x})+1_{\mathcal{X}}(\boldsymbol{x})(1_{\mathbb{R}^n\setminus\mathcal{X}}(\boldsymbol{\phi}_\pi^{\boldsymbol{y}}(0))+\sum_{i\in\mathbb{N}_{\geq 1}}\prod_{j=0}^{i-1}1_{\mathcal{X}}(\boldsymbol{\phi}_\pi^{\boldsymbol{y}}(j))1_{\mathbb{R}^n\setminus\mathcal{X}}(\boldsymbol{\phi}_\pi^{\boldsymbol{y}}(i)))$, we have

$$\begin{aligned}V(\boldsymbol{x})=&1_{\mathbb{R}^n\setminus\mathcal{X}}(\boldsymbol{x})+1_{\mathcal{X}}(\boldsymbol{x})\mathbb{E}_\pi[\\&1_{\mathbb{R}^n\setminus\mathcal{X}}(\boldsymbol{y})+\sum_{i\in\mathbb{N}_{\geq 1}}\prod_{j=0}^{i-1}1_{\mathcal{X}}(\boldsymbol{\phi}_\pi^{\boldsymbol{y}}(j))1_{\mathbb{R}^n\setminus\mathcal{X}}(\boldsymbol{x}(i))\\&]\\=&1_{\mathbb{R}^n\setminus\mathcal{X}}(\boldsymbol{x})+1_{\mathcal{X}}(\boldsymbol{x})\mathbb{E}_{\boldsymbol{\theta}}[\\&1_{\mathbb{R}^n\setminus\mathcal{X}}(\boldsymbol{y})+\mathbb{E}_\pi[\sum_{i\in\mathbb{N}_{\geq 1}}\prod_{j=0}^{i-1}1_{\mathcal{X}}(\boldsymbol{\phi}_\pi^{\boldsymbol{y}}(j))1_{\mathbb{R}^n\setminus\mathcal{X}}(\boldsymbol{x}(i))]\\&]\\=&1_{\mathbb{R}^n\setminus\mathcal{X}}(\boldsymbol{x})+1_{\mathcal{X}}(\boldsymbol{x})\mathbb{E}_{\boldsymbol{\theta}}[V(\boldsymbol{y})]\\=&1_{\mathbb{R}^n\setminus\mathcal{X}}(\boldsymbol{x})+1_{\mathcal{X}}(\boldsymbol{x})\mathbb{E}_{\boldsymbol{\theta}}[V(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}))]\end{aligned}$$

where $\boldsymbol{y}=\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(1)=\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta})$. ■

It is observed that the Bellman equation (3) may have multiple bounded solutions, since

$$V'(\boldsymbol{x}):=V(\boldsymbol{x})+C\mathbb{E}_\pi[\prod_{j\in\mathbb{N}}1_{\mathcal{X}}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(j))]$$

also satisfies the equation (3), where C is a constant and $\mathbb{E}_\pi[\prod_{j\in\mathbb{N}}1_{\mathcal{X}}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(j))]$ equals the liveness probability that the system (1) starting from $\boldsymbol{x}$ will stay within the set $\mathcal{X}$ for all time. Specially, when $C=1$, $V'(\boldsymbol{x})=1$ for $\boldsymbol{x}\in\mathbb{R}^n$ satisfies the Bellman equation (3).

A sufficient and necessary barrier-like condition for certifying lower bounds in the safety verification can be derived via relaxing the Bellman equation (3).

Theorem 1: There exists a function $v(\boldsymbol{x}):\mathbb{R}^n\to\mathbb{R}$ satisfying the following barrier-like condition:

$$\begin{cases}v(\boldsymbol{x}_0)\leq 1-\epsilon_1\\v(\boldsymbol{x})\geq\mathbb{E}_{\boldsymbol{\theta}}[v(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}))] & \forall\boldsymbol{x}\in\mathcal{X}\\v(\boldsymbol{x})\geq 1 & \forall\boldsymbol{x}\in\mathbb{R}^n\setminus\mathcal{X}\\v(\boldsymbol{x})\geq 0 & \forall\boldsymbol{x}\in\mathbb{R}^n\end{cases} \tag{4}$$

if and only if $\mathbb{P}_\pi(S_{\boldsymbol{x}_0})\geq\epsilon_1$.

Proof: 1) We first prove the "only if" part.

Since $v(\boldsymbol{x})$ satisfies (4), we have

$$\begin{aligned}v(\boldsymbol{x})&\geq 1_{\mathbb{R}^n\setminus\mathcal{X}}(\boldsymbol{x})+1_{\mathcal{X}}(\boldsymbol{x})\mathbb{E}_{\boldsymbol{\theta}}[v(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}))]\\&=1_{\mathbb{R}^n\setminus\mathcal{X}}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0))+1_{\mathcal{X}}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0))\mathbb{E}_\pi[v(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(1))]\\&\geq 1_{\mathbb{R}^n\setminus\mathcal{X}}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0))+1_{\mathcal{X}}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0))\mathbb{E}_\pi[1_{\mathbb{R}^n\setminus\mathcal{X}}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(1))\\&\qquad+1_{\mathcal{X}}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(1))\mathbb{E}_{\boldsymbol{\theta}}[v(\boldsymbol{f}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(1),\boldsymbol{\theta}))]]\\&=\mathbb{E}_\pi[1_{\mathbb{R}^n\setminus\mathcal{X}}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0))+1_{\mathcal{X}}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0))1_{\mathbb{R}^n\setminus\mathcal{X}}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(1))]\\&\qquad+1_{\mathcal{X}}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0))\mathbb{E}_\pi[1_{\mathcal{X}}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(1))v(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(2))]\\&\geq\cdots\\&\geq\mathbb{E}_\pi[1_{\mathbb{R}^n\setminus\mathcal{X}}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0))+\sum_{i\in\mathbb{N}_{\geq 1}}\prod_{j=0}^{i-1}1_{\mathcal{X}}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(j))1_{\mathbb{R}^n\setminus\mathcal{X}}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(i))]\\&\geq V(\boldsymbol{x})\end{aligned}$$

for $\boldsymbol{x}\in\mathbb{R}^n$. Therefore, according to Lemma 1,

$$\mathbb{P}_\pi(S_{\boldsymbol{x}_0})=1-V(\boldsymbol{x}_0)\geq\epsilon_1.$$

2) We will prove the "if" part.

If $\mathbb{P}_\pi(S_{\boldsymbol{x}_0})\geq\epsilon_1$, we have $V(\boldsymbol{x}_0)\leq 1-\epsilon_1$ according to Lemma 1, where $V(\cdot):\mathbb{R}^n\to\mathbb{R}$ is the value function in (2). Moreover, according to Proposition 1, $V(\boldsymbol{x})$ satisfies $V(\boldsymbol{x})=\mathbb{E}_{\boldsymbol{\theta}}[V(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}))],\forall\boldsymbol{x}\in\mathcal{X}$, $V(\boldsymbol{x})=1,\forall\boldsymbol{x}\in\mathbb{R}^n\setminus\mathcal{X}$. Moreover, $V(\boldsymbol{x})\geq 0,\forall\boldsymbol{x}\in\mathbb{R}^n$. Thus, $V(\boldsymbol{x})$ satisfies (4). ■

Remark 1: In this study, we consider the safety verification with respect to a fixed initial state $\boldsymbol{x}_0\in\mathcal{X}$. However, if we use an initial set $\mathcal{X}_0$, which is a set of initial states, the barrier-like condition (4), with $v(\boldsymbol{x})\leq 1-\epsilon_1,\forall\boldsymbol{x}\in\mathcal{X}_0$ replacing $v(\boldsymbol{x}_0)\leq 1-\epsilon_1$, is also a sufficient and necessary one for justifying $\mathbb{P}_\pi(S_{\boldsymbol{x}})\geq\epsilon_1,\forall\boldsymbol{x}\in\mathcal{X}_0$, since $\mathbb{P}_\pi(S_{\boldsymbol{x}})\geq\epsilon_1,\forall\boldsymbol{x}\in\mathcal{X}_0$ is equivalent to $V(\boldsymbol{x})\leq 1-\epsilon_1,\forall\boldsymbol{x}\in\mathcal{X}_0$, where $V(\cdot):\mathbb{R}^n\to\mathbb{R}$ is the value function (2).

In addition, the set $\mathbb{R}^n$ in condition (4) can be substituted with a set Ω, which encompasses the reachable set of system (1) starting from the safe set $\mathcal{X}$ within a single step, i.e.,

$$\Omega\supseteq\{\boldsymbol{x}_1\mid\boldsymbol{x}_1=\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}),\forall\boldsymbol{x}\in\mathcal{X},\boldsymbol{\theta}\in\Theta\}\cup\mathcal{X}. \tag{5}$$

The resulting condition also serves as both a sufficient and necessary criterion for certifying lower bounds of liveness probabilities. It is the one (9) in Proposition 3 in [25], which was derived using an auxiliary switched system and Ville's Inequality [19]. In [25], only the sufficiency of the condition for safety verification was demonstrated. In addition, this condition is also a typical instance of condition (3) with $\alpha=1$ and $\beta=0$ in Theorem 1 in [22], which studied finite-time safety verification. ■

IV. Reach-avoid Verification

This subsection presents sufficient and necessary barrier-like conditions for the reach-avoid verification in Definition 4. Two cases are discussed in this section. The first case assumes that the system (1) will either leave the safe set $\mathcal{X}$ or enter the target set $\mathcal{X}_r$ in finite time almost surely. The second case considers the assumption that the specified lower bound ϵ_2 is strictly less than the exact reach-avoid probability $\mathbb{P}_\pi(RA_{\boldsymbol{x}_0})$, i.e., $\epsilon_2<\mathbb{P}_\pi(RA_{\boldsymbol{x}_0})$. These two cases are detailed in Subsection IV-A and IV-B, respectively.

A. Reach-avoid Verification I

The subsection will formulate a sufficient and necessary barrier-like condition for the reach-avoid verification with the assumption that the system (1) will either leave the safe set $\mathcal{X}$ or enter the target set $\mathcal{X}_r$ in finite time almost surely. Like the one in Section III, this condition is also constructed via relaxing a Bellman equation. The Bellman equation is derived from a value function.

Let's start with the value function $V(\cdot):\mathbb{R}^n\to\mathbb{R}$, which characterizes the exact reach-avoid probability $\mathbb{P}_\pi(RA_{\boldsymbol{x}})$ for $\boldsymbol{x}\in\mathbb{R}^n$,

$$V(\boldsymbol{x}) := \mathbb{E}_\pi\big[g(\boldsymbol{x})\big] \tag{6}$$

where

$$g(\boldsymbol{x}) = 1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0)) + \sum_{i\in\mathbb{N}_{\geq 1}}\prod_{j=0}^{i-1} 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(j))1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(i)).$$

Lemma 2: The value function $V(\boldsymbol{x})$ in (6) is equal to the reach-avoid probability $\mathbb{P}_\pi(RA_{\boldsymbol{x}})$, i.e.,

$$V(\boldsymbol{x}) = \mathbb{P}_\pi(RA_{\boldsymbol{x}})$$

for $\boldsymbol{x}\in\mathbb{R}^n$.

Proof: Clearly, $\mathbb{E}_\pi[1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0))] = 1_{\mathcal{X}_r}(\boldsymbol{x})$. In addition, since

$$\begin{aligned}&\mathbb{E}_\pi[\prod_{j=0}^{i-1} 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(j))1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(i))]\\ &= \mathbb{P}_\pi(\wedge_{j=1}^{i-1}[\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(j)\in\mathcal{X}\setminus\mathcal{X}_r]\wedge[\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(i)\in\mathcal{X}_r])\end{aligned}$$

is the probability that the system (1) starting from $\boldsymbol{x}$ will enter the target set $\mathcal{X}_r$ at time $t=i$ while staying within $\mathcal{X}\setminus\mathcal{X}_r$ before i. Thus, $\mathbb{P}_\pi(RA_{\boldsymbol{x}}) = 1_{\mathcal{X}_r}(\boldsymbol{x}) + \sum_{i\in\mathbb{N}_{\geq 1}}\mathbb{P}_\pi(\wedge_{j=1}^{i-1}[\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(j)\in\mathcal{X}\setminus\mathcal{X}_r]\wedge[\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(i)\in\mathcal{X}_r]) = V(\boldsymbol{x})$. ■

We next will show that the value function (6) can be reduced to a solution to a Bellman equation (or, dynamic programming equation) via the dynamic programming principle.

Proposition 2: The value function $V(\cdot):\mathbb{R}^n\to\mathbb{R}$ in (6) satisfies the following Bellman equation

$$V(\boldsymbol{x}) = 1_{\mathcal{X}_r}(\boldsymbol{x}) + 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{x})\mathbb{E}_{\boldsymbol{\theta}}[V(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}))], \forall \boldsymbol{x}\in\mathbb{R}^n. \tag{7}$$

Proof: Since $g(\boldsymbol{x}) = 1_{\mathcal{X}_r}(\boldsymbol{x}) + 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{x})(1_{\mathcal{X}_r}(\boldsymbol{y}) + \sum_{i\in\mathbb{N}_{\geq 1}}\prod_{j=0}^{i-1}1_{\mathcal{X}}(\boldsymbol{\phi}_\pi^{\boldsymbol{y}}(j))1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{y}}(i)))$, we have

$$\begin{aligned}V(\boldsymbol{x}) =& 1_{\mathcal{X}_r}(\boldsymbol{x}) + 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{x})\mathbb{E}_\pi[\\ &\quad 1_{\mathcal{X}_r}(\boldsymbol{y}) + \sum_{i\in\mathbb{N}_{\geq 1}}\prod_{j=0}^{i-1}1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{y}}(j))1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{y}}(i))\\ &\quad]\\ =& 1_{\mathcal{X}_r}(\boldsymbol{x}) + 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{x})\mathbb{E}_{\boldsymbol{\theta}}[\\ &\quad 1_{\mathcal{X}_r}(\boldsymbol{y}) + \mathbb{E}_\pi[\sum_{i\in\mathbb{N}_{\geq 1}}\prod_{j=0}^{i-1}1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{y}}(j))1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{y}}(i))]\\ &\quad]\\ =& 1_{\mathcal{X}_r}(\boldsymbol{x}) + 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{x})\mathbb{E}_{\boldsymbol{\theta}}[V(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}))]\end{aligned}$$

where $\boldsymbol{y} = \boldsymbol{\phi}_\pi^{\boldsymbol{x}}(1) = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta})$. ■

Remark 2: Like the condition (4) in Theorem 1, we can also construct a sufficient and necessary condition for the safety verification scenario in [12], which is certifying upper bounds of the probability that the system eventually enters unsafe sets from an initial state while adhering to state-constrained sets, via relaxing the Bellman equation (7). It is shown in Proposition 3. In this proposition, $\mathcal{X}_r$ is a set of unsafe states and $\mathcal{X}$ is a state-constrained set. This condition is also a typical instance of condition (9) with $\alpha=1$ and $\beta=0$ in Theorem 3 in [22], which provides upper bounds of the reach-avoid probability in the finite-time reach-avoid verification.

Proposition 3: There exists a function $v(\boldsymbol{x}):\mathbb{R}^n\to\mathbb{R}$ satisfying the following barrier-like condition:

$$\begin{cases} v(\boldsymbol{x}_0)\leq \epsilon_1' \\ v(\boldsymbol{x})\geq \mathbb{E}_{\boldsymbol{\theta}}[v(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}))] & \forall \boldsymbol{x}\in\mathcal{X}\setminus\mathcal{X}_r \\ v(\boldsymbol{x})\geq 1 & \forall \boldsymbol{x}\in\mathcal{X}_r \\ v(\boldsymbol{x})\geq 0 & \forall \boldsymbol{x}\in\mathbb{R}^n\setminus\mathcal{X}\end{cases} \tag{8}$$

if and only if $\mathbb{P}_\pi(S'_{\boldsymbol{x}_0})\leq\epsilon_1'$, where $S'_{\boldsymbol{x}_0} = RA_{\boldsymbol{x}_0} = \{\pi \mid \exists k\in\mathbb{N}.\boldsymbol{\phi}_\pi^{\boldsymbol{x}_0}(k)\in\mathcal{X}_r\wedge\forall i\in\mathbb{N}_{\leq k}.\boldsymbol{\phi}_\pi^{\boldsymbol{x}_0}(i)\in\mathcal{X}\}$.

Proof: The proof is shown in Appendix. ■

Like noted in Remark 1, an initial set $\mathcal{X}_0$, which is a set of initial states, can replace the fixed initial state $\boldsymbol{x}_0$ in (8). The resulting condition is also a sufficient and necessary one for justifying $\mathbb{P}_\pi(S'_{\boldsymbol{x}})\leq\epsilon_1', \forall\boldsymbol{x}\in\mathcal{X}_0$, since $\mathbb{P}_\pi(S'_{\boldsymbol{x}})\leq\epsilon_1', \forall\boldsymbol{x}\in\mathcal{X}_0$ is equivalent to $V(\boldsymbol{x})\leq\epsilon_1',\forall\boldsymbol{x}\in\mathcal{X}_0$ according to Lemma 2, where $V(\cdot):\mathbb{R}^n\to\mathbb{R}$ is the value function (6). ■

However, it is generally not feasible to formulate sufficient and necessary conditions for certifying lower bounds in the reach-avoid verification by relaxing the Bellman equation (7). The underlying reason is that the Bellman equation (7) typically does not possess a unique bounded solution. Nevertheless, under specific assumptions, we can guarantee that the Bellman equation (7) possess a unique bounded solution, and thus construct such conditions.

Assumption 1: When the system (1), starting from any state $\boldsymbol{x}\in\mathcal{X}\setminus\mathcal{X}_r$, will leave the set $\mathcal{X}\setminus\mathcal{X}_r$ in finite time almost surely, i.e.,

$$\mathbb{P}_\pi(\forall k\in\mathbb{N}.\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(k)\in\mathcal{X}\setminus\mathcal{X}_r) = 0, \forall\boldsymbol{x}\in\mathcal{X}\setminus\mathcal{X}_r.$$

Proposition 4: Under Assumption 1, the Bellman equation (7) has a unique bounded solution, which is the value function (6).

Proof: As shown in Proposition 2, the value function (6) satisfies the Bellman equation (7).

In the following, we just show that if a bounded function $v(\boldsymbol{x}):\mathbb{R}^n\to\mathbb{R}$ satisfies the Bellman equation (7), $v(\boldsymbol{x}) = V(\boldsymbol{x})$ holds for $\boldsymbol{x}\in\mathbb{R}^n$.

Since $v(\boldsymbol{x})$ satisfies the Bellman equation (7), we have

$$\begin{aligned}v(\boldsymbol{x}) &= 1_{\mathcal{X}_r}(\boldsymbol{x}) + 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{x})\mathbb{E}_{\boldsymbol{\theta}}[v(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}))]\\ &= 1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0)) + 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0))\mathbb{E}_\pi[v(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(1))]\\ &= 1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0)) + 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0))\mathbb{E}_\pi[1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(1))\\ &\qquad + 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(1))\mathbb{E}_{\boldsymbol{\theta}}[v(\boldsymbol{f}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(1),\boldsymbol{\theta}))]]\\ &= \mathbb{E}_\pi[1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0)) + 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0))1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(1))]\\ &\qquad + 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0))\mathbb{E}_\pi[1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(1))v(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(2))]\\ &= \cdots\\ &= \mathbb{E}_\pi[1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0)) + \sum_{i\in\mathbb{N}_{\geq 1}}\prod_{j=0}^{i-1}1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(j))1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(i))]\\ &\qquad + 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{x})\lim_{i\to\infty}h_i(\boldsymbol{x})\\ &= V(\boldsymbol{x}) + 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{x})\lim_{i\to\infty}h_i(\boldsymbol{x})\end{aligned} \tag{9}$$

for $\boldsymbol{x}\in\mathbb{R}^n$, where

$$h_i(\boldsymbol{x}) = \mathbb{E}_\pi[\prod_{j=1}^{i}1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(j))v(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(i+1))].$$

Since $\mathbb{P}_\pi(\forall k\in\mathbb{N}.\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(k)\in\mathcal{X}\setminus\mathcal{X}_r) = 0$ for $\boldsymbol{x}\in\mathcal{X}\setminus\mathcal{X}_r$ and $v(\cdot):\mathbb{R}^n\to\mathbb{R}$ is bounded over $\mathbb{R}^n$, we have $\lim_{i\to\infty}h_i(\boldsymbol{x}) = 0$ for $\boldsymbol{x}\in\mathbb{R}^n$ and consequently, $v(\boldsymbol{x}) = V(\boldsymbol{x})$ over $\mathbb{R}^n$. ■

Under Assumption 1, we can construct sufficient and necessary conditions for certifying lower bounds in the reach-avoid verification via relaxing the Bellman equation (7).

Theorem 2: Under Assumption 1, there exists a function $v(\boldsymbol{x}): \mathbb{R}^n \rightarrow \mathbb{R}$, which is bounded in $\mathcal{X}$ and satisfies the following condition:

$$\begin{cases} v(\boldsymbol{x}_0) \geq \epsilon_2 \\ v(\boldsymbol{x}) \leq \mathbb{E}_{\boldsymbol{\theta}}[v(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}))] & \forall \boldsymbol{x} \in \mathcal{X} \setminus \mathcal{X}_r \\ v(\boldsymbol{x}) \leq 1 & \forall \boldsymbol{x} \in \mathcal{X}_r \\ v(\boldsymbol{x}) \leq 0 & \forall \boldsymbol{x} \in \mathbb{R}^n \setminus \mathcal{X} \end{cases} \tag{10}$$

if and only if $\mathbb{P}_\pi(RA_{\boldsymbol{x}_0}) \geq \epsilon_2$.

Proof: 1) We first prove the "only if" part.

Since $v(\boldsymbol{x})$ satisfies (10), we can obtain, following the induction of (9) by replacing "=" with "$\leq$", that

$$v(\boldsymbol{x}) \leq V(\boldsymbol{x}) + 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{x}) \lim_{i\rightarrow\infty} h_i(\boldsymbol{x})$$

for $\boldsymbol{x} \in \mathbb{R}^n$, where $V(\cdot): \mathbb{R}^n \rightarrow \mathbb{R}$ is the value function defined in (6). Since $v(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(i+1)) \leq 0$ when $\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(i+1) \in \mathbb{R}^n \setminus \mathcal{X}$, we have

$$\begin{aligned} h_i(\boldsymbol{x}) &= \mathbb{E}_\pi[\prod_{j=1}^{i} 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(j)) v(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(i+1))] \\ &\leq \mathbb{E}_\pi[\prod_{j=1}^{i} 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(j)) w_{i+1}(\boldsymbol{x})] \end{aligned},$$

where

$$w_{i+1}(\boldsymbol{x}) = 1_{\mathcal{X}}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(i+1)) v(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(i+1)).$$

Also, since $v(\cdot): \mathbb{R}^n \rightarrow \mathbb{R}$ is bounded over $\mathcal{X}$ and $\mathbb{P}_\pi(\forall k \in \mathbb{N}. \boldsymbol{\phi}_\pi^{\boldsymbol{x}}(k) \in \mathcal{X} \setminus \mathcal{X}_r) = 0$ for $\boldsymbol{x} \in \mathcal{X} \setminus \mathcal{X}_r$, we conclude

$$\lim_{i\rightarrow\infty} h_i(\boldsymbol{x}) = 0$$

for $\boldsymbol{x} \in \mathbb{R}^n$. Consequently, $v(\boldsymbol{x}) \leq V(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathbb{R}^n$.

Thus, $\mathbb{P}_\pi(RA_{\boldsymbol{x}_0}) = V(\boldsymbol{x}_0) \geq v(\boldsymbol{x}_0) \geq \epsilon_2$.

2) We will prove the "if" part.

If $\mathbb{P}_\pi(RA_{\boldsymbol{x}_0}) \geq \epsilon_2$, we have $V(\boldsymbol{x}_0) \geq \epsilon_2$ according to Lemma 2, where $V(\cdot): \mathbb{R}^n \rightarrow \mathbb{R}$ is the value function in (6). Moreover, according to Proposition 2, $V(\boldsymbol{x})$ satisfies $V(\boldsymbol{x}) = \mathbb{E}_{\boldsymbol{\theta}}[V(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}))], \forall \boldsymbol{x} \in \mathcal{X} \setminus \mathcal{X}_r$, $V(\boldsymbol{x}) = 1, \forall \boldsymbol{x} \in \mathcal{X}_r$, and $V(\boldsymbol{x}) = 0, \forall \boldsymbol{x} \in \mathbb{R}^n \setminus \mathcal{X}$. Consequently, $V(\boldsymbol{x})$ satisfies (10). ■

Without Assumption 1, we cannot use condition (10) to justify lower bounds in the reach-avoid verification, since we cannot guarantee

$$\lim_{i\rightarrow\infty} h_i(\boldsymbol{x}) = 0$$

for $\boldsymbol{x} \in \mathcal{X} \setminus \mathcal{X}_r$.

Remark 3: In Theorem 2, we consider the reach-avoid verification with respect to a fixed initial state $\boldsymbol{x}_0 \in \mathcal{X}\setminus\mathcal{X}_r$. However, if we use an initial set $\mathcal{X}_0$, which is a set of initial states, the barrier-like condition (10), with $v(\boldsymbol{x}) \geq \epsilon_2$, $\forall \boldsymbol{x} \in \mathcal{X}_0$ replacing $v(\boldsymbol{x}_0) \geq \epsilon_2$, is also a sufficient and necessary one for justifying $\mathbb{P}_\pi(RA_{\boldsymbol{x}}) \geq \epsilon_2, \forall \boldsymbol{x} \in \mathcal{X}_0$, since $\mathbb{P}_\pi(RA_{\boldsymbol{x}}) \geq \epsilon_2, \forall \boldsymbol{x} \in \mathcal{X}_0$ is equivalent to $V(\boldsymbol{x}) \geq \epsilon_2, \forall \boldsymbol{x} \in \mathcal{X}_0$, where $V(\cdot): \mathbb{R}^n \rightarrow \mathbb{R}$ is the value function (6). ■

B. Reach-avoid Verification II

The subsection will formulate a sufficient and necessary barrier-like condition for the reach-avoid verification without Assumption 1. Instead, another assumption that ϵ_2 is strictly smaller than the exact reach-avoid probability $\mathbb{P}_\pi(RA_{\boldsymbol{x}_0})$ is imposed. Like the one in Subsection IV-A, this condition is constructed via relaxing a Bellman equation, which is derived from a discounted value function.

Let's start with the discounted value function $\tilde{V}_\gamma(\cdot): \mathbb{R}^n \rightarrow \mathbb{R}$,

$$\tilde{V}_\gamma(\boldsymbol{x}) := \mathbb{E}_\pi[\tilde{g}_\gamma(\boldsymbol{x})], \tag{11}$$

where

$$\tilde{g}_\gamma(\boldsymbol{x}) = 1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0)) + \sum_{i\in\mathbb{N}_{\geq 1}} \gamma^i \prod_{j=0}^{i-1} 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(j)) 1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(i))$$

and $\gamma \in [0,1]$ is a user-defined value.

The value $\tilde{V}_\gamma(\boldsymbol{x})$ in (11) is a lower bound of the exact reach-avoid probability $\mathbb{P}_\pi(RA_{\boldsymbol{x}})$ for $\boldsymbol{x} \in \mathbb{R}^n$. Moreover, when γ approaches 1, $\tilde{V}_\gamma(\boldsymbol{x})$ will approach $\mathbb{P}_\pi(RA_{\boldsymbol{x}})$ for $\boldsymbol{x} \in \mathbb{R}^n$.

Lemma 3: For $\boldsymbol{x} \in \mathbb{R}^n$,

$$\tilde{V}_\gamma(\boldsymbol{x}) \leq \mathbb{P}_\pi(RA_{\boldsymbol{x}})$$

and

$$\lim_{\gamma\rightarrow 1^-} \tilde{V}_\gamma(\boldsymbol{x}) = \mathbb{P}_\pi(RA_{\boldsymbol{x}}),$$

where $\tilde{V}(\cdot): \mathbb{R}^n \rightarrow \mathbb{R}$ is the value function in (11).

Proof: The conclusion $\tilde{V}_\gamma(\boldsymbol{x}) \leq \mathbb{P}_\pi(RA_{\boldsymbol{x}})$ can be justified according to $\gamma \in [0,1]$ and Lemma 2.

In the following, we just show $\lim_{\gamma\rightarrow 1^-} \tilde{V}_\gamma(\boldsymbol{x}) = \mathbb{P}_\pi(RA_{\boldsymbol{x}})$.

1) We first show $\tilde{V}_\gamma(\boldsymbol{x})$ is uniformly convergent over $\gamma \in [0,1]$. According to Lemma 2, $\mathbb{P}_\pi(RA_{\boldsymbol{x}}) = V(\boldsymbol{x})$, where $V(\cdot): \mathbb{R}^n \rightarrow \mathbb{R}$ is the value function in (6). Thus, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sum_{k=m+1}^{M} \mathbb{E}_\pi[\prod_{j=0}^{k-1} 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(j)) 1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(k))] < \epsilon, \forall M > m > N.$$

Since

$$\begin{aligned} &\sum_{k=m+1}^{M} \mathbb{E}_\pi[\gamma^k \prod_{j=0}^{k-1} 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(j)) 1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(k))] \\ \leq &\sum_{k=m+1}^{M} \mathbb{E}_\pi[\prod_{j=0}^{k-1} 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(j)) 1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(k))] \end{aligned}$$

holds for $\gamma \in [0,1]$, we have $\tilde{V}_\gamma(\boldsymbol{x})$ is uniformly convergent over $\gamma \in [0,1]$.

In addition, $\mathbb{E}_\pi[\gamma^i \prod_{j=0}^{i-1} 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(j)) 1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(i))]$ is continuous over $\gamma \in [0,1]$, where $i \in \mathbb{N}_{\geq 1}$. Therefore, according to Term-by-term Continuity Theorem, we obtain $\lim_{\gamma\rightarrow 1^-} \tilde{V}_\gamma(\boldsymbol{x}) = \mathbb{E}_\pi[1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0)) + \sum_{i\in\mathbb{N}_{\geq 1}} \lim_{\gamma\rightarrow 1^-} \gamma^i \prod_{j=0}^{i-1} 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(j)) 1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(i))] = V(\boldsymbol{x}) = \mathbb{P}_\pi(RA_{\boldsymbol{x}})$. ■

Proposition 5: When $\gamma \in [0,1)$, the value function (11) $\tilde{V}_\gamma(\cdot): \mathbb{R}^n \rightarrow \mathbb{R}$ in (11) satisfies the following Bellman equation:

$$\tilde{V}_\gamma(\boldsymbol{x}) = 1_{\mathcal{X}_r}(\boldsymbol{x}) + \gamma 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{x}) \mathbb{E}_{\boldsymbol{\theta}}[\tilde{V}_\gamma(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}))], \forall \boldsymbol{x} \in \mathbb{R}^n. \tag{12}$$

Moreover, the Bellman equation (12) possess a unique bounded solution.

Proof: The conclusion that the value function (11) satisfies the Bellman equation (12) can be justified by following the proof of Proposition 2.

In the following, we just show that if a bounded function $v(\boldsymbol{x}): \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies the Bellman equation (12), $v(\boldsymbol{x}) = \tilde{V}_\gamma(\boldsymbol{x})$ holds for $\boldsymbol{x} \in \mathbb{R}^n$.

Since $v(\boldsymbol{x})$ satisfies the Bellman equation (12), we have

$$\begin{aligned}
v(\boldsymbol{x}) &= 1_{\mathcal{X}_r}(\boldsymbol{x}) + \gamma 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{x})\mathbb{E}_{\boldsymbol{\theta}}[v(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}))]\\
&= 1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0)) + \gamma 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0))\mathbb{E}_{\boldsymbol{\theta}}[v(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(1))]\\
&= 1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0)) + \gamma 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0))\mathbb{E}_{\pi}[1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(1))\\
&\qquad + \gamma 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(1))\mathbb{E}_{\boldsymbol{\theta}}[v(\boldsymbol{f}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(1),\boldsymbol{\theta}))]]\\
&= \mathbb{E}_{\pi}[1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0)) + \gamma 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0))1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(1))]\\
&\qquad + \gamma^2 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0))\mathbb{E}_{\pi}[1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(1))v(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(2))]\\
&= \cdots\\
&= \mathbb{E}_{\pi}[1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0)) + \sum_{i\in\mathbb{N}_{\geq 1}}\gamma^i\prod_{j=0}^{i-1}1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(j))1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(i))]\\
&\qquad + 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{x})\lim_{i\to\infty}h_i(\boldsymbol{x})\\
&= \tilde{V}_\gamma(\boldsymbol{x}) + 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{x})\lim_{i\to\infty}h_i(\boldsymbol{x})
\end{aligned} \tag{13}$$

for $\boldsymbol{x}\in\mathbb{R}^n$, where

$$h_i(\boldsymbol{x}) = \gamma^{i+1}\mathbb{E}_\pi[\prod_{j=1}^{i}1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(j))v(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(i+1))].$$

Since $v(\cdot):\mathbb{R}^n\to\mathbb{R}$ is bounded over $\mathbb{R}^n$, we have $\lim_{i\to\infty}h_i(\boldsymbol{x})=0$ for $\boldsymbol{x}\in\mathbb{R}^n$ and consequently, $v(\boldsymbol{x})=\tilde{V}_\gamma(\boldsymbol{x})$ over $\mathbb{R}^n$. ■

We can construct a sufficient and necessary barrier-like condition for the reach-avoid verification in Definition 4 via relaxing the Bellman equation (12), under the assumption that the reach-avoid probability $\mathbb{P}_\pi(RA_{\boldsymbol{x}_0})$ is strictly larger than the threshold ϵ_2. This condition is the stochastic version of the one in Corollary 1 in [26].

Assumption 2: The reach-avoid probability $\mathbb{P}_\pi(RA_{\boldsymbol{x}_0})$ is strictly larger than the threshold ϵ_2, i.e., $\mathbb{P}_\pi(RA_{\boldsymbol{x}_0})>\epsilon_2$.

Theorem 3: Under Assumption 2, there exist a constant $\gamma\in(0,1)$ and a function $v(\boldsymbol{x}):\mathbb{R}^n\to\mathbb{R}$, which is bounded over $\mathcal{X}$ and satisfies the following condition:

$$\begin{cases}
v(\boldsymbol{x}_0)\geq\epsilon_2 & \\
v(\boldsymbol{x})\leq\gamma\mathbb{E}_{\boldsymbol{\theta}}[v(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}))] & \forall\boldsymbol{x}\in\mathcal{X}\setminus\mathcal{X}_r\\
v(\boldsymbol{x})\leq 1 & \forall\boldsymbol{x}\in\mathcal{X}_r\\
v(\boldsymbol{x})\leq 0 & \forall\boldsymbol{x}\in\mathbb{R}^n\setminus\mathcal{X}
\end{cases} \tag{14}$$

if and only if $\mathbb{P}_\pi(RA_{\boldsymbol{x}_0})\geq\epsilon_2$.

Proof: 1) We first prove the "only if" part.

Since $v(\boldsymbol{x})$ satisfies (14), we can obtain, following the induction of (13) by replacing "=" with "$\leq$", that

$$v(\boldsymbol{x})\leq\tilde{V}_\gamma(\boldsymbol{x}) + 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{x})\lim_{i\to\infty}h_i(\boldsymbol{x})$$

for $\boldsymbol{x}\in\mathbb{R}^n$, where $\tilde{V}_\gamma(\cdot):\mathbb{R}^n\to\mathbb{R}$ is the value function defined in (11). Since $v(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(i+1))\leq 0$ when $\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(i+1)\in\mathbb{R}^n\setminus\mathcal{X}$, we have

$$\begin{aligned}
h_i(\boldsymbol{x}) &= \gamma^{i+1}\mathbb{E}_\pi[\prod_{j=1}^{i}1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(j))v(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(i+1))]\\
&\leq\gamma^{i+1}\mathbb{E}_\pi[\prod_{j=1}^{i}1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(j))w_{i+1}(\boldsymbol{x})]
\end{aligned},$$

where

$$w_{i+1}(\boldsymbol{x}) = 1_{\mathcal{X}}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(i+1))v(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(i+1)).$$

Also, since $v(\cdot):\mathbb{R}^n\to\mathbb{R}$ is bounded over $\mathcal{X}$ and $\lim_{i\to\infty}\gamma^{i+1}=0$, we conclude

$$\lim_{i\to\infty}h_i(\boldsymbol{x})=0$$

for $\boldsymbol{x}\in\mathbb{R}^n$. Consequently, $v(\boldsymbol{x})\leq\tilde{V}_\gamma(\boldsymbol{x})$ for $\boldsymbol{x}\in\mathbb{R}^n$.

Thus, $\mathbb{P}_\pi(RA_{\boldsymbol{x}_0})\geq\tilde{V}_\gamma(\boldsymbol{x}_0)\geq v(\boldsymbol{x}_0)\geq\epsilon_2$ according to Lemma 3.

2) We will prove the "if" part.

According to Lemma 3,

$$\lim_{\gamma\to 1}\tilde{V}_\gamma(\boldsymbol{x}_0)=\mathbb{P}_\pi(RA_{\boldsymbol{x}_0})$$

holds. Since $\mathbb{P}_\pi(RA_{\boldsymbol{x}_0})>\epsilon_2$, there exists γ_0 such that $\tilde{V}_{\gamma_0}(\boldsymbol{x}_0)\geq\epsilon_2$ according to Lemma 3. Moreover, according to Proposition 5, $\tilde{V}_{\gamma_0}(\boldsymbol{x})$ satisfies $\tilde{V}_{\gamma_0}=\gamma_0\mathbb{E}_{\boldsymbol{\theta}}[\tilde{V}_{\gamma_0}(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}))],\forall\boldsymbol{x}\in\mathcal{X}\setminus\mathcal{X}_r$, $\tilde{V}_{\gamma_0}(\boldsymbol{x})=1,\forall\boldsymbol{x}\in\mathcal{X}_r$, and $\tilde{V}_{\gamma_0}(\boldsymbol{x})=0,\forall\boldsymbol{x}\in\mathbb{R}^n\setminus\mathcal{X}$. Consequently, $\tilde{V}_{\gamma_0}(\boldsymbol{x})$ satisfies (14). ■

Remark 4: If we consider an initial set $\mathcal{X}_0\subseteq\mathcal{X}\setminus\mathcal{X}_r$, which includes infinite initial states, rather than a fixed initial state $\boldsymbol{x}_0\in\mathcal{X}\setminus\mathcal{X}_r$, we cannot guarantee that there exist a constant $\gamma\in(0,1)$ and a function $v(\boldsymbol{x}):\mathbb{R}^n\to\mathbb{R}$, which is bounded over $\mathcal{X}$ and satisfies the condition (14) with $v(\boldsymbol{x})\geq\epsilon_2,\forall\boldsymbol{x}\in\mathcal{X}_0$ replacing $v(\boldsymbol{x}_0)\geq\epsilon_2$, such that $\mathbb{P}_\pi(RA_{\boldsymbol{x}})\geq\epsilon_2,\forall\boldsymbol{x}\in\mathcal{X}_0$. This is because we cannot guarantee that

$$\lim_{\gamma\to 1^-}\tilde{V}_\gamma(\boldsymbol{x})=\mathbb{P}_\pi(RA_{\boldsymbol{x}})$$

holds uniformly over $\mathcal{X}_0$.

In addition, condition (14) is a typical instance of condition (13) with $\alpha>1$ and $\beta=0$ in Theorem 5 in [22], which offers lower bounds of the reach-avoid probability in the context of finite-time reach-avoid verification. ■

Remark 5: It is worth noting here that we can also construct a sufficient and necessary condition to certify upper bounds of the liveness probability $\mathbb{P}_\pi(\forall k\in\mathbb{N}.\boldsymbol{\phi}_\pi^{\boldsymbol{x}_0}(k)\in\mathcal{X})$ such that the system (1) starting from the initial state $\boldsymbol{x}_0$ will stay within the safe set $\mathcal{X}$ for all time [25], under the assumption that $\mathbb{P}_\pi(\forall k\in\mathbb{N}.\boldsymbol{\phi}_\pi^{\boldsymbol{x}_0}(k)\in\mathcal{X})<1-\epsilon_1$. Under the assumption that $\mathbb{P}_\pi(\forall k\in\mathbb{N}.\boldsymbol{\phi}_\pi^{\boldsymbol{x}_0}(k)\in\mathcal{X})<1-\epsilon_1$, there exist a constant $\gamma\in(0,1)$ and a function $v(\boldsymbol{x}):\mathbb{R}^n\to\mathbb{R}$, which is bounded over $\mathcal{X}$ and satisfies the following condition:

$$\begin{cases}
v(\boldsymbol{x}_0)\geq\epsilon_1 & \\
v(\boldsymbol{x})\leq\gamma\mathbb{E}_{\boldsymbol{\theta}}[v(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}))] & \forall\boldsymbol{x}\in\mathcal{X}\\
v(\boldsymbol{x})\leq 1 & \forall\boldsymbol{x}\in\mathbb{R}^n\setminus\mathcal{X}
\end{cases} \tag{15}$$

if and only if $\mathbb{P}_\pi(\forall k\in\mathbb{N}.\boldsymbol{\phi}_\pi^{\boldsymbol{x}_0}(k)\in\mathcal{X})\leq 1-\epsilon_1$ (or equivalently, $\mathbb{P}_\pi(\exists k\in\mathbb{N}.\boldsymbol{\phi}_\pi^{\boldsymbol{x}_0}\in\mathbb{R}^n\setminus\mathcal{X})\geq\epsilon_1$). Condition (15) is also a typical instance of condition (6) with $\alpha>1$ and $\beta=0$ in Theorem 2 in [22], which offers upper bounds of the liveness probability in the finite-time safety verification. ■

Based on the value function (11), we are able to show the necessity of another sufficient barrier-like condition in [23] for the reach-avoid verification under Assumption 2. The condition is presented below:

$$\begin{cases}
v(\boldsymbol{x}_0)\geq\epsilon_2 & \\
v(\boldsymbol{x})\leq\mathbb{E}_{\boldsymbol{\theta}}[v(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}))] & \forall\boldsymbol{x}\in\mathcal{X}\setminus\mathcal{X}_r\\
v(\boldsymbol{x})\leq\mathbb{E}_{\boldsymbol{\theta}}[w(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}))]-w(\boldsymbol{x}) & \forall\boldsymbol{x}\in\mathcal{X}\setminus\mathcal{X}_r\\
v(\boldsymbol{x})\leq 1 & \forall\boldsymbol{x}\in\mathcal{X}_r\\
v(\boldsymbol{x})\leq 0 & \forall\boldsymbol{x}\in\Omega\setminus\mathcal{X}
\end{cases}, \tag{16}$$

where Ω is a set in (5). If there exist a function $v(\cdot):\Omega\to\mathbb{R}$ and a bounded function $w(\cdot):\Omega\to\mathbb{R}$ satisfying (16), $\mathbb{P}_\pi(RA_{\boldsymbol{x}_0})\geq\epsilon_2$ holds. This conclusion can be justified by following the proof of Corollary 2 in [23]. In the following, we just demonstrate its necessity.

Corollary 1: If $\mathbb{P}_\pi(RA_{\boldsymbol{x}_0})>\epsilon_2$, then there exist a function $v(\cdot):\Omega\to\mathbb{R}$ and a bounded function $w(\cdot):\Omega\to\mathbb{R}$ satisfying (16).

Proof: According to Lemma 3, there exists $\gamma_0 \in (0,1)$ such that $\tilde{V}_{\gamma_0}(\boldsymbol{x}_0) \geq \epsilon_2$ holds. From (12), we can obtain

$$\begin{cases} 1 \geq \tilde{V}_{\gamma_0}(\boldsymbol{x}) \geq 0 & \forall \boldsymbol{x} \in \mathbb{R}^n \\ \tilde{V}_{\gamma_0}(\boldsymbol{x}) = \gamma_0 \mathbb{E}_{\boldsymbol{\theta}}[\tilde{V}_{\gamma_0}(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}))] & \\ \qquad\qquad \leq \mathbb{E}_{\boldsymbol{\theta}}[\tilde{V}_{\gamma_0}(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}))] & \forall \boldsymbol{x} \in \mathcal{X} \setminus \mathcal{X}_r \\ \tilde{V}_{\gamma_0}(\boldsymbol{x}) \leq 1 & \forall \boldsymbol{x} \in \mathcal{X}_r \\ \tilde{V}_{\gamma_0}(\boldsymbol{x}) = 0 & \forall \boldsymbol{x} \in \Omega \setminus \mathcal{X} \end{cases}.$$

Let γ_1 be a constant satisfying $\frac{\gamma_1}{1+\gamma_1} \geq \gamma_0$, and $w(\boldsymbol{x}) := \gamma_1 \tilde{V}_{\gamma_0}(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathbb{R}^n$. Thus, $\frac{\mathbb{E}_{\boldsymbol{\theta}}[w(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}))]-w(\boldsymbol{x})-\tilde{V}_{\gamma_0}(\boldsymbol{x})}{1+\gamma_1} = \frac{\gamma_1\mathbb{E}_{\boldsymbol{\theta}}[\tilde{V}_{\gamma_0}(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}))]-\gamma_1\tilde{V}_{\gamma_0}(\boldsymbol{x})-\tilde{V}_{\gamma_0}(\boldsymbol{x})}{1+\gamma_1} = \frac{\gamma_1}{1+\gamma_1}\mathbb{E}_{\boldsymbol{\theta}}[\tilde{V}_{\gamma_0}(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}))] - \tilde{V}_{\gamma_0}(\boldsymbol{x}) \geq \gamma_0\mathbb{E}_{\boldsymbol{\theta}}[\tilde{V}_{\gamma_0}(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}))] - \tilde{V}_{\gamma_0}(\boldsymbol{x}) = 0$. Thus, the functions $\tilde{V}_{\gamma_0}(\boldsymbol{x})$ and $w(\boldsymbol{x}) := \gamma_1\tilde{V}_{\gamma_0}(\boldsymbol{x})$ satisfy (16). Consequently, there exist a function $v(\cdot): \Omega \to \mathbb{R}$ and a bounded function $w(\cdot): \Omega \to \mathbb{R}$ satisfying (16). ■

V. Conclusion

In this paper, we demonstrated sufficient and necessary barrier-like conditions for safety and reach-avoid verification of stochastic discrete-time systems over the infinite time horizon. These conditions were constructed via relaxing Bellman equations.

In the future, we will develop efficient numerical methods to address the proposed barrier-like constraints for safety and reach-avoid verification.

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VI. Appendix

The proof of Proposition 3: *Proof:* 1) We first prove the "only if" part.

Since $v(\boldsymbol{x})$ satisfies (8), we have

$$\begin{aligned} v(\boldsymbol{x}) &\geq 1_{\mathcal{X}_r}(\boldsymbol{x}) + 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{x})\mathbb{E}_{\boldsymbol{\theta}}[v(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}))] \\ &= 1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0)) + 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0))\mathbb{E}_\pi[v(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(1))] \\ &\geq 1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0)) + 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0))\mathbb{E}_\pi[1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(1)) \\ &\qquad\qquad + 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(1))\mathbb{E}_{\boldsymbol{\theta}}[v(\boldsymbol{f}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(1),\boldsymbol{\theta}))]] \\ &= \mathbb{E}_\pi[1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0)) + 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0))1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(1))] \\ &\qquad\qquad + 1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0))\mathbb{E}_\pi[1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(1))v(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(2))] \\ &\geq \cdots \\ &\geq \mathbb{E}_\pi[1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(0)) + \sum_{i\in\mathbb{N}_{\geq 1}}\prod_{j=0}^{i-1}1_{\mathcal{X}\setminus\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(j))1_{\mathcal{X}_r}(\boldsymbol{\phi}_\pi^{\boldsymbol{x}}(i))] \\ &\geq V(\boldsymbol{x}) \end{aligned}$$

for $\boldsymbol{x} \in \mathbb{R}^n$. Therefore, according to Lemma 2,

$$\mathbb{P}_\pi(S'_{\boldsymbol{x}_0}) = V(\boldsymbol{x}_0) \leq \epsilon'_1.$$

2) We will prove the "if" part.

If $\mathbb{P}_\pi(S'_{\boldsymbol{x}_0}) \leq \epsilon'_1$, we have $V(\boldsymbol{x}_0) \leq \epsilon'_1$ according to Lemma 2, where $V(\cdot): \mathbb{R}^n \to \mathbb{R}$ is the value function in (6). Moreover, according to Proposition 2, $V(\boldsymbol{x})$ satisfies $V(\boldsymbol{x}) = \mathbb{E}_{\boldsymbol{\theta}}[V(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{\theta}))], \forall \boldsymbol{x} \in \mathcal{X} \setminus \mathcal{X}_r$, $V(\boldsymbol{x}) = 1, \forall \boldsymbol{x} \in \mathcal{X}_r$, and $V(\boldsymbol{x}) = 0, \forall \boldsymbol{x} \in \mathbb{R}^n \setminus \mathcal{X}$. Consequently, $V(\boldsymbol{x})$ satisfies (8). ■