Computable analysis and control synthesis over complex dynamical systems via formal verification

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Outline

1. Formal abstractions for verification of complex models
2. Formal verification of stochastic hybrid systems
   - Analysis and control synthesis problems
   - Computable analysis and control synthesis via formal abstractions
3. Formal verification of max-plus linear models
   - Analysis and control synthesis problems
   - Computable analysis and control synthesis via formal abstractions
4. Concluding remarks

Key references will appear here

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4. Concluding remarks

Formal abstractions for verification of complex models

concrete complex model

property, specification, cost or reward

Formal abstractions for verification of complex models

ϵ-quantitative
abstraction

concrete
complex
model

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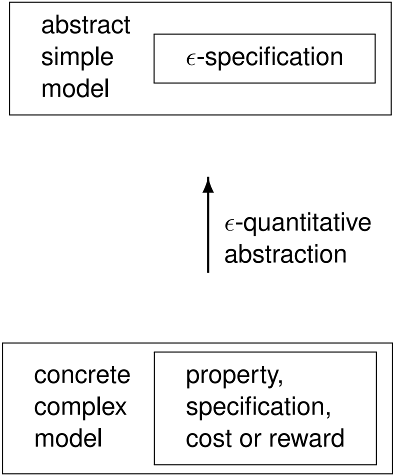
Formal abstractions for verification of complex models



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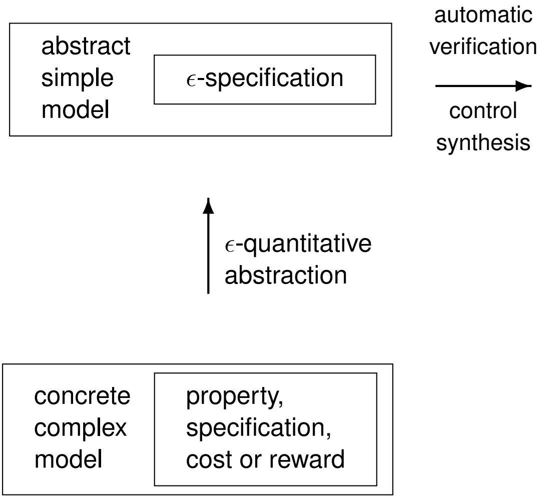

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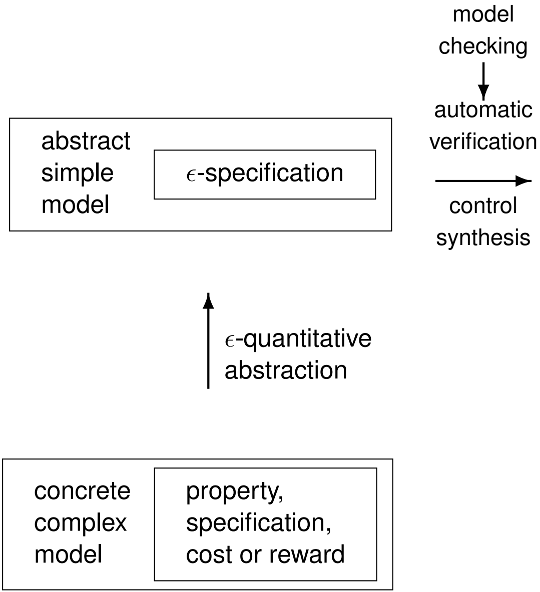

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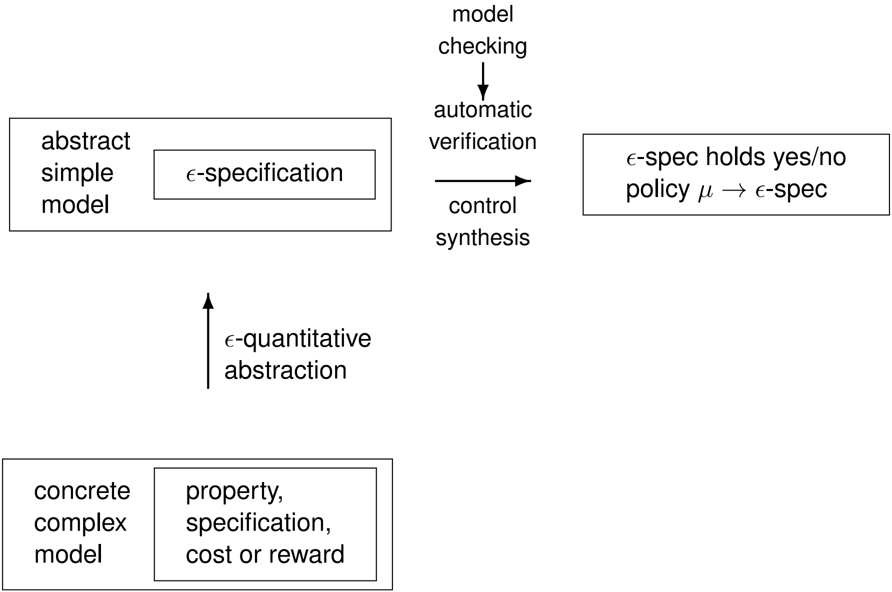

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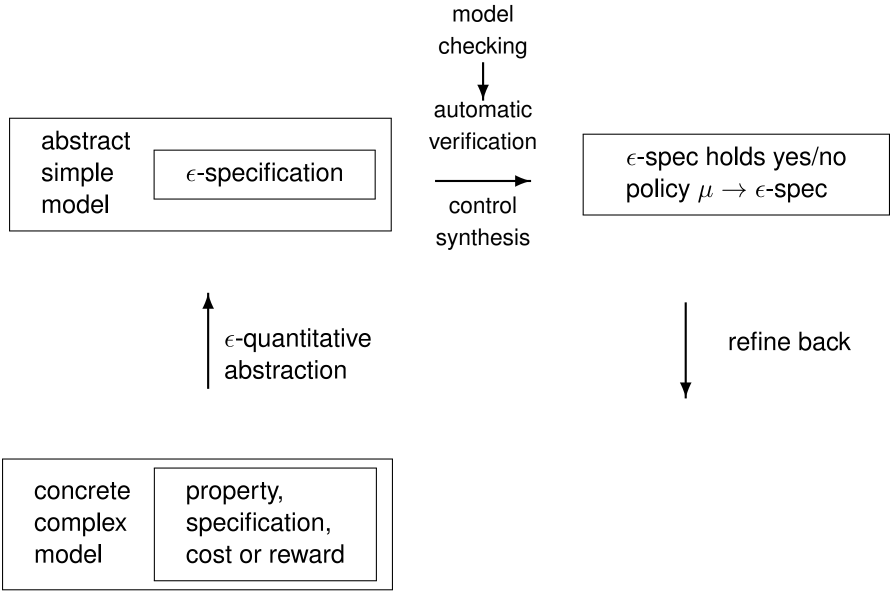

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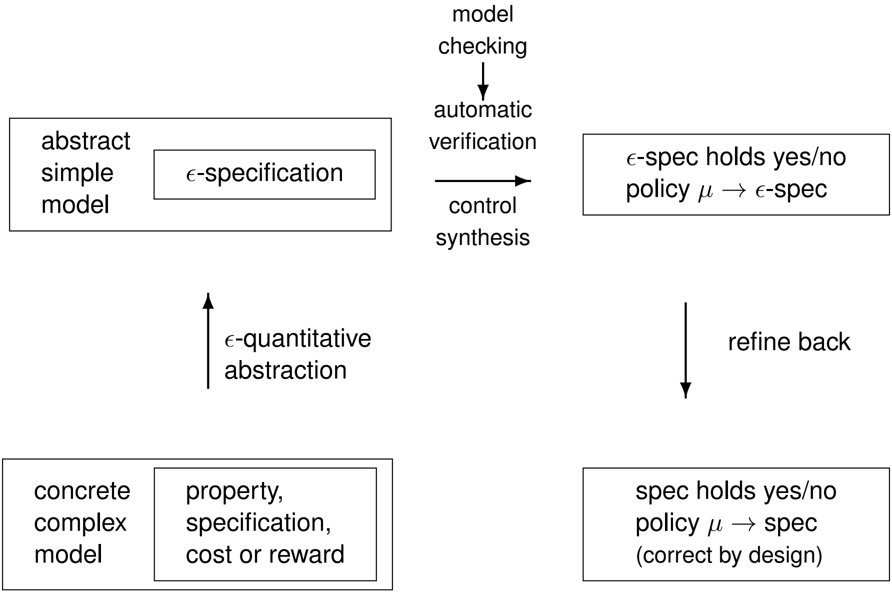

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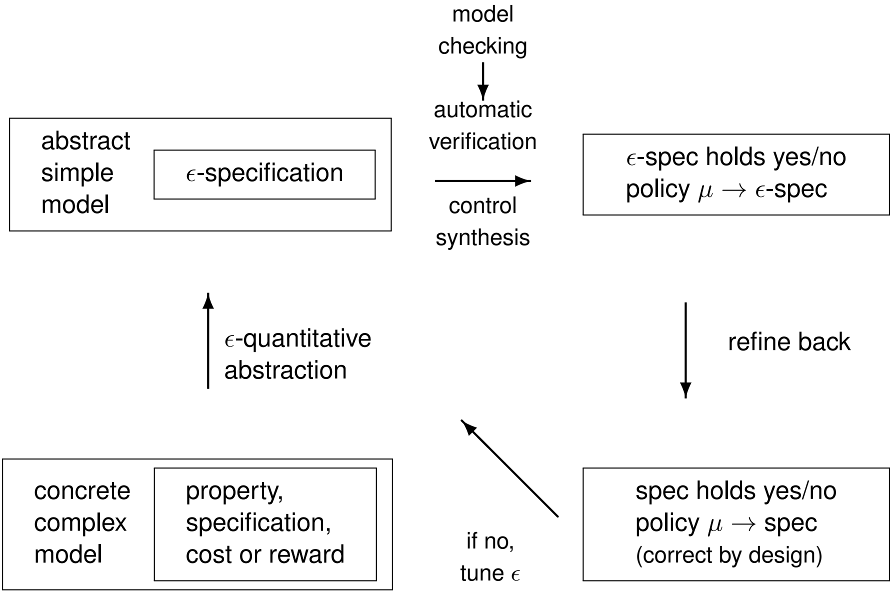

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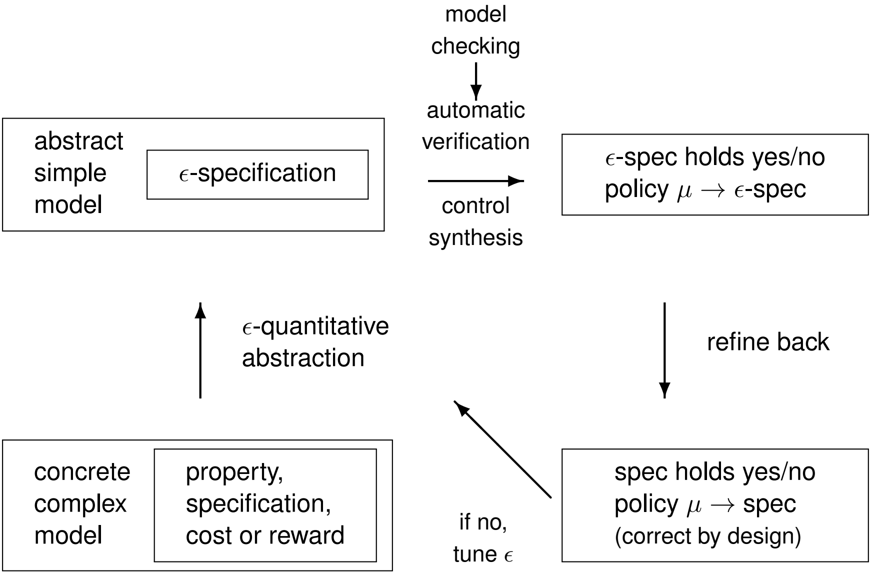

Formal abstractions for verification of dtSHS

PRISM
MRMC

dtMC
dtMDP

relax'd/strenght'd PCTL
inflated LTL – ϵ-spec

prob. model checking

dynamic programming

ϵ-spec holds
policy max/min ϵ-spec

adaptive, sequential abstractions

approximate probabilistic bisimulations

refine back

dtSHS

PCTL
LTL – spec
automata

spec holds
policy max/min spec

Stochastic hybrid (discrete/continuous) systems

PRISM
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prob. model checking

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dtSHS	PCTL LTL – spec automata

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Stochastic hybrid (discrete/continuous) systems

- discrete-time models

finite-space Markov chain

$(\mathcal{Z}, \mathcal{T})$

$\mathcal{Z} = (z_1, z_2, z_3)$

$$\mathcal{T} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

$P(z_1, \{z_2, z_3\}) = p_{12} + p_{13}$

uncountable-space Markov process

$(\mathcal{S}, T_s)$

$\mathcal{S} = \mathbb{R}^2$

$$T_s(x|s) = \frac{e^{-\frac{1}{2}(x-m(s))^T \Sigma^{-1}(s)(x-m(s))}}{\sqrt{2\pi}|\Sigma(s)|^{1/2}}$$

$P(s, A) = \int_A T_s(dx|s), \quad A \in \mathcal{B}(\mathcal{S})$

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$\Rightarrow$ discrete-time, stochastic hybrid systems

Stochastic hybrid (discrete/continuous) systems

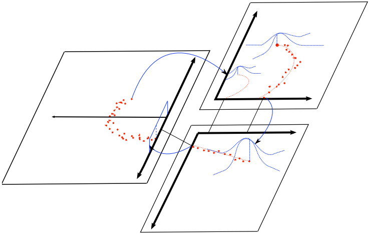
Definition

A discrete-time stochastic hybrid system is a pair $(\mathcal{S}, T_s)$, where

- $\mathcal{S} = \cup_{q \in \mathcal{Q}} (\{q\} \times \mathbb{R}^{n(q)})$, $\mathcal{Q}$ a discrete set of modes, $n : \mathcal{Q} \to \mathbb{N}$
- $T_s : \mathcal{S} \times \mathcal{S} \to [0,1]$ specifies the dynamics of process at point $s = (q, x)$:

$$T_s(ds'|s) = \begin{cases} T_x(dx'|(q,x))\, T_q(q|(q,x)), & \text{if } q' = q \text{ (no transition)} \\ T_r(dx'|(q,x), q')\, T_q(q'|(q,x)), & \text{if } q' \neq q \text{ (transition)} \end{cases}$$

- initial state $\pi : \mathcal{S} \to [0,1]$



[AA et al - Automatica 08]

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- can be control dependent ($u \in \mathcal{U}$):

$$T_s(ds' | s, u) = \begin{cases} T_x(dx' | (q,x), u) T_q(q | (q,x), u), & \text{if } q' = q \text{ (no transition)} \\ T_r(dx' | (q,x), u, q') T_q(q' | (q,x), u), & \text{if } q' \neq q \text{ (transition)} \end{cases}$$

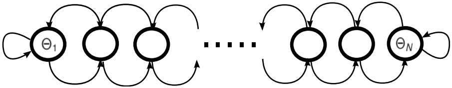
- policy μ: “string” of controls
- equivalent dynamical representation: $s_{k+1} = f(s_k, \xi_k, u_k)$
- related to other models, e.g. LMP

[AA et al - Automatica 08]

Stochastic hybrid systems in risk analysis

$$\begin{cases} Z_{n+1} = g(Z_n, \theta_n) & Z_n \in \mathbb{R}, & \leftarrow \text{ capital} \\ \theta_{n+1} = h(Z_n, \theta_n, \xi_n) & \theta_n \in \{\Theta_1, \ldots, \Theta_N\}, & \leftarrow \text{ interest} \end{cases}$$

where ξ_n i.i.d. random variables; g, h measurable; (Z_0, θ_0) given



[I. Tkachev, AA - CDC 11]

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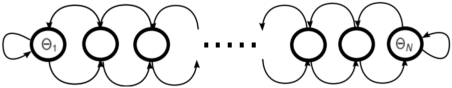

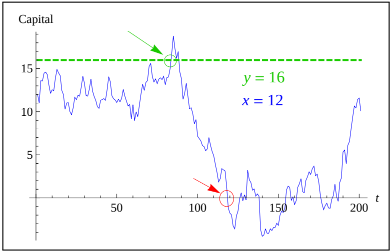

- **objective:** what is the probability that, starting from initial capital $Z_0 = x$, high capitalization y is reached, while company's bankruptcy is avoided

[I. Tkachev, AA - CDC 11]

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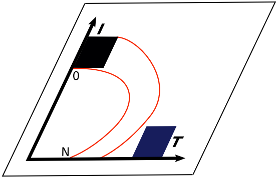
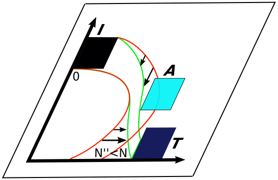
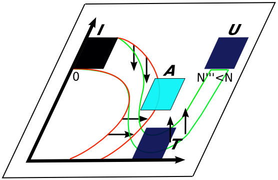


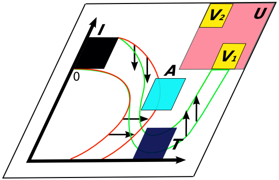
reachability
(safety/invariance)



reach-avoid
(constrained reachability)



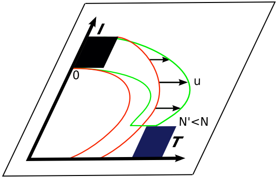
sequential reachability
(trajectory planning)



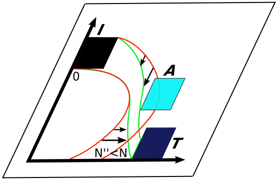
∞-horizon objectives
(i.o., eventually always)

- properties expressed via PCTL, LTL (DFA or Büchi automata)

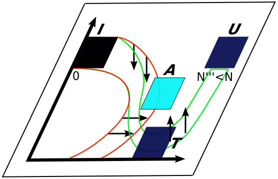
Analysis and control synthesis problems



synthesis for reachability games (2 − 1/2 players)



synthesis for reach-avoid (pursuit evasion games)



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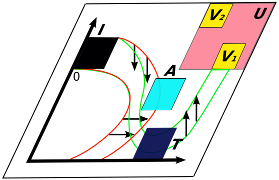

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Probabilistic safety/invariance: characterization

- probabilistic invariance is *the probability that the execution associated with an initial distribution* π *stays in* S *(safe set) during the time horizon* $[0, N]$:

$$\mathcal{P}_{\pi}(S) := P_{\pi}(s_k \in S, \forall k \in [0, N])$$

[AA et al. - Automatica 08]

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- consider realization $s_k \in \mathcal{S}$, $k \in [0, N]$ – then

$$\prod_{k=0}^{N} \mathbf{1}_S(s_k) = \begin{cases} 1, & \text{if } \forall k \in [0, N] : s_k \in S \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow \mathcal{P}_\pi(S) = P_\pi\left(\prod_{k=0}^{N} \mathbf{1}_S(s_k) = 1\right) = E_\pi\left[\prod_{k=0}^{N} \mathbf{1}_S(s_k)\right]$$

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- select $\epsilon \in [0, 1]$ – probabilistic safe/invariant set with safety level ϵ is

$$S(\epsilon) \doteq \{s \in \mathcal{S} : \mathcal{P}_s(S) \geq \epsilon\} \quad (\text{here } \pi = \delta_s)$$

[AA et al. - Automatica 08]

Probabilistic invariance: computation

- computation of $\mathcal{P}_s(S)$ (and thus of $S(\epsilon)$) via dynamic programming: sequential update, backward in time, of multi-stage value function

$$V_k(s) : [0, N] \times \mathcal{S} \to \mathbb{R}^+,$$

accounting for current and expected future rewards – in particular

$$V_N(s) = \mathbf{1}_S(s), \quad V_k(s) = \int_S V_{k+1}(x) T_s(dx|s)$$

$$\boxed{V_0(s) = \mathcal{P}_s(S) \Rightarrow S(\epsilon)}$$

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- control dependent models: find optimal policy μ, optimizing recursively over

$$V_k(s, u) : [0, N] \times \mathcal{S} \times \mathcal{U} \to \mathbb{R}^+$$

[AA et al. - Automatica 08]

Computing probabilistic invariance: issues

- issues
  1. non-standard (max, multiplicative) value functions
  2. continuous control space
  3. hybrid state space

$\Rightarrow$ solution of DP is seldom analytical

Computing probabilistic invariance: issues

- issues
  1. non-standard (max, multiplicative) value functions
  2. continuous control space
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$\Rightarrow$ solution of DP is seldom analytical

- numerical solutions are needed

$\Rightarrow$ problem # 1: difference between real solution and computed solution
(in verification and correct-by-design controller synthesis)

$\Rightarrow$ problem # 2: Bellman's *curse of dimensionality*
(state/control space gridding)

Outline

Dynamical properties as temporal specifications

PRISM
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Approximate model checking of probabilistic invariance

- model $(\mathcal{S}, T_s)$, invariance set $S \in \mathcal{S}$, finite time horizon N, safety level ϵ

[AA et al. - EJC 11]

Approximate model checking of probabilistic invariance

- model $(\mathcal{S}, T_s)$, invariance set $S \in \mathcal{S}$, finite time horizon N, safety level ϵ
- δ-approximate $(\mathcal{S}, T_s)$ with finite-state dt-MC $(\mathcal{Z}, \mathcal{T})$

$\star$ compute approximation error $f(\delta, N)$

- $S \to S_\delta$: define formula Φ_{S_δ} characterizing set S_δ, label states in $\mathcal{Z}$

[AA et al. - EJC 11]

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⇒ probabilistic safe set

$$\begin{aligned} S(\epsilon) &= \{s \in \mathcal{S} : \mathcal{P}_s(S) \geq \epsilon\} \\ &= \{s \in \mathcal{S} : (1 - \mathcal{P}_s(S)) \leq 1 - \epsilon\} \end{aligned}$$

[AA et al. - EJC 11]

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\end{aligned}
$$

can be related to

$$
\begin{aligned}
Z_\delta(\epsilon) &\doteq \mathrm{Sat}\left(\mathbb{P}_{\leq 1-\epsilon}\left(\mathtt{true}\ \mathcal{U}^{\leq N} \neg \Phi_{S_\delta}\right)\right) \\
&= \left\{z \in \mathcal{Z} : z \models \mathbb{P}_{\leq 1-\epsilon}\left(\mathtt{true}\ \mathcal{U}^{\leq N} \neg \Phi_{S_\delta}\right)\right\}
\end{aligned}
$$

[AA et al. - EJC 11]

Approximate model checking of probabilistic invariance

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1. define

$$S(\epsilon) = \{s \in \mathcal{S} : \mathcal{P}_s(S) \geq \epsilon\}$$
$$Z_\delta(\epsilon) = \mathrm{Sat}\left(\mathbb{P}_{\leq 1-\epsilon}\left(\mathtt{true}\ \mathcal{U}^{\leq N} \neg\Phi_{S_\delta}\right)\right)$$

2. select $\eta > 0 : \eta/2 \in (0, 1-\epsilon)$
3. pick $\delta : f(\delta, N) \leq \eta/2$
4. compute $Z_\delta(\epsilon + \eta/2)$
5. define $\hat{S}_\eta(\epsilon) \doteq \{s \in \mathcal{S} \leftrightarrow z \in Z_\delta(\epsilon + \eta/2)\}$

$\Rightarrow$

$$S(\epsilon + \eta) \subseteq \hat{S}_\eta(\epsilon) \subseteq S(\epsilon)$$

[AA et al. - EJC 11]

Verification of over- or under-specifications in PCTL

- any PCTL formula can be expressed via equivalent DP recursions

- consider PCTL formula $\mathbb{P}_{\sim\epsilon}(\Psi)$ on SHS $(\mathcal{S}, T_s)$
- δ-approximate SHS $(\mathcal{S}, T_s)$ as a dt-MC $(\mathcal{Z}, \mathcal{T})$
- compute approximation error $f(\delta, N)$

[D'Innocenzo, AA, J.-P. Katoen - HSCC 12]

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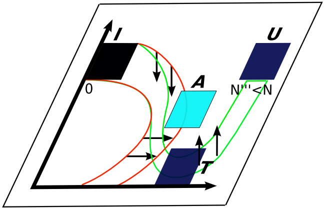
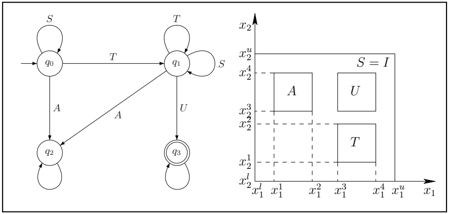
- compute $g(\Psi, f)$, a function based on formula & error
- model check $\mathbb{P}_{\sim\epsilon \pm g(\Psi,f)}(\Psi)$ on $(\mathcal{Z}, \mathcal{T})$

1. if PCTL formula is “robust”, then conclusion holds for $\mathbb{P}_{\sim\epsilon}(\Psi)$ on SHS
2. else refine $\delta \rightarrow$ reduce $f(\delta, N) \rightarrow$ decrease $g(\Psi, f)$

[D’Innocenzo, AA, J.-P. Katoen - HSCC 12]

Approximate model checking of automata specifications



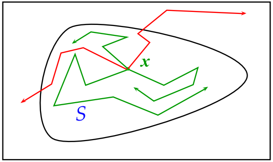


- generalization to “richer” set of properties over dtSHS
- specifications expressed as a DFA or a Büchi automata

- probabilistic reachability-like computation over product construction
- recent extensions to controller synthesis

[AA et al. - HSCC 11; I. Tkachev et al. - HSCC13]

Characterization & computation of ∞-horizon properties



- consider target set T; invariant set $S = T^c = \mathcal{S} \setminus T$; $\forall s \in \mathcal{S}$:

$$P_s(\forall n \geq 0 : s_n \in S) \quad \leftrightarrow \quad 1 - P_s(\texttt{true}\, \mathcal{U}\, T)$$

[I. Tkachev, AA - CDC 11, HSCC 12, CDC12,TCS 13]

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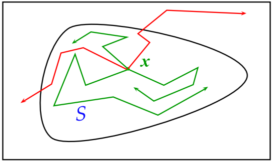

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$$P_s(\forall n \geq 0 : s_n \in S) \quad \leftrightarrow \quad 1 - P_s(\mathtt{true}\, \mathcal{U}\, T)$$

- existence and computation of absorbing set $B : \forall x \in B, T_s(B|x) = 1$

- characterization – study of existence/uniqueness of (non-trivial) solutions of Bellman equations

  convergence of Bellman recursions, contractivity of operators

- computation – formal reduction to finite-horizon problems

[I. Tkachev, AA - CDC 11, HSCC 12, CDC12,TCS 13]

On the approximation error $f(\delta, N)$

PRISM
MRMC

dtMC
dtMDP

relax'd/strenght'd PCTL
inflated LTL – ϵ-spec

prob. model checking

dynamic programming

ϵ-spec holds
policy max/min ϵ-spec

adaptive, sequential abstractions

approximate probabilistic bisimulations

refine back

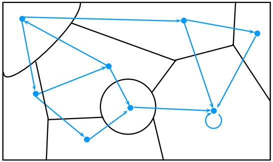
dtSHS

PCTL
LTL – spec
automata

spec holds
policy max/min spec

On the approximation error $f(\delta, N)$

- approximation via δ-partitioning: $S = \cup_{i=1,\ldots,m} \times S^i$



- under Lip-continuity assumptions on density of kernel T_s,

$$h(i,j), \qquad i,j = 1,\ldots,m$$

- for any $z_q^i \in S_\delta$, $\forall s : s \wedge z^i \in S^i$, error is

$$f(\delta, N) \doteq |\mathcal{P}_s(S) - \mathcal{P}_{z^i}(S_\delta)| \leq \max_{i=1,\ldots,m} N\delta_i \sum_{j=1,\ldots,m} h(i,j),$$

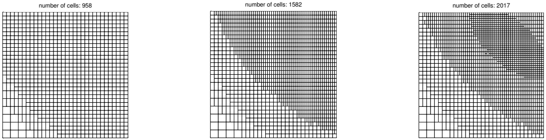
$\delta = \max_{i=1,\ldots,m} \delta_i$, $\delta_{q,i} = \text{diam}\,(S^i)$

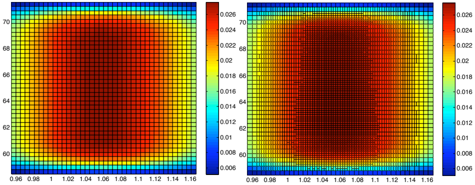
error is linear in N, δ_i and depends on local constants $h(i,j) \rightarrow$ local tuning

[AA et al. - EJC 11, S. Soudjani, AA - QEST 11, TAC 13]

On the approximation error $f(\delta, N)$

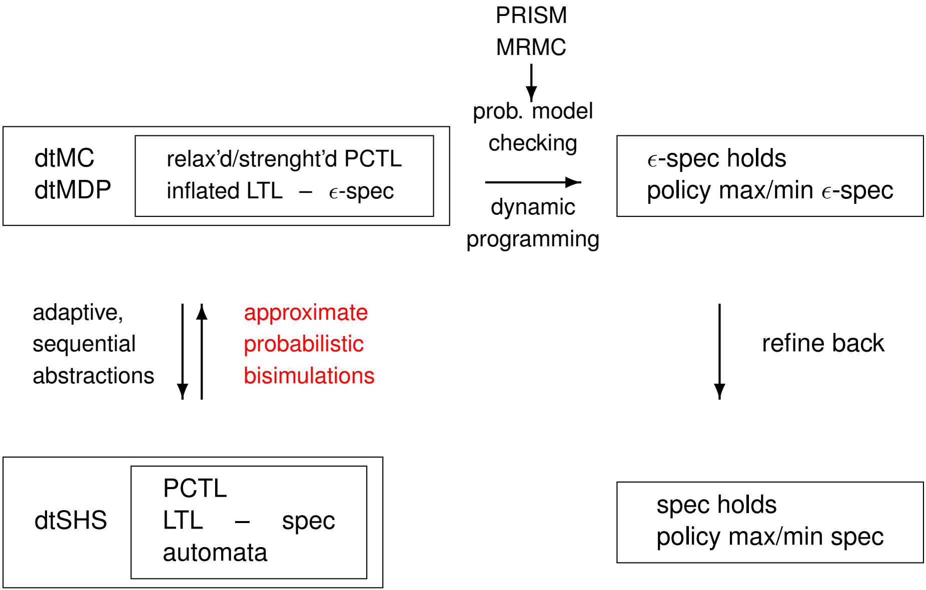
- formula-based abstractions
- software (in the making) for sequential, adaptive grid generation based on approximation error
- from MATLAB/Simuling model to MRMC/PRISM input





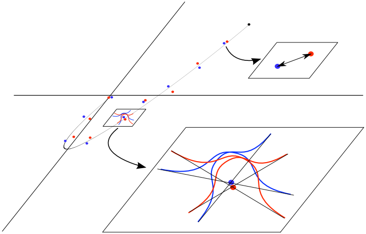
[S. Soudjani, AA - QEST 11, HSCC 12, ATVA12, SIAM 13]

Approximate probabilistic bisimulations



Approximate probabilistic bisimulations

- above abstraction leads to approximate probabilistic bisimulation *[Larsen & Skou, 91]* - alternatively . . .



- consider models $(T_{s,i}, \mathcal{S}_i)$ with solution processes $s_i(k), i = 1, 2, k \geq 0$
- parallel composition of models with output $s_{1,2}(k) = s_1(k) - s_2(k)$

[AA - ENTCS 13; I. Tkachev, AA - HSCC 13]

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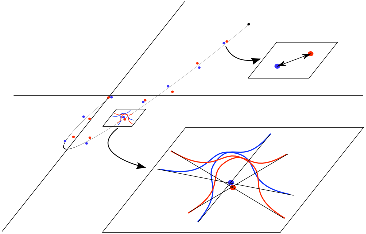

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- parallel composition of models with output $s_{1,2}(k) = s_1(k) - s_2(k)$

Definition

A function $\psi : \mathcal{S}_1 \times \mathcal{S}_2 \to \mathbb{R}^+$ is a probabilistic bisimulation function if $\psi(s_{1,2}) \geq \|s_1 - s_2\|^2$ and if $\psi_{s_0}(s_{1,2}(k))$ is a *supermartingale*.

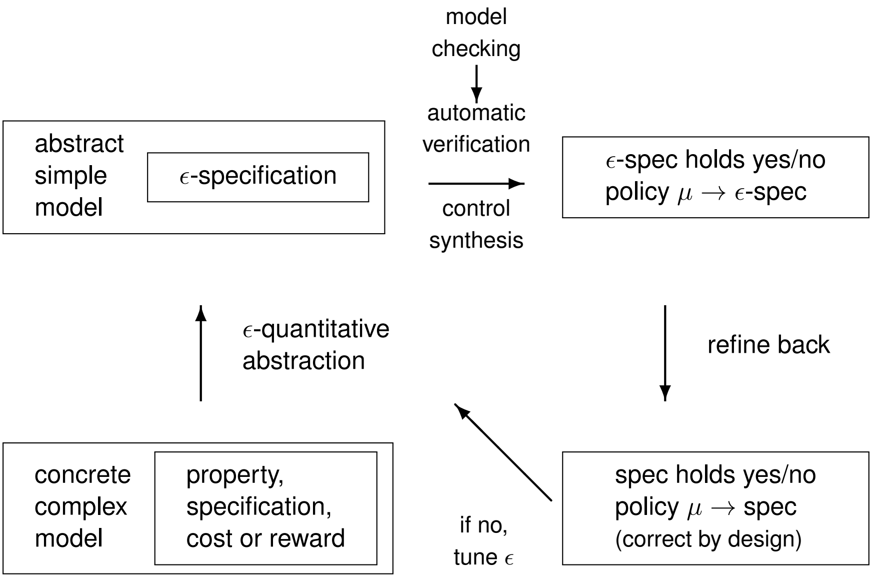
- ψ is an upper bound on the distance btw solutions of two models: $P_{s_0}\left(\sup_{k \geq 0} \|s_1(k) - s_2(k)\|^2 \geq \epsilon\right) \leq \psi_{s_0}(s_{1,2}(0))/\epsilon$

[AA - ENTCS 13; I. Tkachev, AA - HSCC 13]

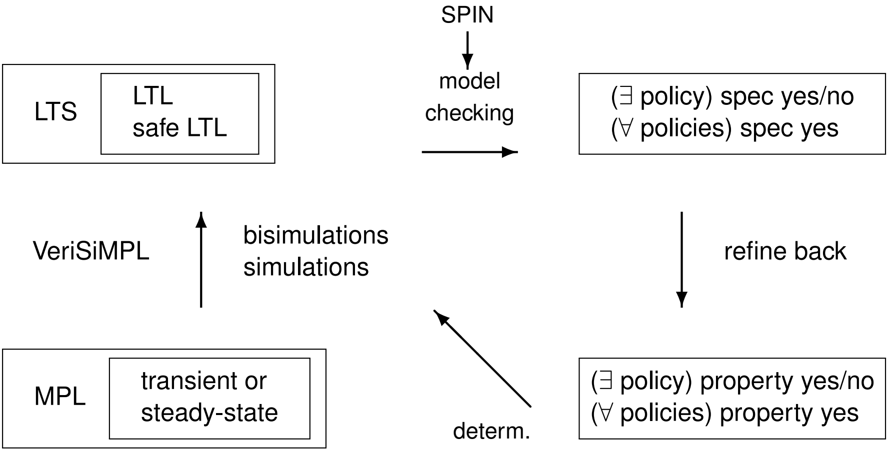
Outline

1. Formal abstractions for verification of complex models
2. Formal verification of stochastic hybrid systems
   - Analysis and control synthesis problems
   - Computable analysis and control synthesis via formal abstractions
3. Formal verification of max-plus linear models
   - Analysis and control synthesis problems
   - Computable analysis and control synthesis via formal abstractions
4. Concluding remarks

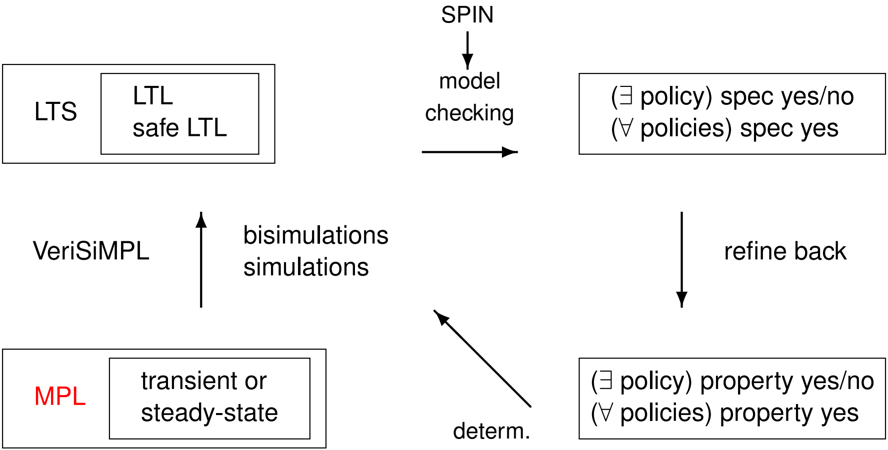
Formal abstractions for verification of complex models



Formal abstractions for verification of MPL models



Introduction to MPL systems



Introduction to MPL systems

- Max-Plus-Linear (MPL) systems are event-driven models
- applications: railway scheduling, planning of production lines, network calculus

- $x(k)$ is the time of k-th event, $k \in \mathbb{N} \cup \{0\}$
- timing updates: maximization ($\oplus$) and addition ($\otimes$) operations

$\rightarrow$ max-plus algebra

- $\epsilon = -\infty, \quad \mathbb{R}_\epsilon = \mathbb{R} \cup \{\epsilon\}, \quad \alpha, \beta \in \mathbb{R}_\epsilon$
- $\alpha \oplus \beta := \max(\alpha, \beta), \quad \alpha \otimes \beta := \alpha + \beta,$ and matrix operations

Max-plus-linear models

Definition (Autonomous MPL model)

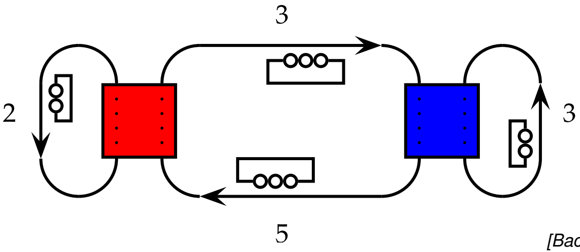
$$x(k+1) = A \otimes x(k),$$

where $A \in \mathbb{R}_{\epsilon}^{n \times n}$ and $k \in \mathbb{N} \cup \{0\}$

Example

A simple railway model *[Heidergott, 06]*

$$x(k+1) = \begin{bmatrix} 2 & 5 \\ 3 & 3 \end{bmatrix} \otimes x(k), \quad \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \max\{2 + x_1(k), 5 + x_2(k)\} \\ \max\{3 + x_1(k), 3 + x_2(k)\} \end{bmatrix}$$



[Baccelli et al., 92]

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Definition (Non-autonomous MPL model)

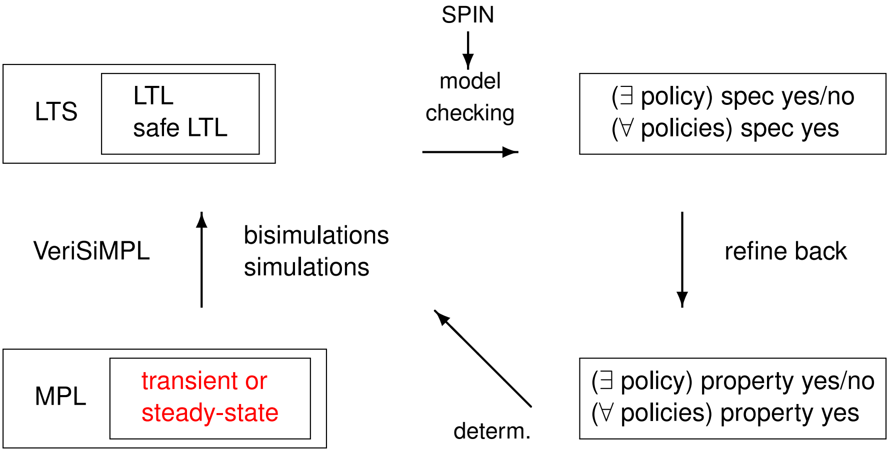
$$x(k+1) = A \otimes x(k) \oplus B \otimes u(k),$$

where $B \in \mathbb{R}_\epsilon^{n \times m}$ and $u \in \mathbb{R}^m$ (synthesis = scheduling)

[Baccelli et al., 92]

Outline

Classical analysis of MPL models



Classical analysis of MPL models

- study of transient and periodic regimes, of asymptotics
- classical analysis based on algebraic or geometric properties

Definition

1. max-plus eigenvector $x \in \mathbb{R}^n$: $A \otimes x = \lambda \otimes x \Rightarrow x(k+1) = \lambda \otimes x(k)$
2. cycles on precedence graph $\Rightarrow$ periodic regime with period c:
   $\forall k \geq k_0, x(k+c) = \lambda^{\otimes^c} \otimes x(k)$

Example

1. eigenspace (periodic regime with period 1 and $\lambda = 4$):

$$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}5\\4\end{bmatrix}, \begin{bmatrix}9\\8\end{bmatrix}, \begin{bmatrix}13\\12\end{bmatrix}, \begin{bmatrix}17\\16\end{bmatrix}, \begin{bmatrix}21\\20\end{bmatrix}, \begin{bmatrix}25\\24\end{bmatrix}, \begin{bmatrix}29\\28\end{bmatrix}, \begin{bmatrix}33\\32\end{bmatrix}, \begin{bmatrix}37\\36\end{bmatrix}, \begin{bmatrix}41\\40\end{bmatrix}, \begin{bmatrix}45\\44\end{bmatrix}, \ldots$$

2. periodic regime with period $c = 2$ (transient $k_0 = 3$):

$$\begin{bmatrix}4\\0\end{bmatrix}, \begin{bmatrix}6\\7\end{bmatrix}, \begin{bmatrix}12\\10\end{bmatrix}, \begin{bmatrix}15\\15\end{bmatrix}, \begin{bmatrix}20\\18\end{bmatrix}, \begin{bmatrix}23\\23\end{bmatrix}, \begin{bmatrix}28\\26\end{bmatrix}, \begin{bmatrix}31\\31\end{bmatrix}, \begin{bmatrix}36\\34\end{bmatrix}, \begin{bmatrix}39\\39\end{bmatrix}, \begin{bmatrix}44\\42\end{bmatrix}, \begin{bmatrix}47\\47\end{bmatrix}, \ldots$$

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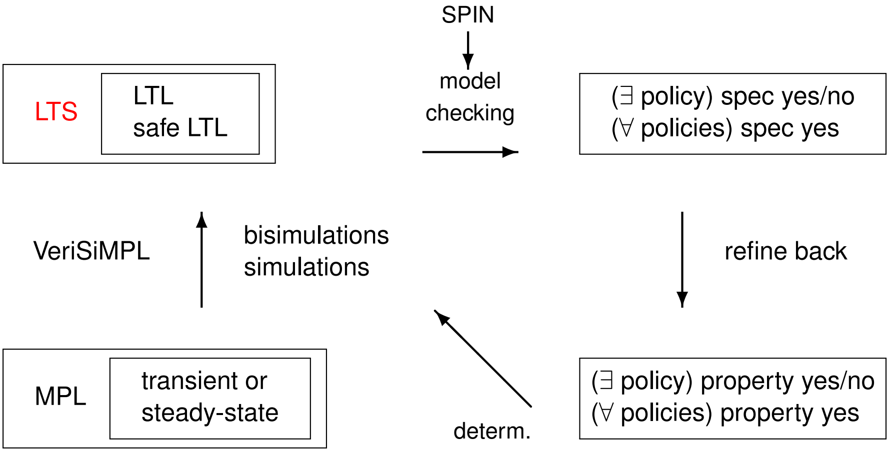
Labeled transition system (LTS)



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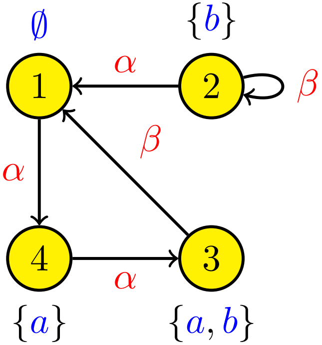
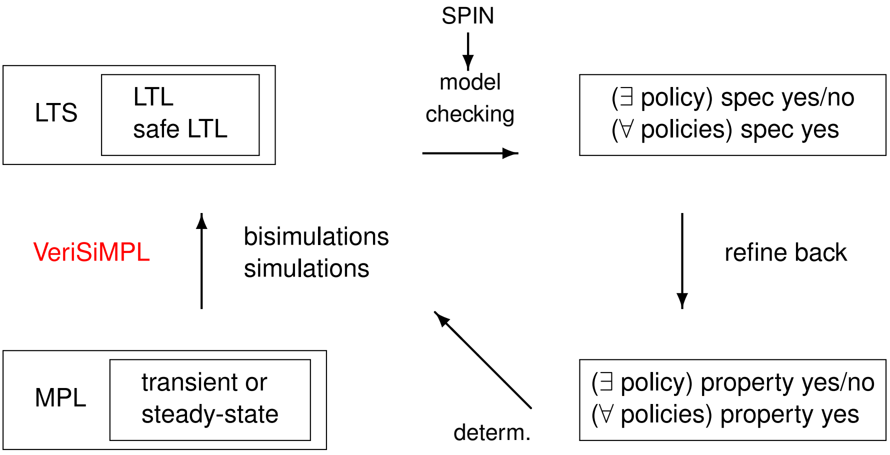


- set of states $S = \{1, 2, 3, 4\}$
- set of inputs $Act = \{\alpha, \beta\}$
- transitions $\longrightarrow = \{(1, \alpha, 4), (4, \alpha, 3), \ldots\}$
- set of outputs $AP = \{a, b\}$ and output map $L(1) = \emptyset$, $L(2) = \{b\}, \ldots$

- labels can be defined over states or transitions
- LTS can be deterministic vs non-deterministic
- LTS can be infinite vs finite

[Baier & Katoen, 08]

Finite LTS as abstractions of MPL models



- procedure: need to compute
  1. S: states of LTS
  2. $\rightarrow$: LTS transitions
  3. L: LTS labels

LTS states: partitioning of state space

- state space $\mathbb{R}^n$ is partitioned in finitely many polytopic regions
- partition is not arbitrary, it is adapted to underlying dynamics
- obtained state-space partition defines states of LTS
- partition can be possibly refined (*determinization* – more later)

Example

- we obtain a total of 5 regions:

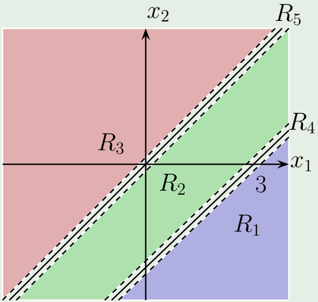
$$R_1 = \{x \in \mathbb{R}^2 : x_1 - x_2 < 0\}$$

$$R_2 = \{x \in \mathbb{R}^2 : x_1 - x_2 = 0\}$$

$$R_3 = \{x \in \mathbb{R}^2 : x_1 - x_2 > 3\}$$

$$R_4 = \{x \in \mathbb{R}^2 : x_1 - x_2 = 3\}$$

$$R_5 = \{x \in \mathbb{R}^2 : 0 < x_1 - x_2 < 3\}$$



Difference-bound matrices (DBM)

Definition (DBM)

A difference-bound matrix in $\mathbb{R}^n$ is the finite intersection of sets defined by

$$x_i - x_j \simeq_{i,j} \alpha_{i,j},$$

where $\simeq_{i,j} \in \{<, \leq\}$, $\alpha_{i,j} \in \mathbb{R} \cup \{+\infty\}$, for $1 \leq i \neq j \leq n$

- DBM allow compact matrix representation
- DBM are easy to manipulate (projections, emptiness and inclusion check)

[Dill, 90]

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- DBM allow compact matrix representation
- DBM are easy to manipulate (projections, emptiness and inclusion check)
- closure: image/inverse image of DBM over MPL dynamics is again a DBM

[Dill, 90]

LTS transitions: one-step reachability

- consider any two TS states (partitioning regions) R, R'
- $R \rightarrow R'$ iff there exists a $x(k) \in R$ such that $x(k+1) \in R'$: check

$$R' \cap \{x(k+1) : x(k) \in R\} \neq \emptyset$$

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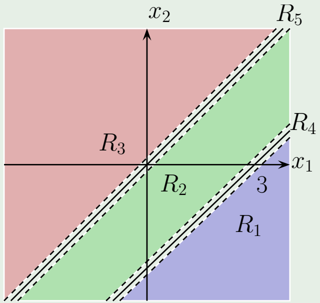
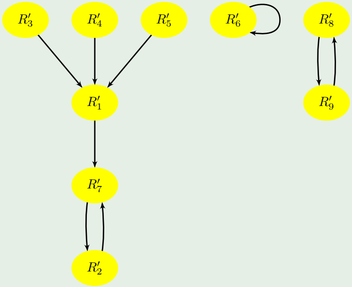
- computation of transitions:

  use region representation via DBM, DBM forward-mapping via PWA dynamics, DBM emptiness check
- transitions are stored on sparse Boolean matrix

LTS transitions, an example

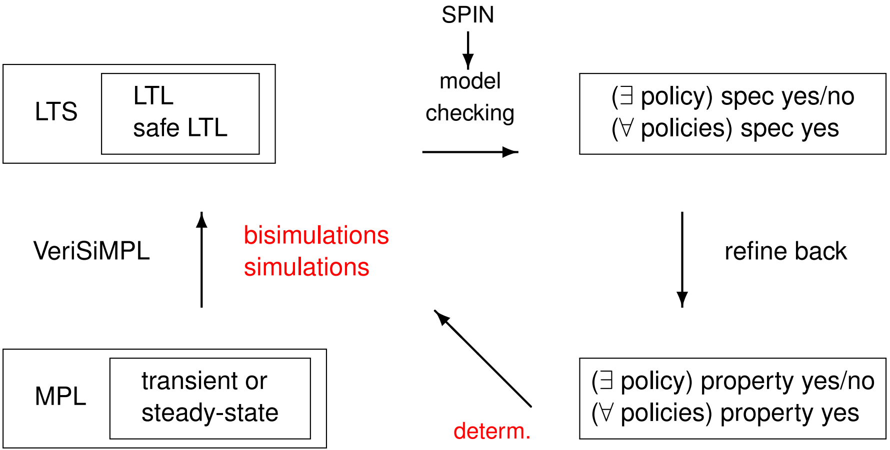
Example





- determinism vs non-determinism of obtained TS
- above R_i - original partitions, R'_i - refined partitions (determinization)

Relationship between LTS and MPL



Relationship between LTS and MPL

Theorem

- *TS simulates the original MPL model*
- *TS bisimulates the MPL model if and only if it is deterministic*

- non-deterministic TS can be “determinized” by refining partitioning regions
- however, refinement procedure may not terminate

Theorem

- *if TS is deterministic over the periodic regime, then TS is globally deterministic*
- *every irreducible MPL model admits finite deterministic TS abstraction*

LTS labels

Definition

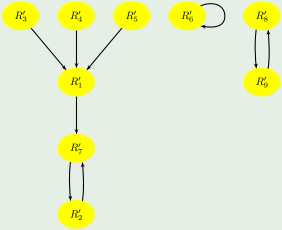
- state labels:
  all possible values of $x_i(k) - x_j(k)$, for $1 \leq i < j \leq n$
  time difference of same-event variables
- transition labels:
  all possible values of $x_i(k+1) - x_i(k)$, for $1 \leq i \leq n$
  time difference of successive events

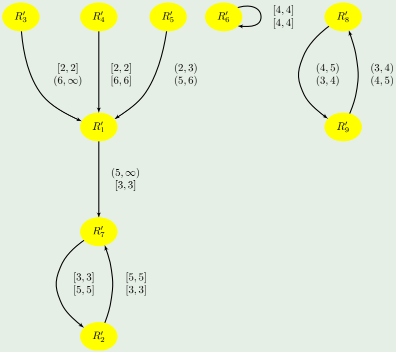
- labels are vectors of intervals, can be represented as DBM

LTS labels, an example

Example

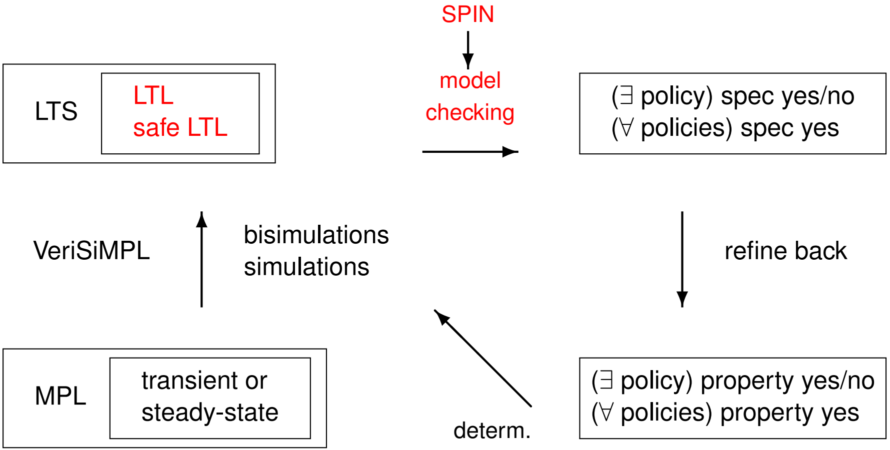
- LTS transition labels





Formal analysis of MPL models is now "very simple"

VeriSiMPL – Verification via biSimulation of MPL models



Formal analysis of MPL models is now “very simple”

VeriSiMPL – Verification via biSimulation of MPL models

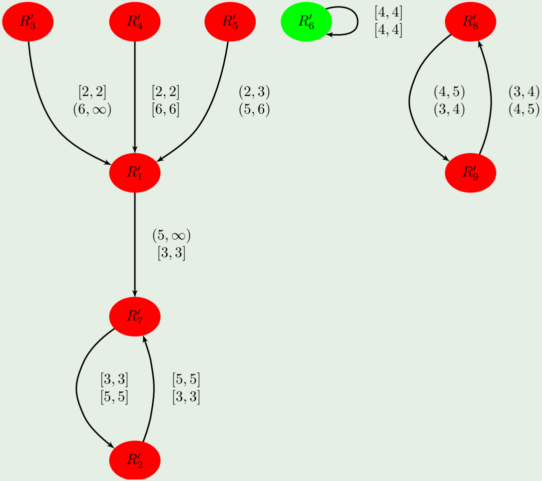
- abstract MPL model as LTS (in MATLAB)
- export LTS abstraction (as PROMELA script) into SPIN model checker
- consider properties in LTL logic
- verify property via SPIN over LTS and export outcome back to MPL model

http://sourceforge.net/projects/verisimpl

MPL verification in practice

Example

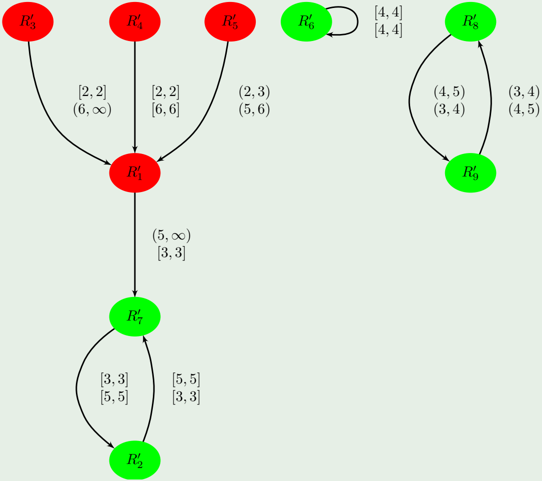
- automatically identify MPL eigenspace: $\bigvee_{\varphi \in L=AP}(\Box\varphi \ \wedge \ |\varphi| = 0)$



MPL verification in practice

Example

- automatically identify MPL periodic regime: $\Psi = \bigvee_{\varphi \in L = AP} \Box(\varphi \wedge \bigcirc^c \varphi)$



Computational benchmark for abstraction

- coded in MATLAB, run over 12-core Intel Xeon, 3.47 GHz, 24 GB
- A randomly generated with elements taking values between 1 and 100
- 10 independent experiments per dimension – mean values are displayed:

size of MPL model	time for generation of states	time for generation of transitions	time for generation of labels	total number of LTS states	total number of LTS transitions
3	0.1 [s]	0.4 [s]	0.1 [s]	3.6	4.3
5	0.2 [s]	0.4 [s]	0.1 [s]	8.6	13.8
7	0.9 [s]	0.5 [s]	0.3 [s]	37.2	289.3
9	4.1 [s]	0.8 [s]	1.6 [s]	120.0	$1.7 \cdot 10^3$
11	24.8 [s]	15.2 [s]	16.1 [s]	613.2	$1.9 \cdot 10^4$
13	3.5 [m]	5.5 [m]	2.8 [m]	$1.9 \cdot 10^3$	$1.9 \cdot 10^5$
15	53.6 [m]	2.0 [h]	39.4 [m]	$7.4 \cdot 10^3$	$2.0 \cdot 10^6$

- bottleneck: generation of transitions

Computational benchmark for reachability analysis

- *A* randomly generated with elements taking values between 1 and 100
- set of initial conditions is selected as the unit hypercube
- 10 independent experiments per dimension – mean values are displayed:

size of MPL model	time for generation of abstract TS	number of regions of abstract TS	time for generation of reach tube
3	0.09 [s]	5	0.09 [s]
10	4.73 [s]	700	8.23 [s]
19	67.07 [m]	$3.48 \cdot 10^5$	7.13 [h]

- generation time for reach tube of 10-dimensional MPL model, different time horizons
- comparison VeriSiMPL vs MPT (multi-parametric tool, ETH Zürich):

time horizon	20	40	60	80	100
VeriSiMPL	11.02 [s]	17.94 [s]	37.40 [s]	51.21 [s]	64.59 [s]
MPT	47.61 [m]	1.19 [h]	2.32 [h]	3.03 [h]	3.73 [h]

Stochastic Max-plus-linear models

Definition (Deterministic MPL model)

$$x(k+1) = A \otimes x(k),$$

where $A \in \mathbb{R}_{\epsilon}^{n \times n}$ and $k \in \mathbb{N} \cup \{0\}$

Definition (Stochastic MPL model)

$$x(k+1) = A \otimes x(k),$$

where $A(k) = [a_{ij}(k)]_{i,j} \in \mathbb{R}_{\epsilon}^{n \times n}$, $\{a_{ij}(k)\}_k$ are i.i.d. random processes with pdf $t_{ij}(\cdot)$, and $k \in \mathbb{N} \cup \{0\}$

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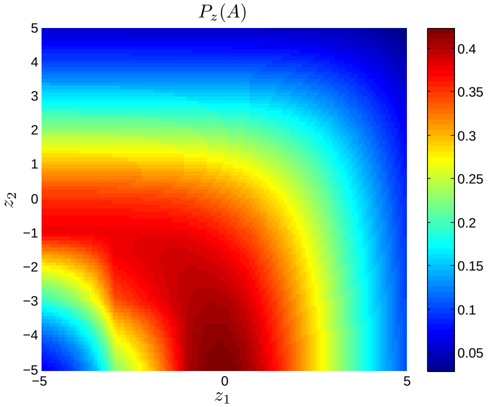
- abstraction of SMPL models as Markov chains
- can be obtained in two possible ways:
  1. leveraging theory above, under continuity assumptions on kernels $t_{ij}(\cdot)$
  2. by symbolic approach over distributions that are closed under max-plus algebra operations
- error quantification

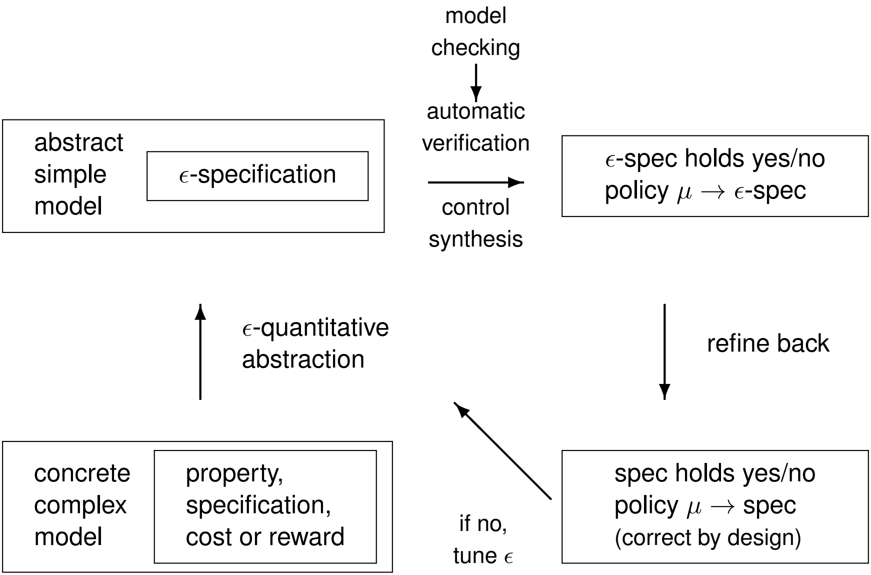
Simulations over 2D SMPL model

- exponential distributions (rates btw $1/3$ and 1) for the entries of $2D$ matrix A
- pick time horizon $N = 5$, safe set $\mathcal{A} = [-5, 5]^2$
- select $(3700, 2900)$ bins per dimension, partition uniformly
- abstraction error results in $E = 32.5\delta < 0.1$



Outline

Formal abstractions for verification of complex models



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- topics: stochastic hybrid systems, max-plus linear models

Thanks for your attention!

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