Compression of enumerations and gain

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Recursion Theory and its Applications
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Joint work with George Barmpalias, Bohua Zhan

For $\sigma \in 2^{<\omega}$, let $C(\sigma)$ (or $K(\sigma)$) be its plain (or prefix-free) Komogorov complexity

$$\min\{|\rho| : \rho \in 2^{<\omega}, U(\rho) \downarrow = \sigma\}$$

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Given a set $A \subset \omega$, let $A \upharpoonright_n$ be the first n bits of A. This can be regarded as the set $A \cap [0, n)$, or the string

$$A(0)A(1) \cdots A(n-1).$$

Roughly speeking, $n \mapsto C(A \upharpoonright_n)$ describes how the complexity of A grows.

Definition

Given $A, B \subset \omega$.

- $A \leq_C B$ if $C(A\restriction_n) \leq C(B\restriction_n) + O(1)$.
- $A \leq_K B$ if $K(A\restriction_n) \leq K(B\restriction_n) + O(1)$.
- $A \leq_{rK} B$ if $K(A\restriction_n | B\restriction_n) \leq O(1)$.
- $A \leq_{cl} B$ if $A \leq_T B$ with oracle use bounded by $n \mapsto n + O(1)$.
- $A \leq_{ibT} B$ if $A \leq_T B$ with oracle use bounded by indentity.

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Fact

$$ibT \Rightarrow cl \Rightarrow rK \Rightarrow C, K$$

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Given $A, B \subset \omega$. $A \leq_{rK} B$ if and only if there is a partial recursive function $f : 2^{<\omega} \times \omega \to 2^{<\omega}$ and a constant k such that one of

$$f(B\restriction_n, 0), f(B\restriction_n, 1), \cdots, f(B\restriction_n, k)$$

halts and outputs $A\restriction_n$.

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For rK, K and C-degrees, the question is open.

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Given r.e. sets A and B with $B < A$. Construct r.e. D which contains "part" of information in A. We would want $B < B \oplus D < A$, so we need

- $D \leq A$,
- $B \oplus D \not\leq B$,
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However, we don't know a "join" (least upper bound) operator for r.e. rK-degrees.

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Fact

Given r.e. set A and B. If $A' \oplus \varnothing, B' \oplus \varnothing$ are strong gainless compression of A, B, respectively, then $A' \oplus B'$ is a least upper bound of A and B in rK-degrees.

Theorem

Given r.e. set A and B with $B <_r A$. If both A, B have strong gainless compression, then there is a r.e. C such that $B <_r C <_r A$, where r is any of rK, K or C.

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Proof for the *rK* case.

Take $A' \oplus \varnothing, B' \oplus \varnothing$ to be strong gainless compressions of A, B.
We shall enumerate $D' \subset A'$, and wish that $B <_{rK} B' \oplus D' <_{rK} A$.
For the e^{th} rK functional Φ_e, define the length of agreement

$$p_s(e) = \max\{\ell : B' \oplus D' \upharpoonright_\ell \in \Phi_e(B \upharpoonright_\ell) \text{ at some stage } t \leq s\}$$

$$q_s(e) = \max\{\ell : A \upharpoonright_\ell \in \Phi_e(B' \oplus D' \upharpoonright_\ell) \text{ at some stage } t \leq s\}$$

Proof for the rK case.

When a is enumerated into A' at stage s,

- if $a < p_s(e)$ then the condition P_e wants to enumerate a in D',
- if $a < q_s(e)$ then the condition N_e wants to skip a.

Let the conditions act according to the following priority

$$P_0 > N_0 > P_1 > N_1 > \cdots > P_e > N_e > \cdots .$$

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$$A \equiv_{rK} \varnothing \oplus A' \leq_{rK} \varnothing \oplus D' \leq_{rK} B' \oplus D' \leq_{rK} B,$$

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$$A \leq_{rK} B' \oplus D' \leq_{rK} B' \oplus \varnothing \equiv_{rK} B.$$

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A useful Lemma:

Lemma

Given r.e. set A, D. $A \leq_{rK} D$ if and only if there is recursive enumerations $(A_s), (D_s)$ and a constant k, such that for all $s > t$ and all n,

$$|A_s - A_t \upharpoonright_n| \geq k \Rightarrow |D_s - D_t \upharpoonright_n| \geq 1.$$

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$$|A_s - A_t \restriction_n| \geq k \Rightarrow |D_s - D_t \restriction_n| \geq 1.$$

Roughly speeking, within the same range $[0, n]$, for each k elements enumerated into A, we need at least 1 element enumerated into D.

Theorem

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Proof.

For every 8 numbers of $[0, 2^{n+2})$ enumerated into A, enumerated an even number of $[2^n, 2^{n+1})$ into D.

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Verification.

- $A \leq_{rK} D$,
- D does not run out of number.

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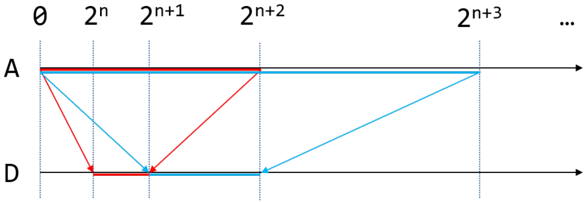

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Theorem (Barmpalias, Hölzl, Lewis, Merkle, 2013)

Every rK-complete set has a strong gainless compression.

Definition

Given r.e. set A. We call a r.e. set D a **compression** of A if $A \leq_{rK} D$.

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Theorem

Each r.e. set A has a weak gainless compression D.

Game

Player A and D play on counters $a_0, a_1, \cdots, a_n, \cdots$, which are initially 0. *In each round (stage),*

- *Player A enumerates n. For all $i \geq n$, set negative counters to* 0 *and add* 1 *to the positive counters.*
- *Player D may enumerate m. For all $i \geq m$, set positive counters to* 0 *and substract* 1 *to the negative counters.*

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Fact

- *If* (a_i) *has an uniform upper bound, then* $A \leq_{rK} D$*.*
- *If* (a_i) *has an uniform lower bound, then* $D \leq_{rK} A$*.*

Theorem

Each r.e. set A has a weak gainless compression D.

Proof.

When some $a_n \geq 8$, find the smallest m such that

$$a_i \geq 4 \text{ for all } m \leq i \leq n$$

and enumerate it into D.

Verification.

- Each number is enumerated into D only once.
- $0 \leq a_n \leq 8$ throughout the construction.
- $|D \restriction_n| \leq |A \restriction_n|/2$.

□

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If (a_n) *is uniformly bounded then player D wins. Otherwise player A wins.*

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Game

Consider game G_k with player A and D. In each round (stage),

- *Player A chooses k numbers $n_1 < n_2 < \cdots < n_k$.*
- *Player D chooses an even number in $[n_1, n_k]$.*

Each player cannot choose the numbers he has chosen. If player D runs out of number, then player A wins. Otherwise player D wins.

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Player A wins G_2 and G_3.

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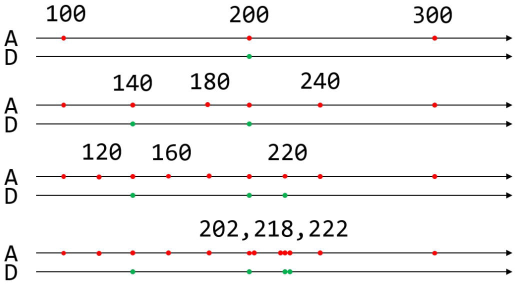
Question

Is there a k such that player D wins G_k? And what is the winning strategy?

Winning strategy for player A in G_3.

- *Player A: 100, 200, 300. Player D: 200.*
- *Player A: 140, 180, 240. Player D: 140.*
- *Player A: 120, 160, 220. Player D: 220.*
- *Player A: 202, 218, 222. Player D: 222.*
- *Player A: 219, 221, 223.*



Game

Consider game G'_k where player A and D play on a single string s, which is initially empty. In each round (stage),

- *Player A insert k many O into the string.*
- *Player D change one of the O in the string to X.*
- *Player A may cut s at both ends.*

If $k-1$ successive X appears, player A wins. Otherwise Player D wins.

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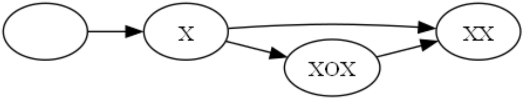
The above argument showed that player A wins G'_3.



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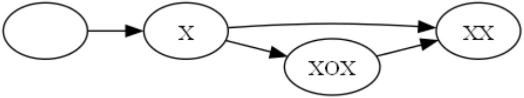
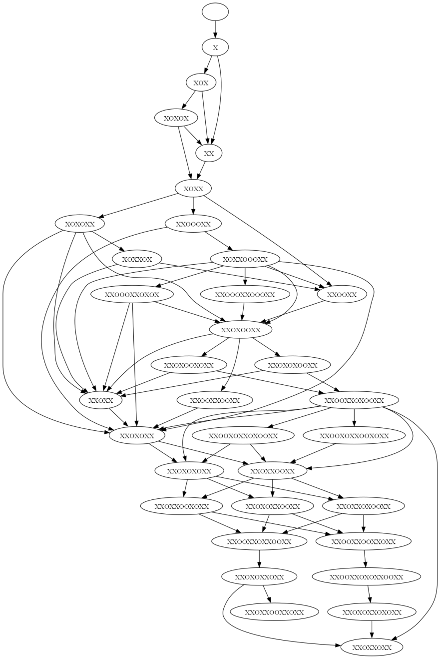


If player D wins G'_k, then he wins G_k.

Fact

Player A wins G'_4.



To summarize,

- the structures of r.e. rK, K, C degrees are dense among sets that have strong gainless compression.
- Each r.e. set has a strong compression.
- Each r.e. set has a weak gainless compression.
- Simply presented games related to a strong gainless compression.

Thanks!