Dimensionality and Randomness

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Joint work with George Barmpalias

Notations. We use σ, τ to denote finite binary sequences (strings). Let 2^n be the set of strings of length n.
We use x, y to denote infinite binary sequences (reals).
Let $x \upharpoonright_n$ be the first n bits of x. Let $\prec$ be the prefix relation among strings and reals.

Definition

Let U be an universal prefix-free Turing machine. The **Kolmogorov Complexity** of a string σ is the length of the shortest program that outputs σ, i.e.

$$K(\sigma) := \min\{|\tau| : U(\tau) = \sigma\}.$$

The **deficiency** of σ is the number of bits it can be best compressed, i.e. $\mathrm{d}(\sigma) := |\sigma| - K(\sigma)$.
The **deficiency** of a set of strings V is $\mathrm{d}(V) = \sup_{\sigma \in V} \mathrm{d}(\sigma)$.
A set of strings V is **incompressible** if $\mathrm{d}(V) < \infty$.

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Definition

A real x is **random** if $K(x \upharpoonright_n) \geq^+ n$.
Equivalently, x is **random** if $\{x \upharpoonright_n : n \in \omega\}$ is incompressible.

Observation

*Among incomplete incompressible sets, a **thin** one cannot compute a **fat** one.*

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Example

Let $x = 0011\ 0011\ 0001\ 0110\ \cdots$ be an incomplete random real, let

$x_0 = 1011\ 0011\ 0001\ 0110\ \cdots$

$x_1 = 0111\ 0011\ 0001\ 0110\ \cdots$

$x_2 = 0001\ 0011\ 0001\ 0110\ \cdots$

Let $V = \{x \upharpoonright_n : n \in \omega\}$, $W_i = \{x_i \upharpoonright_n : n \in \omega\}$ and $W = \bigcup W_i$.

V is "thin" in the sense that $|V \cap 2^n| = 1$;

W is "fat" in the sense that $|W \cap 2^n| = n$;

Each x_i is random, so V and W_i are incompressible.

However, W cannot be incompressible.

An **order** is a non-decreasing unbounded function.

Theorem (Barmpalias, Z., 2024)

An incomplete random real z cannot compute an incompressible set V with $n \mapsto |V \cap 2^n|$ being an order.

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Proof Sketch.

Suppose otherwise, let Φ be such that $\Phi(z)$ is incompressible by constant k and $n \mapsto |\Phi(z) \cap 2^n|$ is an order.
Let $P = \{x : \mathrm{d}(\Phi(x)) \leq k\}$ and $\Phi^{-1}(\sigma) = \{x : \sigma \in \Phi(x)\}$.
By a Lemma, $\mu(P \cap \Phi^{-1}(\sigma)) \leq^{\times} 2^{-|\sigma|}$.
Let $Q_n^m = \{x : |\Phi(x) \cap 2^n| \geq 2^m\}$, then $\mu(P \cap Q_n^m) \leq^{\times} 2^{-m}$.
P is Π^0_1 and Q_n^m is uniformly Σ^0_1.
$z \in P$ and for each m there is n with $z \in Q_n^m$.
Use $P \cap Q_n^m$ to carefully build a difference-test. □

Theorem (Barmpalias, Z., 2024)

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Proof Sketch.

Use z as a random oracle.
Randomly choose $l_n \in [2^{n-1}, 2^n]$, randomly choose n strings of length l_n and put them in V.
Probability that we fail at level $n \leq^{\times} 2^{-n}$. Since z is random we only fail at finitely many n, so V is incompressible. □

A set V is g-**fat** if $\sup_{i \leq n} |V \cap 2^i| \geq g(n)$.

Theorem (Barmpalias, Z., 2024)

If g is a computable order with $\lim_n n/g(n) = 0$, then an incomplete random real cannot compute an g-fat incompressible set.

Theorem (Barmpalias, Z., 2024)

Every random real z computes an $n/(\log n)^2$-fat incompressible set.

A **tree** is a non-empty set of strings T such that $\sigma \prec \tau$ and $\tau \in T$ then $\sigma \in T$. A real x is a **path** of T if $x \upharpoonright_n \in T$ for all n. Let $[T]$ be the set of paths through T. A tree T is

- **pruned** if each $\sigma \in T$ has an extension in T;

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All trees are assumed to be pruned. A tree T is

- **proper** if $|T \cap 2^n|$ is unbounded;
- **perfect** if each $\sigma \in T$ has two incomparable extensions in T;
- **positive** if $|T \cap 2^n| \geq c \cdot 2^n$ for some $c > 0$.

Corollary (Barmpalias, Z., 2024)

An incomplete random real z cannot compute a proper incompressible tree.

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Theorem (Barmpalias, Wang, 2023)

There is a perfect incompressible tree which cannot compute any positive incompressible tree.

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Consider the following principles in RCA:

- WKL: every infinite tree has a path;
- P^+: every positive tree has a positive perfect subtree;
- P: every positive tree has a perfect subtree;
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Theorem (Barmpalias, Wang, 2023)

Each of the following extensions of RCA *has an* ω*-model:*

- WWKL $+ \neg P^-$;
- $P + \neg P^+$;
- $P^+ + \neg$WKL.

Question

Is there an ω-model of $\mathrm{RCA} + \mathrm{P}^- + \neg P$, i.e. an ω-model where every positive tree has a countable family of paths, but some positive tree does not have a perfect subtree?

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Conjecture

There is a proper incompressible tree which cannot compute any perfect incompressible tree.

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A tree is T **skeletal** if there is $x \in [T]$ such that $\sigma \in T$ has two incomparable extensions in T if and only if $\sigma \prec x$.

Conjecture

A skeletal incompressible tree cannot compute a perfect incompressible tree unless it is complete.

Let $\mathbf{M}$ be the universal left-c.e. continuous semi-measure. For a Π^0_1 class P with associated co-c.e. tree T, P is **deep** if there is computable f such that

$$\mathbf{M}(T \cap 2^{f(n)}) \leq 2^{-n}.$$

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- The class of effectively proper incompressible trees.
- The class of complete extensions of PA.

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- The class of complete extensions of PA.

Theorem (Bienvenu, Porter, 2017)

An incomplete random cannot compute any member of any deep Π^0_1 class.

Fact

The class of effectively proper incompressible trees is a deep Π^0_1 class.
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Theorem (Barmpalias, Z., 2024)

There exists a perfect incompressible tree which is not a member of any deep Π^0_1 class.

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Theorem (Barmpalias, Z., 2024)

There exists a perfect incompressible tree which is not a member of any deep Π^0_1 class.

Question

Is there a reasonable way to define a deep Π^0_2 class?

Thanks!